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Optimal Domain and Integral Extension of Operators

Acting in Function Spaces

Susumu Okada
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Dedicated to our three pillars of strength:
Eileen, Margit and María José

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Preface

Operator theory and functional analysis have a long tradition, initially being guided by problems from mathematical physics and applied mathematics. Much of the work in Banach spaces from the 1930s onwards resulted from investigating how much real (and complex) variable function theory might be extended to functions taking values in (function) spaces or operators acting in them. Many of the first ideas in geometry, basis theory and the isomorphic theory of Banach spaces have vector measure-theoretic origins and can be credited (amongst others) to N. Dunford, I.M. Gelfand, B.J. Pettis and R.S. Phillips. Somewhat later came the penetrating contributions of A. Grothendieck, which have pervaded and influenced the shape of functional analysis and the theory of vector measures/integration ever since.

Today, each of the areas of functional analysis/operator theory, Banach spaces, and vector measures/integration is a strong discipline in its own right. However, it is not always made clear that these areas grew up together as cousins and that they had, and still have, enormous influences on one another. One of the aims of this monograph is to reinforce and make transparent precisely this important point.

The monograph itself contains mostly new material which has not appeared elsewhere and is directed at advanced research students, experienced researchers in the area, and researchers interested in the interdisciplinary nature of the mathematics involved. We point out that the monograph is self-contained with complete proofs, detailed references and (hopefully) clear explanations. There is an emphasis on many and varied examples, with most of them having all the explicit details included. They form an integral part of the text and are designed and chosen to enrich and illuminate the theory that is developed. Due to the relatively new nature of the material and its research orientation, many open questions are posed and there is ample opportunity for the diligent reader to enter the topic with the aim of "pushing it further". Indeed, the theory which is presented, even though rather complete, is still in its infancy and its potential for further development and applications to concrete problems is, in our opinion, quite large. As can be gleaned from the above comments, the reader is assumed to have some proficiency in certain aspects of general functional analysis, linear operator theory, measure theory and the theory of Banach spaces, particularly function spaces. More specialized material is introduced and developed as it is needed.

Chapter 1 should definitely be read first and in its entirety. It explains completely and in detail what the monograph is all about: its aims, contents, philosophy and the whole structure and motivation that it encompasses. The actual details and the remainder of the text then follow in a natural and understandable way.

Chapter 2 treats certain aspects of the class of spaces on which the linear operators in later chapters will be defined. These are the quasi-Banach *function* spaces (over a finite measure space), with the emphasis on $\mathbb{C}$-valued functions. This is in contrast to much of the existing literature which is devoted to spaces over $\mathbb{R}$. Of particular importance is the notion of the *p-th power* $X_{[p]}$, $0 < p < \infty$, of a given quasi-Banach function space X. This associated family of quasi-Banach function spaces $X_{[p]}$, which is intimately connected to the base space X, is produced via a procedure akin to that which produces the Lebesgue L^p-spaces from L^1 and plays a crucial role in the sequel.

Let X be a quasi-Banach function space and $T : X \to E$ be a continuous linear operator, with E a Banach space. Then $m_T : A \mapsto T(\chi_A)$, for A a measurable set, defines a finitely additive E-valued vector measure with the property that $T(s) = \int s \, dm_T$ for each $\mathbb{C}$-valued simple function $s \in X$. Under appropriate conditions on X and/or T, it follows that m_T is actually σ-additive and so $\int f \, dm_T \in E$ is defined for each m_T-integrable function $f \in L^1(m_T)$. The crucial point is that the linear map $I_{m_T} : f \mapsto \int f \, dm_T$ is then typically defined for many more functions $f \in L^1(m_T)$ than just those coming from the domain X of T (which automatically satisfies $X \subseteq L^1(m_T)$). That is, I_{m_T} is an E-valued *extension* of T. This is one of the crucial points that pervades the entire monograph. In order to fully develop this idea, it is necessary to make a detailed study of the Banach lattice $L^1(m_T)$ and the related spaces $L^p(m_T)$, for $1 \leq p \leq \infty$, together with various properties of the integration operator $I_{m_T} : L^1(m_T) \to E$. This is systematically carried out in Chapter 3, not just for m_T but, for general E-valued vector measures ν. Of course, there is already available a vast literature on the theory of vector measures and integration, of which we surely make good use. However, for applications in later chapters we need to further develop many new aspects concerning the spaces $L^p(\nu)$ and the integration operators $I_\nu^{(p)} : L^p(\nu) \to E$, for $1 < p < \infty$, in particular, their ideal properties in relation to compactness, weak compactness and complete continuity.

Chapter 4 presents a detailed and careful analysis of the particular extension $I_{m_T} : L^1(m_T) \to E$ of an operator $T : X \to E$. In addition to always existing, the remarkable feature is that $L^1(m_T)$ and I_{m_T} turn out to be *optimal*, in the sense that if Y is any σ-order continuous quasi-Banach function space (over the same measure space as for X) for which $X \subseteq Y$ continuously and such that there exists a continuous linear operator $T_Y : Y \to E$ which coincides with T on X, then necessarily Y is continuously embedded in the *Banach* function space $L^1(m_T)$

and T_Y coincides with I_{m_T} restricted to Y. The remainder of the monograph then develops and applies the many consequences that follow from the existence of this optimal extension.

For instance, a continuous linear operator $T : X \to E$ is called *p-th power factorable*, for $1 \le p < \infty$, if there exists a continuous linear operator $T_{[p]} : X_{[p]} \to E$ which coincides with T on $X \subseteq X_{[p]}$. There is no a priori reason to suspect any connection between the p-th power factorability of T and its associated E-valued vector measure m_T. The purpose of Chapter 5 is to show that such a connection does indeed exist and that it has some far-reaching consequences. The spaces $L^p(m_T)$, for $1 \le p < \infty$, which are treated in Chapter 3, also exhibit certain optimality properties and play a vital role in Chapter 5. Many operators of interest coming from various branches of analysis are p-th power factorable; quite some effort is devoted to studying these in detail.

According to the results of Chapter 5, a p-th power factorable operator $T : X \to E$ factorizes through $L^p(m_T)$. However, whenever possible, the factorization of T through a *classical* L^p-space (i.e., of a scalar measure) is preferable. The Maurey–Rosenthal theory provides such a factorization, albeit under various assumptions. In Chapter 6 we investigate norm inequalities in the following sense. Let $0 < q < \infty$ and $1 \le p < \infty$ be given and X be a q-convex quasi-Banach function space. Then T is required to satisfy

$$\Big(\sum_{j=1}^{n} \|T(f_j)\|_E^{q/p}\Big)^{1/q} \le C\Big\|\Big(\sum_{j=1}^{n} |f_j|^{q/p}\Big)^{1/q}\Big\|_X \tag{0.1}$$

for some constant $C > 0$ and all finite collections $f_1, \dots, f_n$ of elements from X. This inequality already encompasses the main geometrical condition of p-th power factorability. In fact, we will treat a more general inequality which can be applied even when X is not q-convex, namely

$$\Big(\sum_{j=1}^{n} \|T(f_j)\|_E^{q/p}\Big)^{1/q} \le C\Big\|\sum_{j=1}^{n} |f_j|^{q/p}\Big\|_{\mathrm{b},X_{[q]}}^{1/q}, \tag{0.2}$$

where $\|\cdot\|_{b,X_{[q]}}$ is a suitable *seminorm* which is continuous and defined on the quasi-Banach function space $X_{[q]}$. If X is q-convex, then (0.1) and (0.2) are equivalent; if, in addition, $p = 1$, then (0.2) reduces to q-concavity of T. In this case, we are in the (abstract) setting of the Maurey–Rosenthal Theorem, a version due to A. Defant. The main aim of Chapter 6 is to establish various equivalences with (0.2), thereby providing factorizations for a *subclass* of the p-th power factorable operators for which the required hypotheses of the Maurey–Rosenthal Theorem are not necessarily fulfilled. In view of the fact that the spaces $L^p(m_T)$ are always p-convex, this has some interesting consequences for the associated integration operator $I_{m_T}^{(p)} : L^p(m_T) \to E$.

Chapter 7 is the culmination of all of the ideas in this monograph applied to two important classes of operators arising in classical harmonic analysis (over a compact abelian group G). One class of operators consists of the *Fourier transform* F from $X = L^p(G)$, for $1 \leq p < \infty$, into either $E = c_0(\Gamma)$ or $E = \ell^{p'}(\Gamma)$, where Γ is the dual group of G and $\frac{1}{p} + \frac{1}{p'} = 1$. Concerning the optimal domain $L^1(m_F)$ and an analysis of the extension $I_{m_F} : L^1(m_F) \to E$, the situation depends rather dramatically on whether $p = 1$ or $1 < p < \infty$ and whether $E = c_0(\Gamma)$ or $E = \ell^{p'}(\Gamma)$. This is particularly true of the ideal properties of I_{m_F} and whether or not the inclusion $L^p(G) \subseteq L^1(m_F)$ is proper. The second class of operators consists of the *convolution operators* $C_\lambda : L^p(G) \to L^p(G)$ defined via $f \mapsto \lambda * f$, where λ is any $\mathbb{C}$-valued, regular Borel measure on G. Again there are significant differences depending on whether $p = 1$ or $1 < p < \infty$ and whether λ is absolutely continuous or not with respect to Haar measure on G. Also relevant are those measures λ whose Fourier–Stieltjes transform $\widehat{\lambda}$ belongs to $c_0(\Gamma)$. There is also a difference for certain features exhibited between the two classes of operators. For instance, with $E = \ell^{p'}(\Gamma)$ the Fourier transform map $F : L^p(G) \to E$ is (for certain groups G) never q-th power factorable for any $1 \leq q < \infty$. On the other hand, the q-th power factorability of the convolution operator $C_\lambda : L^p(G) \to L^p(G)$ is completely determined as to whether or not λ is a so-called *L^r-improving measure* (a large class of measures introduced by E.M. Stein). Many other questions are also treated, e.g., when is $L^1(m_{C_\lambda}) = L^1(G)$ as large as possible, when is $L^1(m_{C_\lambda}) = L^p(G)$ as small as possible, what happens if the vector measure m_F or m_{C_λ} has finite variation, and so on?

During the preparation of this monograph (circa 3 years!) we have been helped by many colleagues. A partial list of those to whom we are especially grateful for their special mathematical effort is: O. Blasco, G.P. Curbera, A. Defant, P.G. Dodds, F. Galaz-Fontes, G. Mockenhaupt, B. de Pagter and L. Rodríguez-Piazza. We also thank J. Calabuig for his invaluable technical assistance in relation to LaTeX and "matters computer". The first author (S. Okada) also wishes to acknowledge the support of the Katholische Universität Eichstätt-Ingolstadt (via the Maximilian Bickhoff-Stiftung and a 19 month Visiting Research Professorship), the Universidad Politécnica de Valencia (for a 12 month extended research period via grants CTESIN-2005-025, Generalitat Valenciana, MTM2006-11690-C02-01, Ministerio de Educación y Ciencia, Spain, FEDER, and 2488-I+D+I-2007-UPV), the Universidad de Sevilla (for a 12 month extended research period via the Spanish Government Grant DGU # SAB 2004-0206) and the Centre for Mathematics and its Applications at the Australian National University in Canberra. And, of course, for the real driving force, motivation and moral support needed to undertake and complete such an extensive project, we are all totally indebted to our wonderful partners Eileen O'Brien, Margit Kollmann and María José Arnau, respectively.

The Authors November, 2007

Chapter 1

Introduction

Let $g \in L^2([0,1])$ and fix $2 < r < \infty$. The corresponding *multiplication operator* $T : L^r([0,1]) \to L^1([0,1])$ defined by $T : f \mapsto gf$, for $f \in L^r([0,1])$, is surely continuous. Indeed, if $\frac{1}{r} + \frac{1}{r'} = 1$, then $g \in L^{r'}([0,1])$ and so, by Hölder's inequality

$$\begin{aligned} \|Tf\|_{L^1([0,1])} &= \int_0^1 |f(t)| \cdot |g(t)| \, dt \\ &\leq \|g\|_{L^{r'}([0,1])} \|f\|_{L^r([0,1])}, \end{aligned}$$

for $f \in L^r([0,1])$. Accordingly, $\|T\| \leq \|g\|_{L^{r'}([0,1])}$. Somehow, this estimate awakens a "feeling of discontent" concerning T. It seems more natural to interpret T as being defined on the larger domain space $L^2([0,1])$, that is, still as $f \mapsto gf$ (with values in $L^1([0,1])$) but now for all $f \in L^2([0,1])$. Again Hölder's inequality ensures that the extended operator $T : L^2([0,1]) \to L^1([0,1])$ is continuous with $\|T\| = \|g\|_{L^2([0,1])} \geq \|g\|_{L^{r'}([0,1])}$. Of course, for each $2 \leq p \leq r$, we can also consider the $L^1([0,1])$-valued operator T as being defined on $L^p([0,1])$. In the event that $g \notin \bigcup_{2<q<\infty} L^q([0,1])$, the space $L^2([0,1])$ is the "best choice" of domain space for T amongst all $L^p([0,1])$-spaces in the sense that it is the *largest* one that T maps into $L^1([0,1])$. For, suppose that there exists $p \in [1,2]$ such that $gf \in L^1([0,1])$ for all $f \in L^p([0,1])$, in which case $T : f \mapsto gf$ is continuous by the Closed Graph Theorem. Since $\xi : h \mapsto \int_0^1 h(t)\, dt$ is a continuous linear functional on $L^1([0,1])$ it follows that the composition $\xi \circ T : f \mapsto \int_0^1 f(t)g(t)\, dt$ is a continuous linear functional on $L^p([0,1])$ and so $g \in L^{p'}([0,1])$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Since $2 \leq p' \leq \infty$, this is only possible if $p = 2$ (by the assumption on g). Can there exist another function space Z (over $[0,1]$), even larger than $L^2([0,1])$, which contains $L^r([0,1])$ continuously and such that the initial operator $T : L^r([0,1]) \to L^1([0,1])$ has a continuous $L^1([0,1])$-valued extension to Z? If so, then we have just seen that Z cannot be an $L^p([0,1])$-space.

Consider now the *kernel operator* $T : L^2([0,1]) \to L^2([0,1])$ given by

$$Tf : t \mapsto \int_0^t f(s)\,ds, \qquad t \in [0,1], \quad f \in L^2([0,1]). \tag{1.1}$$

For each $t \in [0,1]$, Hölder's inequality gives

$$\left| \int_0^t f(s)\,ds \right| \le \int_0^1 |f(s)|\,ds \le \|f\|_{L^2([0,1])} \tag{1.2}$$

and so $\left(\int_0^1 |(Tf)(t)|^2\,dt\right)^{1/2} \le \|f\|_{L^2([0,1])}$; this shows that T is continuous with $\|T\| \le 1$. For $f \in L^1([0,1])$, we see from (1.2) that $|(Tf)(t)| \le \|f\|_{L^1([0,1])}$ for all $t \in [0,1]$ and so $\left(\int_0^1 |(Tf)(t)|^2\,dt\right)^{1/2} \le \|f\|_{L^1([0,1])}$. Hence, the operator T can be continuously extended to the larger domain space $L^1([0,1]) \supseteq L^2([0,1])$ while maintaining its values in $L^2([0,1])$. Now, the function $f(s) := 1/(1-s)$, for $s \in [0,1)$, does *not* belong to $L^1([0,1])$. However, for each $0 < t < 1$ we have $\int_0^t f(s)\,ds = -\ln(1-t)$ and an integration by parts (twice) shows that

$$\int_0^1 \left| \int_0^t f(s)\,ds \right|^2 dt = \int_0^1 \ln^2(1-t)\,dt = 2 < \infty.$$

Accordingly, Tf is defined pointwise on $[0,1)$ by (1.1) and satisfies $Tf \in L^2([0,1])$. So, the vector space Z consisting of all measurable functions f on $[0,1]$ such that $T|f|$ is defined pointwise a.e. on $[0,1]$ (via (1.1)) and belongs to $L^2([0,1])$, has the property that it contains both $L^2([0,1])$ and $L^1([0,1])$ and is *genuinely larger* than $L^1([0,1])$. Can we put a topology on Z so that Z becomes complete and $T : Z \to L^2([0,1])$ becomes continuous? If so, then again Z is, in some sense, the largest function space which contains the domain space of the original operator $T : L^2([0,1]) \to L^2([0,1])$ and such that T has a continuous $L^2([0,1])$-valued extension to Z.

The phenomena exhibited by the two operators T above, namely the possibility to extend T to some sort of maximal or optimal domain (but, keeping the codomain space fixed) has been studied in various contexts within the general theory of kernel operators, differential operators etc. for some time; see for example [8], [24], [50], [93], [114], [126], [155], [156] and the references therein. In more recent years this idea has also turned out to be fruitful in the investigation of other large classes of operators (in addition to kernel and differential operators), such as convolutions, the Fourier transform and the Sobolev embedding; see [25], [28], [29], [113], [123], [124], for example. Let us formulate the basic idea more precisely. Given are a function space $X(\mu)$ equipped with a linear topology ($\mu \ge 0$ is some finite measure), a Banach space E and a continuous linear operator $T : X(\mu) \to E$. One seeks the largest function space $Z(\mu)$ over μ, usually within a given class, for which $X(\mu) \subseteq Z(\mu)$ with a continuous inclusion and such that there exists a continuous linear operator from $Z(\mu)$ into E (again denoted by T) which coincides with the initial operator T on $X(\mu)$. If it exists, $Z(\mu)$ is called the *optimal domain* for T (within the given class). The aim is then to answer certain natural questions: which properties of T on $X(\mu)$ are transferred (or not) to the extended operator T on $Z(\mu)$, can one identify $Z(\mu)$ more concretely, what properties does $Z(\mu)$ have, does the space $Z(\mu)$ always exist, is it always genuinely larger than $X(\mu)$, and so on? Of course, one hopes to better understand the initial operator T on $X(\mu)$, whenever one has "good information" about its maximal extension $T : Z(\mu) \to E$. These aims form one of the central themes of this monograph and are worth elaborating on in more detail. In particular, it will (hopefully!) become apparent that the ideas are not as abstract as the first impression above may suggest, that they have some worthwhile and far-reaching consequences, and that they apply to a large, diverse and interesting class of operators which arise in various branches of analysis.

So, what is this "mysterious" optimal domain space? To describe it more explicitly, let (Ω, Σ, μ) be a positive, finite measure space and $X(\mu)$ be a complete space of $\mathbb{C}$-valued functions on Ω (relative to some lattice quasi-norm $\|\cdot\|_{X(\mu)}$ for the pointwise μ-a.e. order) such that the space $\operatorname{sim}\Sigma$, consisting of all Σ-simple functions, is contained in $X(\mu)$ and the quasi-norm $\|\cdot\|_{X(\mu)}$ is *σ-order continuous* (i.e., $\lim_{n\to\infty} \|f_n\|_{X(\mu)} = 0$ whenever $f_n \downarrow 0$ in $X(\mu)$). As a consequence of the σ-order continuity of $\|\cdot\|_{X(\mu)}$, the space $\operatorname{sim}\Sigma$ is necessarily dense in $X(\mu)$. Let E be a (complex) Banach space and $T : X(\mu) \to E$ be a continuous linear operator. Then the finitely additive set function $m_T : \Sigma \to E$ defined by

$$m_T(A) := T(\chi_A), \qquad A \in \Sigma, \tag{1.3}$$

is actually σ-additive, again due to the σ-order continuity of $\|\cdot\|_{X(\mu)}$, that is, m_T is an E-valued *vector measure*. Moreover, for each $s \in \operatorname{sim}\Sigma$, it follows from (1.3) that

$$\int_\Omega s\, dm_T = T(s) \tag{1.4}$$

with the integral $\int_\Omega s\, dm_T \in E$ defined in the obvious way. Assume now that μ and m_T have the same null sets, a mild restriction in practise. Then it is possible to extend (1.4) from $\operatorname{sim}\Sigma$ to the space of all *m_T-integrable functions*, denoted by $L^1(m_T)$ and equipped with an appropriate lattice norm $\|\cdot\|_{L^1(m_T)}$ (see (1.9) below with $\nu := m_T$), in such a way that:

(T-1) $L^1(m_T)$ is a Banach function space (over μ), with σ-order continuous norm and containing $X(\mu)$ continuously and densely,

(T-2) T has an E-valued, continuous linear extension to $L^1(m_T)$, namely the *integration operator* $I_{m_T} : f \mapsto \int_\Omega f\, dm_T$, and

(T-3) if $Y(\mu)$ is any quasi-Banach function space (over μ) with a σ-order continuous quasi-norm and containing $X(\mu)$ continuously such that T has an E-valued, continuous extension from $X(\mu)$ to $Y(\mu)$, then $Y(\mu) \subseteq L^1(m_T)$ continuously.

Remarkably, under the mild assumptions on $X(\mu)$ and T mentioned above, we automatically have the *existence* (unique up to isomorphism) of the optimal domain space of T, namely $L^1(m_T)$ and, simultaneously, also the *maximal extension* of T, namely I_{m_T}. As a bonus, much of the general theory of vector measures and integration needed is already available and rather well developed, [42], [86]. However, to apply it explicitly in the above context requires us to further develop some new aspects of the theory. This is the aim of Chapter 3. In particular, we require the theory of the spaces

$$L^p(m_T) := \{f\ \Sigma\text{-measurable}\ : |f|^p \in L^1(m_T)\}, \tag{1.5}$$

for $1 \le p < \infty$ (needed in Chapter 5 and later chapters), equipped with the norm

$$\|f\|_{L^p(m_T)} := \big\| |f|^p \big\|_{L^1(m_T)}^{1/p}, \qquad f \in L^p(m_T). \tag{1.6}$$

We also develop important criteria related to various ideal properties of integration operators, such as compactness, weak compactness and complete continuity.

Another important notion that is mentioned above, if only in passing, is that of a *quasi-Banach function space.* The general theory of quasi-Banach spaces is by now well established; see [83], [87], [88], [135], for example, and the references therein. However, we need its specialization to *function spaces*, where the existing literature is rather sparse. Moreover, those results which are actually available, are often only for spaces over $\mathbb{R}$. For applications occurring later in the book, we really need the theory over $\mathbb{C}$. Accordingly, a major effort is invested in Chapter 2 with the aim of developing the theory of quasi-Banach function spaces over $\mathbb{C}$, at least to the extent needed in later chapters. Perhaps the most important concept in this chapter is that of the *p-th power*

$$X(\mu)_{[p]} := \{f\ \Sigma\text{-measurable}\ : |f|^{1/p} \in X(\mu)\} \tag{1.7}$$

of a given quasi-Banach function space $X(\mu)$, defined for $0 < p < \infty$ and equipped with the quasinorm

$$\|f\|_{X(\mu)_{[p]}} := \big\| |f|^{1/p} \big\|_{X(\mu)}^{p}, \qquad f \in X(\mu)_{[p]}. \tag{1.8}$$

These spaces, introduced in [61, Definition 1.9] and [30, p. 156], are crucial for the factorization results of Chapters 5–7. It is clear from (1.5) and (1.6) that the Banach function spaces $L^p(m_T) = \big(L^1(m_T)\big)_{[1/p]}$ for $1 \le p < \infty$ are particular examples of such spaces.

As also noted above, one is under the assumption that μ and m_T have the same null sets. This property assures that $L^1(m_T)$ is a Banach function space relative to the *same* scalar measure μ over which the domain space $X(\mu)$ of T is a quasi-Banach function space; when this property holds, we say that the operator $T : X(\mu) \to E$ is *μ-determined.* For a general vector measure $\nu : \Sigma \to E$ it is difficult to identify the Banach function space $L^1(\nu)$ in a more concrete way, other than that due simply to its definition. Namely, a Σ-measurable function $f : \Omega \to \mathbb{C}$ is *ν-integrable* whenever $\int_\Omega |f|\, d|\langle \nu, x^*\rangle| < \infty$ for each x^* belonging to the dual space E^*, of E (the space of such functions is denoted by $L^1_{\mathrm{w}}(\nu)$) and, additionally, for each $A \in \Sigma$ there exists a vector in E, denoted by $\int_A f\, d\nu$, satisfying

$$\Big\langle \int_A f\, d\nu,\ x^* \Big\rangle = \int_A f\, d\langle \nu, x^*\rangle, \qquad x^* \in E^*.$$

Here, $\langle \nu, x^*\rangle$ denotes the complex measure $A \mapsto \langle \nu(A), x^*\rangle$ and $|\langle \nu, x^*\rangle|$ is its variation measure. The norm in $L^1(\nu)$ is then given by

$$\|f\|_{L^1(\nu)} := \sup_{\|x^*\|_{E^*} \le 1} \int_\Omega |f|\, d|\langle \nu, x^*\rangle|, \qquad f \in L^1(\nu). \tag{1.9}$$

However, for vector measures of the particular form $\nu = m_T$, for some μ-determined operator $T : X(\mu) \to E$, it is often possible to identify $L^1(m_T)$ more explicitly, which then provides the possibility to study the maximal extension $I_{m_T} : L^1(m_T) \to E$, of T, in some detail. Chapter 4 is devoted to an analysis of the optimal domain spaces $L^1(m_T)$ and the extended operators I_{m_T} in the setting of μ-determined operators T. Many (and varied) examples are presented, each one exhibiting its own particular features. For instance, for operators T of the form

$$(Tf)(x) := \int_0^1 K(x,y) f(y)\, d\mu(y), \quad x \in [0,1],$$

acting on functions $f : [0,1] \to \mathbb{R}$ coming from some Banach function space $X(\mu)$, where μ is Lebesgue measure in $[0,1]$ and $K : [0,1] \times [0,1] \to \mathbb{R}$ is a suitable kernel, it is possible to characterize precisely when T is μ-determined in terms of certain properties of K. Moreover, in this case, it is often also possible (e.g., if $X(\mu)$ is rearrangement invariant) to describe $L^1(m_T)$ rather precisely via interpolation theory (e.g., using the **K**-functional of Peetre), [24]. A particular instance is the Volterra operator (i.e., the kernel operator T given by (1.1)), studied in [129]. An important example from classical analysis, which fits precisely into the above context, arises from the Sobolev kernel (introduced in [50]); see [25]. The approach to analyzing this Sobolev kernel operator via the methods suggested above, that is, based on an investigation of its optimal domain, its maximal extension operator and vector integration, has already been carried out with significant consequences for compactness and other properties of the Sobolev embedding in rearrangement invariant function spaces over bounded domains in $\mathbb{R}^n$, [28], [29]. A further important class of operators, arising in harmonic analysis and which is susceptible to our methods, consists of the Fourier transform maps from $L^p(G)$ to $\ell^{p'}(\Gamma)$ and convolution operators from $L^p(G)$ to $L^p(G)$ for $1 \le p < \infty$, where G is a compact abelian group with dual group Γ; these are treated separately and in depth in Chapter 7. The main feature of Chapter 4 is, of course, the "Optimality Theorem" itself, as formulated via (T-1)–(T-3) above. But, this is not all:

a natural question is whether the optimal domain $L^1(m_T)$ is always larger than $X(\mu)$, that is, whether the integration operator I_{m_T} is a *genuine* extension of T? In general, no! For instance, whenever $X(\mu)$ is a Banach function space and the μ-determined operator $T : X(\mu) \to E$ is *semi-Fredholm*, then it is shown that $L^1(m_T) = X(\mu)$, that is, T is already defined on its optimal domain and no further E-valued extension is possible. This result is applied to the finite Hilbert transform in $L^p((-1,1))$ for $p \neq 2$, certain Volterra type operators, and multiplication operators which arise as the "evaluation of a spectral measure", for example. It is also used in Chapter 7 to show that various convolution operators are already defined on their optimal domain (e.g., all translation operators in $L^p(G)$ for $1 \le p < \infty$). In the last part of Chapter 4 there is also a discussion of operators $T : X(\mu) \to E$ which are not μ-determined. It is shown that these can actually be reduced to μ-determined operators if we are prepared to work with the restriction of T to a closed (even complemented) subspace of $X(\mu)$. Curiously, there also exist non-trivial quasi-Banach function spaces $X(\mu)$ on which *no* μ-determined operators exist at all (into any non-zero Banach space E)! For instance, this is the case if $X(\mu)$ has trivial dual.

Chapter 5 is, perhaps, the core of the monograph. Let $X(\mu)$ be a quasi-Banach function space (over a finite, positive measure space (Ω, Σ, μ)), henceforth assumed to have a σ-order continuous quasi-norm $\|\cdot\|_{X(\mu)}$. Its p-th powers $X(\mu)_{[p]}$ (see (1.7)) satisfy $X(\mu) \subseteq X(\mu)_{[p]}$ for $1 \le p < \infty$ and $X(\mu)_{[p]} \subseteq X(\mu)$ for $0 < p \le 1$, with continuous inclusions; see Chapter 2. Moreover, the σ-order continuity of the quasi-norm $\|\cdot\|_{X(\mu)_{[p]}}$ (see (1.8)) is inherited from that of $X(\mu)$. Let $1 \le p < \infty$ and E be a Banach space. A continuous linear operator $T : X(\mu) \to E$ is called *p-th power factorable* if there exists a continuous linear operator $T_{[p]} : X(\mu)_{[p]} \to E$ which coincides with T on $X(\mu) \subseteq X(\mu)_{[p]}$. In other words, $T_{[p]}$ is an E-valued continuous linear extension of T to the quasi-Banach function space $X(\mu)_{[p]}$ (in which $X(\mu)$ is always dense). Hence, the following diagram commutes:

$$
\begin{tikzcd}
X(\mu) \arrow[rr, "T"] \arrow[dr, "i_{[p]}"'] & & E \\
& X(\mu)_{[p]} \arrow[ur, "T_{[p]}"'] &
\end{tikzcd}
\tag{1.10}
$$

where $i_{[p]} : X(\mu) \to X(\mu)_{[p]}$ denotes the natural (and continuous) injection. Of course, (1.10) is equivalent to the existence of a constant $C > 0$ such that

$$
\|Tf\|_E \le C\,\|f\|_{X(\mu)_{[p]}} = C\,\big\|\,|f|^{1/p}\big\|_{X(\mu)}^p, \qquad f \in X(\mu). \tag{1.11}
$$

It should be mentioned that there exist operators $T : X(\mu) \to E$ which are not p-th power factorable for any $1 < p < \infty$ and others which are p-th power factorable for some, but not all p. Since $X(\mu)_{[q]} \subseteq X(\mu)_{[p]}$ continuously whenever $q \in [1,p]$, it follows that T is q-th power factorable for all $q \in [1,p]$ whenever it is p-th power factorable.

There is no a priori reason to suspect any connection whatsoever between the p-th power factorability of an operator $T : X(\mu) \to E$ and its associated vector measure m_T as specified by (1.3). So, it is somewhat remarkable that there is indeed a connection between the two and, with some far-reaching consequences. Now, there surely exist p-th power factorable operators which are not μ-determined; our primary concern is, however, those which *are*. So, given $1 \le p < \infty$, let us denote by $\mathcal{F}_{[p]}(X(\mu), E)$ the set of all μ-determined, p-th power factorable operators from $X(\mu)$ into E. Consider now a μ-determined operator $T : X(\mu) \to E$. It was already noted earlier that the Banach function spaces $L^p(m_T)$ satisfy $L^p(m_T) = L^1(m_T)_{[1/p]}$ for all $1 \le p < \infty$ or, equivalently, that $L^1(m_T) = L^p(m_T)_{[p]}$. Moreover, the restriction map $I^{(p)}_{m_T} : L^p(m_T) \to E$ of the integration operator $I_{m_T} : L^1(m_T) \to E$ to $L^p(m_T) \subseteq L^1(m_T)$ satisfies $I^{(p)}_{m_T} \in \mathcal{F}_{[p]}(L^p(m_T), E)$. Indeed, $S := I^{(p)}_{m_T}$ is μ-determined, because its associated vector measure $m_S : \Sigma \to E$ coincides with m_T, and S admits a continuous linear extension to $L^p(m_T)_{[p]} = L^1(m_T)$, namely the integration operator $S_{[p]} := I_{m_T}$. So, for every $1 \le p < \infty$, we *always* have $I^{(p)}_{m_T} \in \mathcal{F}_{[p]}(L^p(m_T), E)$ together with the commutative diagram:

$$\begin{tikzcd} L^p(m_T) \arrow[rr, "I^{(p)}_{m_T}"] \arrow[dr, "i_{[p]}"'] & & E . \\ & L^p(m_T)_{[p]} = L^1(m_T) \arrow[ur, "I_{m_T}"'] & \end{tikzcd} \tag{1.12}$$

However, even though (1.12) always holds, it does *not* follow in general that the map $I^{(p)}_{m_T} : L^p(m_T) \to E$ is an extension of the original operator T. The problem is that the containments $X(\mu) \subseteq L^1(m_T)$ and $L^p(m_T) \subseteq L^1(m_T)$, which always hold, do *not* necessarily imply that $X(\mu) \subseteq L^p(m_T)$. However, *if* we are in the situation that $X(\mu) \subseteq L^p(m_T)$ does hold, then we should be rather pleased! Firstly, T would have an E-valued extension to $L^p(m_T)$, a space which is intimately connected to the operator T itself (indeed, it is actually constructed from T), whereas $X(\mu)_{[p]}$ has no features that it inherits directly from T. Secondly, $L^p(m_T)$ has the advantage that it is a *Banach* function space, whereas $X(\mu)_{[p]}$ may only be a quasi-Banach function space, a class of spaces which is typically not so well understood as Banach spaces and whose duality theory may be severely restricted. Thirdly, $L^p(m_T)$ is always a p-convex Banach lattice with σ-order continuous norm and possessing further desirable properties (see Chapter 3). So, operators defined on it would be expected to have additional properties. Furthermore, the particular extension of T that we are dealing with (when it exists) is the *integration operator* $I^{(p)}_{m_T}$ corresponding to a vector measure (of course, restricted to $L^p(m_T) \subseteq L^1(m_T)$), a class of operators about which many detailed properties are known; see Section 3.3 of Chapter 3. This detailed information is then available for a thorough analysis of T itself. As a sample: for each $1 < p < \infty$ it is known that the operator $I^{(p)}_{m_T} : L^p(m_T) \to E$ is always weakly compact (a result in Chapter 3) and so, *if* T factors via $I^{(p)}_{m_T}$, that is, if the commutative diagram

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, "J^{(p)}_T"'] & & E \\ & L^p(m_T) \arrow[ur, "I^{(p)}_{m_T}"'] & \end{tikzcd}$$

is valid, where $J^{(p)}_T$ denotes the (assumed) inclusion of $X(\mu)$ into $L^p(m_T)$, then T is necessarily weakly compact! For all of these reasons one of the main aims of Chapter 5 is to establish the following important result: *for* $1 \le p < \infty$, *a* μ*-determined operator* $T : X(\mu) \to E$ *is* p*-th power factorable if and only if* $X(\mu) \subseteq L^p(m_T)$ *or, equivalently, if and only if* $I^{(p)}_{m_T} : L^p(m_T) \to E$ *is an extension of* T. That is, $T \in \mathcal{F}_{[p]}(X(\mu), E)$ if and only if $X(\mu)_{[p]} \subseteq L^1(m_T)$ and the following commutative diagram is valid:

$$\begin{tikzcd} & X(\mu)_{[p]} \arrow[dr, "T_{[p]}"] & \\ X(\mu) \arrow[ur, "i_{[p]}"] \arrow[rr, "T"] \arrow[dr, "J^{(p)}_T"'] & & E\,. \\ & L^p(m_T) \arrow[ur, "I^{(p)}_{m_T}"'] & \end{tikzcd} \tag{1.13}$$

The remainder of Chapter 5 is then devoted to developing various consequences of this result, many of which play a vital role in Chapters 6 and 7. It should be pointed out that the spaces $L^p(m_T)$ also possess an important *optimality* property: this property, for $p > 1$, is somewhat more involved than for $p = 1$, as we now explain. So, let $T \in \mathcal{F}_{[p]}(X(\mu), E)$. Suppose that $Y(\mu)$ is a σ-order continuous quasi-Banach function space such that $X(\mu) \subseteq Y(\mu)$ continuously and that there exists a continuous linear operator $T_{Y(\mu)} : Y(\mu) \to E$ which coincides with T on $X(\mu)$ and belongs to $\mathcal{F}_{[p]}(Y(\mu), E)$. Then necessarily $Y(\mu) \subseteq L^p(m_T)$, with a continuous inclusion, and

$$T_{Y(\mu)}(f) = I^{(p)}_{m_T}(f) = \int_\Omega f\, dm_T, \qquad f \in Y(\mu) \subseteq L^p(m_T).$$

In view of the main result mentioned above, this can be reformulated as follows: if $T \in \mathcal{F}_{[p]}(X(\mu), E)$, then $L^p(m_T)$ is maximal amongst all σ-order continuous quasi-Banach function spaces $Y(\mu)$ which continuously contain $X(\mu)$ and such that T has a linear and continuous E-valued extension $T_{Y(\mu)} : Y(\mu) \to E$ which is itself p-th power factorable.

A few final comments concerning Chapter 5 are in order. Firstly, if Σ contains a sequence of pairwise disjoint, non-μ-null sets, then the containment $L^p(m_T) \subseteq L^1(m_T)$ is *proper* for all $1 < p < \infty$; this is a general result in Chapter 3. Accordingly, the factorization of T through $L^p(m_T)$, when possible, rather than through $L^1(m_T)$ (always possible), genuinely provides *additional* information about T. For instance, as noted above, T must be weakly compact. As a further sample, we mention (for $1 < p < \infty$) that if $T \in \mathcal{F}_{[p]}(X(\mu), E)$, if its associated vector measure m_T has σ-finite variation, and if the integration operator $I_{m_T} : L^1(m_T) \to E$ is weakly compact (automatic if E is reflexive, say), then T is actually compact. Or, if E is a Banach lattice and $T : X(\mu) \to E$ is positive and p-th power factorable, then T is a p-convex operator. There is also a careful treatment of the special case when the variation measure $|m_T| : \Sigma \to [0, \infty]$ of m_T is *finite*. For instance, in this case a μ-determined operator $T : X(\mu) \to E$ has an extension to the *classical* space $L^p(|m_T|)$, rather than $L^p(m_T)$, if and only if $X(\mu) \subseteq L^p(|m_T|)$ continuously or, equivalently, if the Radon–Nikodým derivative $\frac{d|m_T|}{d\mu}$ belongs to the Köthe dual of the quasi-Banach function space $X(\mu)_{[p]}$. As in other chapters, many detailed and illuminating examples are also presented.

The Maurey–Rosenthal factorization theory establishes a link between norm inequalities for operators and their factorization through classical L^p-spaces. Although there exist some earlier related results, this theory essentially has its roots in the work of Krivine, Maurey and Rosenthal, [92], [99], [106], [137]. In the notation of the previous paragraphs, let $1 \leq p < \infty$ and $T \in \mathcal{F}_{[p]}(X(\mu), E)$. According to the results of Chapter 5, the operator T then factorizes through $L^p(m_T)$. Of course, whenever available, the factorization of T through a *classical* L^p-space (i.e., of a scalar measure) is preferable. Under certain conditions, such as $|m_T|(\Omega) < \infty$, it was noted that T can indeed be factored through such a classical L^p-space, namely through $L^p(|m_T|)$. However, many interesting and important μ-determined operators T fail to satisfy $|m_T|(\Omega) < \infty$. Accordingly, there is a need to seek a different type of factorization for such operators than that offered by the results of Chapter 5 alone. The Maurey–Rosenthal theory provides such a factorization, albeit under various assumptions. One version can be abstractly formulated as follows (cf. [30, Corollary 5]). Let $0 < q < \infty$ and $X(\mu)$ be a σ-order continuous q-*convex* quasi-Banach function space with q-convexity constant $\mathbf{M}^{(q)}[X(\mu)]$. Consider a Banach space E and a q-*concave operator* $T : X(\mu) \to E$ with q-concavity constant $\mathbf{M}_{(q)}[T]$. Then there exists $g \in L^1(\mu)$ satisfying

$$\sup_{\|f\|_{X(\mu)} \leq 1} \left(\int_\Omega |f|^q g \, d\mu \right)^{1/q} \leq \mathbf{M}^{(q)}[X(\mu)] \cdot \mathbf{M}_{(q)}[T]$$

and the operator T satisfies

$$\|T(f)\|_E \leq \left(\int_\Omega |f|^q g \, d\mu \right)^{1/q}, \qquad f \in X(\mu),$$

an inequality which should be compared with (1.11). The factorization version of these inequalities for T can be formulated via the commutative diagram

$$\begin{tikzcd}
X(\mu) \arrow[rr, "T"] \arrow[dr, "M_{g^{1/q}}"'] & & E, \\
& L^q(\mu) \arrow[ur, "S"'] &
\end{tikzcd}$$

where S is a continuous linear operator and $M_{g^{1/q}}$ is the continuous $L^q(\mu)$-valued *multiplication operator* $f \mapsto g^{1/q}f$ defined on $X(\mu)$ and satisfying the inequality $\|M_{g^{1/q}}\| \cdot \|S\| \leq \mathbf{M}^{(q)}[X(\mu)] \cdot \mathbf{M}_{(q)}[T]$.

Recall now the paradigm of a p-th power factorable operator. Namely, for any finite measure $\mu \geq 0$ and each $q > 1$, set $X(\mu) := L^q(\mu)$ and $E := L^1(\mu)$ and let $T := i_q$ denote the natural inclusion of $X(\mu)$ into E. If $1 \leq p \leq q$, then $T \in \mathcal{F}_{[p]}(X(\mu), E)$ with $L^p(m_T) = L^p(\mu)$ and $X(\mu)_{[p]} = L^{q/p}(\mu)$ where $1 \leq \frac{q}{p} \leq q$. Then $|m_T|(\Omega) < \infty$ and the factorization given by (1.13) reduces to

$$\begin{tikzcd}
& X(\mu)_{[p]} = L^{q/p}(\mu) \arrow[dr, "(i_q)_{[p]}=i_{q/p}"] & \\
X(\mu) = L^q(\mu) \arrow[ur, "i_{[p]}"] \arrow[rr, "T=i_q"] \arrow[dr, "J_T^{(p)}"'] & & E = L^1(\mu)\ . \\
& L^p(m_T) = L^p(\mu) \arrow[ur, "I_{m_T}^{(p)}=i_p"'] &
\end{tikzcd}$$

That is, we produce the trivial factorizations of i_q via the inclusion maps through $L^p(\mu)$ and $L^{q/p}(\mu)$. However, these factorizations actually imply *stronger* properties for $T = i_q$ than those coming merely from the condition that $i_q \in \mathcal{F}_{[p]}(X(\mu), E)$. Indeed, if $f_1, \ldots, f_n \in X(\mu) = L^q(\mu)$, then combining with Hölder's inequality yields

$$\begin{aligned}
\sum_{j=1}^{n} \left\| i_q(f_j) \right\|_{L^1(\mu)}^{q/p} \ &\leq\ \big(\mu(\Omega)\big)^s \Big(\sum_{j=1}^{n} \int_\Omega |f_j|^{q/p} \, d\mu \Big) \\
&=\ \big(\mu(\Omega)\big)^s \Big\| \Big(\sum_{j=1}^{n} |f_j|^{q/p} \Big)^{1/q} \Big\|_{L^q(\mu)}^q,
\end{aligned}$$

where $s = q/(pr)$ and r satisfies $(q/p)^{-1} + r^{-1} = 1$.

In Chapter 6 we investigate inequalities of the previous kind for general spaces $X(\mu)$ and E and operators $T : X(\mu) \to E$ (in place of $i_q : L^q(\mu) \to L^1(\mu)$). More precisely, we consider vector norm inequalities in the following sense. Let $0 < q < \infty$ and $1 \leq p < \infty$ be given and $X(\mu)$ be a q-convex quasi-Banach function space. Then T is required to satisfy

$$\Big(\sum_{j=1}^{n}\|T(f_j)\|_E^{q/p}\Big)^{1/q} \le C\Big\|\Big(\sum_{j=1}^{n}|f_j|^{q/p}\Big)^{1/q}\Big\|_{X(\mu)} \tag{1.14}$$

for some constant $C > 0$ and all finite collections $f_1, \dots, f_n$ of elements from $X(\mu)$. This inequality already encompasses the main geometrical condition of p-th power factorability. In fact, we will treat a still more general inequality which has the advantage that it can be applied even when $X(\mu)$ is not q-convex, namely

$$\Big(\sum_{j=1}^{n}\big\|T(f_j)\big\|_E^{q/p}\Big)^{1/q} \le C\Big\|\sum_{j=1}^{n}\big|f_j\big|^{q/p}\Big\|_{b,X(\mu)_{[q]}}^{1/q}; \tag{1.15}$$

here $\|\cdot\|_{b,X(\mu)_{[q]}}$ is a suitable *seminorm* which is continuous and defined on the quasi-Banach function space $X(\mu)_{[q]}$. This seminorm and its properties are treated in Chapter 2. If $X(\mu)$ is q-convex, then (1.14) and (1.15) are actually equivalent; if, in addition, $p = 1$, then (1.15) reduces to q-concavity of the operator T. In this case, we are back to the abstract version of the Maurey–Rosenthal Theorem mentioned above.

The main aim of Chapter 6 is to establish various equivalences with (1.15), thereby providing factorizations for a subclass of operators $T \in \mathcal{F}_{[p]}(X(\mu), E)$ for which the required assumptions of the Maurey–Rosenthal theory are not necessarily fulfilled. It turns out that these factorizations still pass through classical L^r-spaces and via multiplication operators. So, the main result can be viewed as a significant extension and abstraction of the Maurey–Rosenthal theory and, simultaneously, incorporates the factorization features associated with p-th power factorability. In view of the fact that the spaces $L^p(m_T)$ are always p-convex, this result also has some interesting consequences for the integration operator $I_{m_T}^{(p)} : L^p(m_T) \to E$ associated to each $T \in \mathcal{F}_{[p]}(X(\mu), E)$. For instance, suppose that E is a *Banach lattice* and T is a *positive operator*. Then, for $p > 1$, the integration operator $I_{m_T}^{(p)} : L^p(m_T) \to E$ is p-concave if and only if

$$\Big(\sum_{j=1}^{n}\big\|T(s_j)\big\|_E^{p}\Big)^{1/p} \le C\Big\|T\Big(\sum_{j=1}^{n}\big|s_j\big|^{p}\Big)\Big\|_E^{1/p}$$

for some constant $C > 0$ and all finite collections $s_1, \dots, s_n$ of elements from $\operatorname{sim}\Sigma$. Applications to other classes of operators are also given.

Chapter 7 is the culmination of all of the ideas in this monograph applied to various important operators arising in classical harmonic analysis. It highlights the fruitfulness of the notions and concepts involved, provides a detailed analysis of the operators concerned, and provides myriad problems worthy of further research.

To fix the setting, let G be any (infinite) *compact abelian group* with *dual group* Γ. Normalised Haar measure in G is denoted by μ and is defined on the

σ-algebra $\Sigma := \mathcal{B}(G)$ of all Borel subsets of G. The corresponding spaces $L^p(\mu)$ are indicated rather by $L^p(G)$, for $1 \le p \le \infty$. Since Γ is a discrete space, its associated L^p-spaces are denoted by the more traditional notation $\ell^p(\Gamma)$. As usual, $c_0(\Gamma)$ denotes the Banach space of all functions on Γ which vanish at infinity and is a proper closed subspace of $\ell^\infty(\Gamma)$. For each $\gamma \in \Gamma$, the value of the *character* γ at a point $x \in G$ is denoted by (x, γ). Finally, the space of all $\mathbb{C}$-valued, regular measures defined on $\mathcal{B}(G)$ is denoted by $M(G)$ and is a Banach space relative to its usual total variation norm $\|\lambda\|_{M(G)} := |\lambda|(G)$. The *Fourier–Stieltjes transform* $\widehat{\lambda} : \Gamma \to \mathbb{C}$ of a measure $\lambda \in M(G)$ is defined by

$$\widehat{\lambda}(\gamma) := \int_G \overline{(x, \gamma)}\, d\lambda(x), \qquad \gamma \in \Gamma,$$

in which case $\widehat{\lambda} \in \ell^\infty(\Gamma)$. If $\lambda \ll \mu$ is absolutely continuous, then it follows from the Radon–Nikodým Theorem that there exists $h \in L^1(G)$ whose indefinite integral

$$\mu_h : A \mapsto \int_A h\, d\mu, \qquad A \in \mathcal{B}(G), \tag{1.16}$$

coincides with λ. In this case, $\widehat{\lambda}$ is equal to the *Fourier transform* $\widehat{h}$ of h defined by

$$\widehat{h}(\gamma) := \int_G \overline{(x, \gamma)}\, h(x)\, d\mu(x), \qquad \gamma \in \Gamma,$$

and the Riemann–Lebesgue Lemma ensures that $\widehat{h} \in c_0(\Gamma)$.

The Fourier transform map $F_1 : L^1(G) \to \ell^\infty(\Gamma)$ defined by $f \mapsto \widehat{f}$ is linear and continuous. If we wish to consider F_1 as being $c_0(\Gamma)$-valued rather than $\ell^\infty(\Gamma)$-valued, then we will denote it by $F_{1,0}$. Since

$$\|\widehat{f}\|_{c_0(\Gamma)} \le \|f\|_{L^1(G)} \le \|f\|_{L^p(G)}, \qquad f \in L^p(G),$$

for each $1 \le p < \infty$, it follows that the Fourier transform map is also defined, simply by restriction, on each space $L^p(G)$ and maps it continuously into $c_0(\Gamma)$. We denote this operator by $F_{p,0} : L^p(G) \to c_0(\Gamma)$. Actually, more is true: it turns out that $\widehat{f} \in \ell^{p'}(\Gamma)$, whenever $1 \le p \le 2$ and $f \in L^p(G)$, with

$$\|\widehat{f}\|_{\ell^{p'}(\Gamma)} \le \|f\|_{L^p(G)}, \qquad f \in L^p(G),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. This is the *Hausdorff–Young inequality.* The continuous linear operator $f \mapsto \widehat{f}$ so-defined is denoted by $F_p : L^p(G) \to \ell^{p'}(\Gamma)$, for each $1 \le p \le 2$.

Section 7.1 of Chapter 7 is devoted to an analysis of the Fourier transform maps described above. That is, in the notation of earlier chapters, either $T := F_{p,0}$ for $1 \le p < \infty$ with $X(\mu) := L^p(G)$ and $E := c_0(\Gamma)$ or, $T := F_p$ for $1 \le p \le 2$ with $X(\mu) := L^p(G)$ and $E := \ell^{p'}(\Gamma)$. In all cases T is μ-determined. For $T = F_{p,0}$

with $1 \le p < \infty$ arbitrary but fixed for now, it turns out that its associated vector measure m_T has finite variation (actually, $|m_T| = \mu$). Moreover, concerning its optimal domain space we have

$$L^1_{\mathrm{w}}(m_T) = L^1(m_T) = L^1(|m_T|) = L^1(G)$$

and the maximal extension of $T : L^p(G) \to c_0(\Gamma)$ is precisely the Fourier transform map $F_{1,0} : L^1(G) \to c_0(\Gamma)$. In particular, $L^q(m_T) = L^q(G)$ for all $1 \le q < \infty$. The maximal extension $I_{m_T} = F_{1,0}$ is neither weakly compact nor completely continuous. Finally, for $1 < r < 2$ fixed, the operator $F_{r,0} : L^r(G) \to c_0(\Gamma)$ is p-th power factorable if and only if $1 \le p \le r$. If we change the codomain space from $c_0(\Gamma)$ to $\ell^{p'}(\Gamma)$, that is, we now consider $T = F_p$ with $1 < p \le 2$ fixed, then the situation *changes dramatically* and is significantly more involved. For instance, even though the vector measure $m_T : \mathcal{B}(G) \to \ell^{p'}(\Gamma)$ has the same null sets as μ, it now has infinite variation (not even σ-finite). Moreover, for $1 < p < 2$, the optimal domain space $L^1(m_T)$ is no longer an $L^r(G)$-space for any $1 \le r \le \infty$, yet it is a proper subspace of $L^1(G)$ and, for certain groups G, contains $L^p(G)$ as a proper subspace, [113]. The maximal extension $I_{m_T} : L^1(m_T) \to \ell^{p'}(\Gamma)$ of T is neither compact nor completely continuous (it is weakly compact as $\ell^{p'}(\Gamma)$ is reflexive). An interesting and alternate description of the optimal domain space is also presented, namely

$$L^1(m_T) = \big\{ f \in L^1(G) : (f\chi_A)\widehat{\ } \in \ell^{p'}(\Gamma) \ \text{ for all } A \in \mathcal{B}(G) \big\}.$$

It turns out that $L^1(m_T)$ is always weakly sequentially complete and hence, according to the results of Chapter 3, we have that $L^r_{\mathrm{w}}(m_T) = L^r(m_T)$ for all $1 \le r < \infty$ and that $L^r(m_T)$ is reflexive for all $1 < r < \infty$. If G is metrizable, then $L^1(m_T)$ is separable. Most importantly, these optimal domain spaces, even though they are not $L^r(G)$-spaces, are nevertheless ideally suited to harmonic analysis: they are *homogeneous Banach spaces* in the sense of Y. Katznelson, [84], and so translation, convolution and Fourier transform maps act continuously in them. For $p = 2$, the operator $T : L^2(G) \to \ell^2(\Gamma)$ is an isomorphism (due to the Plancherel Theorem) and so, by the results of Chapter 4, $L^2(m_T) = L^2(G)$, that is, no further $\ell^2(\Gamma)$-valued extension of F_2 is possible as it is already defined on its optimal domain. Unlike for the operators $F_{r,0} : L^r(G) \to c_0(\Gamma)$, if $1 < r < 2$, then for certain groups G the operator $F_r : L^r(G) \to \ell^{r'}(\Gamma)$ fails to be p-th power factorable for *every* $1 \le p < \infty$.

The remainder of Chapter 7 deals with *convolution operators* $C^{(p)}_\lambda : L^p(G) \to L^p(G)$ defined by $f \mapsto f * \lambda$, for $1 \le p < \infty$ and $\lambda \in M(G)$. Because of the well-known inequality

$$\|f * \lambda\|_{L^p(G)} \le \|\lambda\|_{M(G)} \|f\|_{L^p(G)}, \qquad f \in L^p(G),$$

each operator $C^{(p)}_\lambda$ is continuous. If $\lambda := \delta_a$ is the Dirac measure at a point $a \in G$, then $C^{(p)}_{\delta_a}$ is precisely the operator $\tau_a : L^p(G) \to L^p(G)$ of *translation* by a, that is, for each $f \in L^p(G)$ we have

$$\tau_a f : x \mapsto f(x-a), \qquad x \in G.$$

The case $p = 1$ is treated separately. In the notation of earlier chapters, for $\lambda \in M(G)$ fixed, we consider $T = C_\lambda^{(1)}$ with $X(\mu) = L^1(G)$ and $E = L^1(G)$. Then $|m_T|$ is a finite measure (actually, a multiple of μ) and so $L^1(|m_T|) = L^1(G)$ for $\lambda \neq 0$. Actually, it turns out that

$$L^1_{\mathrm{w}}(m_T) = L^1(m_T) = L^1(|m_T|) = L^1(G)$$

for *every* $\lambda \in M(G)\setminus\{0\}$. Hence, $C_\lambda^{(1)}$ is already defined on its optimal domain and $I_{m_T} = T$. The interest in the much studied class of operators $\{C_\lambda^{(1)} : \lambda \in M(G)\}$ arises from the fact that a continuous linear operator $S : L^1(G) \to L^1(G)$ satisfies $S \circ \tau_a = \tau_a \circ S$ for all $a \in G$ if and only if $S = C_\lambda^{(1)}$ for some $\lambda \in M(G)$. Our particular interest here is to identify certain ideal properties of T, which separate into essentially two cases. For the case $\lambda \ll \mu$ (see (1.16)), a result of Akemann, [1], states that this is equivalent to $C_\lambda^{(1)}$ being compact which, in turn, is equivalent to $C_\lambda^{(1)}$ being weakly compact. By Costé's Theorem, [42, pp. 90–92], these conditions are also equivalent to $C_\lambda^{(1)}$ having an integral representation via a *Bochner μ-integral density* $F_\lambda : G \to L^1(G)$. We present some further equivalences of a rather different nature, mostly in terms of the function $K_\lambda : G \to M(G)$ defined by

$$K_\lambda : x \mapsto \delta_x * \lambda, \qquad x \in G,$$

but also in terms of a *Pettis μ-integrable density* $H_\lambda : G \to L^1(G)$. Here we mention only two equivalences to $\lambda \ll \mu$: (i) the range of K_λ is a norm compact subset of $M(G)$ and, (ii) there exists $A \in \mathcal{B}(G)$ with $\mu(A) > 0$ such that its image $K_\lambda(A)$ is norm separable in $M(G)$. It is to be noted that $M(G)$ itself is always non-separable. The second main result concerning the operators $C_\lambda^{(1)}$ involves the proper subalgebra

$$M_0(G) := \{\lambda \in M(G) : \widehat{\lambda} \in c_0(\Gamma)\}$$

of $M(G)$; it is known that $M_0(G)$ in turn contains $L^1(G)$ as a proper subalgebra. Our result presents several equivalences to the condition that $\lambda \in M_0(G)$, of which we again mention just two: (i) the operator $T = C_\lambda^{(1)}$ is completely continuous and, (ii) the range of the associated vector measure $m_T : \mathcal{B}(G) \to L^1(G)$ is a relatively compact set. The two mentioned theorems provide non-trivial and important examples of continuous linear operators in Banach spaces (indeed, of integration operators $I_\nu : L^1(\nu) \to E$ corresponding to a vector measure ν) which are completely continuous but, fail to be either compact or weakly compact. Namely, all operators $C_\lambda^{(1)}$ with $\lambda \in M_0(G) \setminus L^1(G)$ are of this kind, which is a non-trivial statement as $M_0(G) \setminus L^1(G)$ is always non-empty.

A large part of Chapter 7 also deals with the operators $C_\lambda^{(p)}$ for $1 < p < \infty$. In this setting, for p fixed and $\lambda \in M(G)$, we consider $T = C_\lambda^{(p)}$ with $X(\mu) = L^p(G)$ and $E = L^p(G)$. Again m_T and μ are mutually absolutely continuous (for $\lambda \neq 0$) and the natural inclusions

$$L^p(G) \;\subseteq\; L^1(m_T) \;=\; L^1_{\mathrm{w}}(m_T) \;\subseteq\; L^1(G) \tag{1.17}$$

hold and are continuous. The question of m_T having finite variation or not is now a major result. Namely, $|m_T|(G) < \infty$ if and only if $\lambda = \mu_h$ (see (1.16)) for some $h \in L^p(G)$ which, in turn, is equivalent to $I_{m_T} : L^1(m_T) \to L^p(G)$ being a compact operator and this, in turn, is also equivalent to the optimal domain space $L^1(m_T) = L^1(G)$ being as large as possible! What about measures $\lambda \notin L^p(G)$? Another major result provides some insight: $\lambda \in M_0(G)$ if and only if the original operator $T = C_\lambda^{(p)}$ is compact or, equivalently, the range of the vector measure $m_T : \mathcal{B}(G) \to L^p(G)$ is a relatively compact set. Since the space $M_0(G)$ is independent of p, the previous equivalences hold for all $1 < p < \infty$ as soon as they hold for some p. A rather concrete description of the optimal domain space, for arbitrary $\lambda \in M(G)$, is also given, namely

$$L^1(m_T) \;=\; \big\{ f \in L^1(G) : (f\chi_A) * \lambda \in L^p(G) \quad \text{for all } A \in \mathcal{B}(G) \big\}$$

with the norm of $f \in L^1(m_T)$ given by

$$\|f\|_{L^1(m_T)} \;=\; \sup\Big\{ \int_G |f| \cdot |\varphi * \widetilde{\lambda}| \, d\mu \;:\; \varphi \in L^{p'}(G),\ \|\varphi\|_{L^{p'}(G)} \le 1 \Big\};$$

here $\widetilde{\lambda} : A \mapsto \lambda(-A)$ denotes the reflection of the measure λ. For measures $\lambda \in M_0(G) \setminus \{0\}$ it turns out that the first inclusion in (1.17), that is, $L^p(G) \subseteq L^1(m_T)$ is always *proper* so that $I_{m_T} : L^1(m_T) \to L^p(G)$ is a genuine extension of $T = C_\lambda^{(p)}$. On the other hand, the optimal domain space $L^1(m_T)$ is the *largest possible*, that is, equals $L^1(G)$, precisely when $\lambda = \mu_h$ for some $h \in L^p(G)$. Accordingly, the second inclusion in (1.17), namely $L^1(m_T) \subseteq L^1(G)$ is *proper* whenever $\lambda \in M(G) \setminus L^p(G)$. In particular, for every $\lambda \in M_0(G) \setminus L^p(G)$ it follows that

$$L^p(G) \subsetneqq L^1(m_T) \subsetneqq L^1(G). \tag{1.18}$$

A major effort is also invested in identifying measures λ for which $L^1(m_T) = L^p(G)$, that is, T is already defined on its optimal domain (by the criteria for (1.18) we must have $\lambda \notin M_0(G)$ in this case). As examples of measures λ for which $L^1(m_T) = L^p(G)$ we mention:

- $\lambda = \delta_a + \eta$ where $\eta \in M(G)$ satisfies $\mathrm{supp}(\eta) \neq G$ and $a \notin \mathrm{supp}(\eta)$.
- $\lambda = \sum_{j=1}^\infty \beta_j \delta_{a_j}$ where $\{a_j\}_{j=1}^\infty \subseteq G$ is a sequence converging to the identity element of G and $\{\beta_j\}_{j=1}^\infty \in \ell^1$.
- $\lambda \in M(G)$ is a positive measure satisfying $\lambda(\{a\}) \neq 0$ for some $a \in G$.

Let $1 \leq q < \infty$. A measure $\lambda \in M(G)$ is called *L^q-improving* if there exists $r \in (q, \infty)$ such that

$$\lambda * f \in L^r(G), \qquad f \in L^q(G),$$

briefly, $\lambda * L^q(G) \subseteq L^r(G)$. If λ is L^q-improving for some $1 < q < \infty$, then it is L^q-improving for every $1 < q < \infty$. Such measures were introduced by E.M. Stein, [152], and have played an important role in harmonic analysis ever since. The final section of Chapter 7 is devoted to a consideration of such measures which, from the viewpoint of this monograph, lead to an extensive and non-trivial collection of p-th power factorable operators. For instance, if $1 \leq p < \infty$ and $u \in (1, p)$ satisfies $\frac{1}{u} + \frac{1}{r} = \frac{1}{p} + 1$ for a given $1 < r < p$, then the convolution operator $T = C^{(p)}_{\mu_h}$ is L^q-improving and $\frac{p}{u}$-th power factorable and satisfies

$$L^p(G) \underset{\neq}{\subsetneq} L^u(G) \subseteq L^1(m_T),$$

for every $h \in L^r(G) \setminus L^p(G)$. Or, let $\lambda \in M(G)$ be an L^q-improving measure and $1 < r < \infty$ be arbitrary. Then there exists $1 < s < r$ such that $C^{(r)}_\lambda$ is p-th power factorable for all $1 < p < s$. One of the main results states that $\lambda \in M(G)$ is L^q-improving if and only if for every $1 < r < \infty$ there exists $1 < p < \infty$ such that $C^{(r)}_\lambda$ is p-th power factorable. If a measure $\lambda \in M(G)$ *fails* to be L^q-improving then, for each $1 < r < \infty$, it turns out that $L^r(G)$ is the *largest* amongst the spaces $\{L^u(G)\}_{1<u<\infty}$ which is contained in $L^1(m_T)$, where $T = C^{(r)}_\lambda$. For such a measure λ, additionally satisfying $\lambda \in M_0(G)$ (such a λ exists!), it is the case that $L^r(G) \underset{\neq}{\subsetneq} L^1(m_T)$. So, even though λ does *not* convolve $L^u(G)$ into $L^r(G)$ for any $u < r$, it *does* convolve $L^1(m_T)$, which is *genuinely larger* than $L^r(G)$, into $L^r(G)$. Accordingly, for the class of convolution operators, the optimality of the spaces $L^1(m_T)$ has an alternate formulation: *given any $\lambda \in M(G)$ and $1 < r < \infty$, the space $L^1(m_T)$ corresponding to $T = C^{(r)}_\lambda$ is the largest Banach function space $X(\mu)$ with σ-order continuous norm which contains $L^r(G)$ and such that $\lambda * X(\mu) \subseteq L^r(G)$.* Hence, the notions of optimal domain and p-th power factorable operator, when applied to particular concrete operators, may reduce to important classical and well-established concepts. This provides us with the opportunity to conclude this introductory chapter to the monograph by recalling a comment made at its beginning but now with more justification, perhaps: … "it will (hopefully!) become apparent that the ideas are not as abstract as the first impression may suggest, that they have some worthwhile and far-reaching consequences, and that they apply to a large, diverse and interesting class of operators which arise in various branches of analysis". If this is indeed the case, then we have achieved our aim: so, enjoyable reading and stimulating thoughts.

Chapter 2

Quasi-Banach Function Spaces

Quasi-Banach spaces are an important class of metrizable topological vector spaces (often, not locally convex), [70], [83], [87], [88], [105], [135]; for quasi-Banach lattices we refer to [82, pp. 1116–1119] and the references therein. In the past 20 years or so, the subclass of quasi-Banach *function* spaces has become relevant to various areas of analysis and operator theory; see, for example, [29], [30], [32], [50], [59], [61], [87], [126], [152] and the references therein. Of particular importance is the notion of the *p-th power* $X_{[p]}$, $0 < p < \infty$, of a given quasi-Banach function space X. This associated family of quasi-Banach function spaces $X_{[p]}$, which is intimately connected to the base space X, is produced via a procedure akin to that which produces the Lebesgue L^p-spaces from L^1 (or more generally, produces the p-convexification of Banach lattices (of functions), [9], [99, pp. 53–54], [157]).

Whereas the theory of real Banach function spaces is well developed, this is not always the case for such spaces over $\mathbb{C}$; often one reads "the case for complex spaces is analogous" with no further details provided, or no mention of the complex setting is made at all. Similarly, the literature on quasi-Banach function spaces is rather sparse, both over $\mathbb{R}$ and $\mathbb{C}$. The natural setting of this monograph concerns spaces over $\mathbb{C}$ which are often only equipped with a quasi-norm rather than a norm and operators acting in such spaces (e.g., we have chapters on vector integration, harmonic analysis and factorization of operators). Accordingly, we devote quite an effort in this chapter to carefully developing those aspects of the theory of quasi-Banach function spaces over $\mathbb{C}$ (with detailed proofs) which are needed in later chapters. In particular, this includes aspects of the theory of q-convex operators and q-concave operators acting between such spaces. We have also included several examples to illustrate the differences that occur when passing from the normed to the quasi-normed setting. It is time to begin.

2.1 General theory

Let Z be a complex vector space. A function $\|\cdot\| : Z \to [0, \infty)$ is called a *quasi-norm*, [88, §15.10], if

(Q1) $\|z\| = 0$ if and only if $z = 0$.

(Q2) $\|\alpha z\| = |\alpha| \cdot \|z\|$ for $\alpha \in \mathbb{C}$ and $z \in Z$, and

(Q3) there is a constant $K \geq 1$ such that $\|z_1 + z_2\| \leq K\big(\|z_1\| + \|z_2\|\big)$ for all $z_1, z_2 \in Z$.

In this case, Z is called a *quasi-normed space*; it admits a countable base of neighbourhoods of 0, namely, $\{z \in Z : \|z\| < 1/n\}$ for $n \in \mathbb{N}$. So, Z is a metrizable topological vector space, [88, §15,11.(1)]. The closed unit ball $\{z \in Z : \|z\| \leq 1\}$ of Z is denoted by $\mathbf{B}[Z]$. A subset of Z is bounded (in the sense of topological vector spaces) if and only if it is contained in a multiple of $\mathbf{B}[Z]$. Of course, if we can take $K = 1$ in (Q3), then $\|\cdot\|$ is a norm, i.e., Z is a normed space. Every quasi-normed space is necessarily *locally bounded* as a topological vector space, that is, it possesses a bounded neighbourhood of 0, [88, p. 159].

The given quasi-norm on a quasi-normed space Z is usually denoted by $\|\cdot\|_Z$ unless stated otherwise. If we need to emphasize the quasi-norm, then we may say that $(Z, \|\cdot\|_Z)$ is a quasi-normed space. The classical examples of such spaces are the Hardy spaces H^p for $0 < p \leq \infty$ (not normable if $0 < p < 1$), [83, Ch. III], [153], and the Lebesgue spaces ℓ^p and $L^p([0,1])$ for $0 < p \leq \infty$ (not normable if $0 < p < 1$), [83, Ch. II], [88, §15.9].

A linear map $T : Z \to W$ between quasi-normed spaces is continuous if and only if

$$\|T\| := \sup\big\{ \|Tz\|_W : z \in \mathbf{B}[Z]\big\} < \infty, \tag{2.1}$$

[83, p. 8]. It follows from (Q2) that $\|Tz\|_W \leq \|T\| \cdot \|z\|_Z$ for all $z \in Z$. The collection of all continuous linear maps from Z into W is denoted by $\mathcal{L}(Z, W)$. We will write $\mathcal{L}(Z) := \mathcal{L}(Z, Z)$. If W happens to be a normed space, then (2.1) defines a norm in $\mathcal{L}(Z, W)$, which we call the operator norm. If, in addition, W happens to be a Banach space, then $\mathcal{L}(Z, W)$ is also a Banach space for the operator norm. For Z and W *real* quasi-normed spaces, the collection of all continuous linear operators from Z to W is still denoted by $\mathcal{L}(Z, W)$.

Throughout this chapter we fix a *positive, finite* measure space (Ω, Σ, μ); that is, μ is a finite, non-negative measure defined on a σ-algebra Σ of subsets of a non-empty set Ω. It is always assumed that μ is non-trivial, that is, its range is an infinite set. The characteristic function of a set $A \in \Sigma$ is denoted by χ_A. Let $\operatorname{sim}\Sigma$ denote the vector space of all $\mathbb{C}$-valued Σ-simple functions. By $L^0(\mu)$ we denote the space of all (equivalence classes of) $\mathbb{C}$-valued Σ-measurable functions modulo μ-null functions. Unless otherwise stated explicitly, we identify an individual Σ-measurable function with its equivalence class in $L^0(\mu)$. The space $L^0(\mu)$ is a complex vector lattice. Indeed, it is the complexification of the real vector lattice

$L^0_{\mathbb{R}}(\mu) := \{f \in L^0(\mu) : f$ is $\mathbb{R}$-valued μ-a.e.$\}$ equipped with its μ-a.e. pointwise order. The positive cone of $L^0(\mu)$ is denoted by $L^0(\mu)^+$. The real vector lattice $L^0_{\mathbb{R}}(\mu)$ is *super Dedekind complete* in the sense that every bounded set $W \subseteq L^0_{\mathbb{R}}(\mu)$ has a supremum in $L^0_{\mathbb{R}}(\mu)$ and contains a countable subset W_0 with $\sup W = \sup W_0$, [102, Definition 23.1, Example 23.3(iv)].

Relative to the translation invariant metric

$$d(f,g) := \int_\Omega \frac{|f-g|}{1+|f-g|}\,d\mu, \qquad\qquad f,g \in L^0(\mu),$$

the space $L^0(\mu)$ is a complete, metrizable, topological vector space with the property that a sequence $\{f_n\}_{n=1}^\infty \subseteq L^0(\mu)$ converges to $f \in L^0(\mu)$ if and only if it converges in μ-measure to f; see [88, pp. 157–158] and [160, Ch. 5, §6], for example. Accordingly, we also refer to the topology in $L^0(\mu)$ as the "topology of convergence in measure". Define $\|\!|\cdot|\!\| : L^0(\mu) \to [0,\infty)$ by

$$\|\!|f|\!\| :=\; d(f,0) = \int_\Omega \frac{|f|}{1+|f|}d\mu, \qquad\qquad f \in L^0(\mu).$$

It is routine to verify that $\|\!|\cdot|\!\|$ is an F-*norm* on $L^0(\mu)$, in the sense of [88, p. 163]. That is, it satisfies the following axioms:

(F1) $\|\!|f|\!\| \geq 0$,
(F2) $f = 0$ if and only if $\|\!|f|\!\| = 0$,
(F3) $\|\!|\lambda f|\!\| \leq \|\!|f|\!\|$ whenever $|\lambda| \leq 1$,
(F4) $\|\!|f+g|\!\| \leq \|\!|f|\!\| + \|\!|g|\!\|$,
(F5) $\|\!|\lambda f_n|\!\| \to 0$ whenever $\|\!|f_n|\!\| \to 0$, and
(F6) $\|\!|\lambda_n f|\!\| \to 0$ whenever $|\lambda_n| \to 0$.

The collection of sets $B_\varepsilon := \{f \in L^0(\mu) : \|\!|f|\!\| < \varepsilon\}$, with $\varepsilon > 0$ arbitrary, forms a base of neighbourhoods of 0 in $L^0(\mu)$. The following result shows that the topology of $L^0(\mu)$ cannot be given by any quasi-norm.

Lemma 2.1. *The complete, metrizable, topological vector space $L^0(\mu)$ is not locally bounded.*

Proof. We need to show that B_ε fails to be a bounded set for every $\varepsilon > 0$. So, fix $\varepsilon > 0$. It is required to exhibit a neighbourhood U_ε of 0 such that $B_\varepsilon \not\subseteq \rho U_\varepsilon$ for every $\rho > 0$. That is, for every $\rho > 0$ there exists $f_\rho \in B_\varepsilon$ with $\rho^{-1} f_\rho \notin U_\varepsilon$.

The first claim is that there exist $\delta(\varepsilon) > 0$ and a set $A \in \Sigma$ satisfying

$$2\delta(\varepsilon) \;<\; \mu(A) \;\leq\; \min\bigl\{\varepsilon, \mu(\Omega)\bigr\}. \tag{2.2}$$

Indeed, for μ non-atomic, if we let $\delta(\varepsilon) := 4^{-1}\min\{\varepsilon,\mu(\Omega)\}$, then the inequalities $2\delta(\varepsilon) < \min\{\varepsilon,\mu(\Omega)\} \leq \mu(\Omega)$ hold, from which it is clear that a set $A \in \Sigma$ satisfying (2.2) exists. If μ is purely atomic, then μ has countably many atoms, say $A_1, A_2, \ldots$. Since $\sum_{n=1}^\infty \mu(A_n) = \mu(\Omega) < \infty$, there exists $N \in \mathbb{N}$ such that

$\mu(A_N) \le \min\{\varepsilon, \mu(\Omega)\}$. Choose any $\delta(\varepsilon) \in \big(0,\, 2^{-1}\mu(A_N)\big)$. Then $A := A_N$ satisfies (2.2). For μ not purely atomic, let ν denote the non-atomic part of μ. Then there exists $\Omega_{\mathrm{na}} \in \Sigma$ such that $\Omega_{\mathrm{a}} := \Omega \setminus \Omega_{\mathrm{na}}$ is the union of all the atoms of μ with $\nu(\Omega_{\mathrm{a}}) = 0$ and $\nu(F) = \mu(F)$ for all $F \in \Omega_{\mathrm{na}} \cap \Sigma$. By the previous argument there exist $A \in \Omega_{\mathrm{na}} \cap \Sigma$ and $\delta(\varepsilon) > 0$ such that $2\delta(\varepsilon) < \nu(A) \le \min\{\varepsilon, \nu(\Omega_{\mathrm{na}})\}$. Since $\mu(A) = \nu(A)$ and $\nu(\Omega_{\mathrm{na}}) \le \mu(\Omega)$ we again have (2.2). This establishes the stated claim.

For any $\rho > 0$, define $f_\rho := \rho\, \chi_A$ with A and $\delta(\varepsilon)$ as given by (2.2). Then (2.2) yields $|||f_\rho||| = \left(\frac{\rho}{1+\rho}\right)\mu(A) < \mu(A) \le \varepsilon$, that is, $f_\rho \in B_\varepsilon$. On the other hand, (2.2) also gives $|||\rho^{-1} f_\rho||| = |||\chi_A||| = \mu(A)/2 > \delta(\varepsilon)$, that is, $\rho^{-1} f_\rho \notin B_{\delta(\varepsilon)}$. So, the neighbourhood $U_\varepsilon := B_{\delta(\varepsilon)}$ of 0 has the required property. □

Let $X(\mu)$ be an *order ideal* of the vector lattice $L^0(\mu)$, that is, a vector subspace such that $f \in X(\mu)$ whenever $f \in L^0(\mu)$ satisfies $|f| \le |g|$ for some $g \in X(\mu)$. Note that $X_{\mathbb{R}}(\mu) := L^0_{\mathbb{R}}(\mu) \cap X(\mu)$ is an order ideal of the real vector lattice $L^0_{\mathbb{R}}(\mu)$ and that $X(\mu)$ is the complexification of $X_{\mathbb{R}}(\mu)$. That is, $X(\mu) = X_{\mathbb{R}}(\mu) + iX_{\mathbb{R}}(\mu)$ and $|f| = \bigvee_{0\le\theta<2\pi} \big|(\cos\theta)\cdot \mathrm{Re}\,(f) + (\sin\theta)\cdot \mathrm{Im}\,(f)\big|$, [165, Ch. 12, §91]. The positive cone of $X(\mu)$ is then denoted by $X(\mu)^+ := X(\mu) \cap L^0(\mu)^+$. We say that a positive function $e \in X(\mu)$ is a *weak order unit* of $X(\mu)$ if $f \wedge (ne) \uparrow f$ for every $f \in X(\mu)^+$, that is, e is a weak order unit of the real vector lattice $X_{\mathbb{R}}(\mu)$; see, for example, [2, p. 36]. A quasi-norm $\|\cdot\|_{X(\mu)}$ on $X(\mu)$ is said to be a *lattice quasi-norm* if

$$\|f\|_{X(\mu)} \le \|g\|_{X(\mu)} \qquad \text{whenever} \quad f, g \in X(\mu) \quad \text{and} \quad |f| \le |g|. \tag{2.3}$$

In particular, $\|f\|_{X(\mu)} = \|\overline{f}\|_{X(\mu)} = \big\|\,|f|\,\big\|_{X(\mu)}$ for all $f \in X(\mu)$, where $\overline{f}$ is the complex conjugate function of f. The restriction of $\|\cdot\|_{X(\mu)}$ to $X_{\mathbb{R}}(\mu)$ is also a lattice quasi-norm and $\|\cdot\|_{X(\mu)}$ is the complexification of the (real) lattice quasi-norm $\|\cdot\|_{X_{\mathbb{R}}(\mu)}$ so that $\|f\|_{X(\mu)} = \big\|\,|f|\,\big\|_{X_{\mathbb{R}}(\mu)}$ for all $f \in X(\mu)$. In this case, $(X(\mu), \|\cdot\|_{X(\mu)})$, or simply $X(\mu)$, is called a *quasi-normed function space* based on (Ω, Σ, μ). If the lattice quasi-norm $\|\cdot\|_{X(\mu)}$ happens to be a norm, then $X(\mu)$ is, of course, called a *normed* function space. As we now show, the quasi-normed function space $X(\mu)$ is continuously embedded into $L^0(\mu)$ whenever $\operatorname{sim}\Sigma \subseteq X(\mu)$. Our proof relies on a well-known construction of a particular F-norm arising from a given quasi-normed space $(Z, \|\cdot\|_Z)$. Indeed, let $K \ge 1$ satisfy (Q3) and choose $r > 0$ according to $2^{1/r} = K$. Then the formula

$$|||z||| := \inf\left\{ \sum_{j=1}^{n} \|z_j\|_Z^r \,:\, z = \sum_{j=1}^{n} z_j,\ z_j \in Z\,(j = 1, \ldots, n),\ n \in \mathbb{N} \right\}, \qquad z \in Z, \tag{2.4}$$

defines an F-norm on Z generating the same topology as $\|\cdot\|_Z$, and satisfying

$$4^{-1}\|z\|_Z^r \le |||z||| \le \|z\|_Z^r, \qquad z \in Z, \tag{2.5}$$

[83, Theorem 1.2].

Proposition 2.2. *Let $(X(\mu), \|\cdot\|_{X(\mu)})$ be a quasi-normed function space based on (Ω, Σ, μ) such that* $\operatorname{sim}\Sigma \subseteq X(\mu)$.

(i) *The natural inclusion map from $(X(\mu), \|\cdot\|_{X(\mu)})$ into $L^0(\mu)$, equipped with its topology of convergence in measure, is continuous.*

(ii) *Every Cauchy sequence in $(X(\mu), \|\cdot\|_{X(\mu)})$ admits a subsequence converging μ-a.e.*

(iii) *Let $\{f_n\}_{n=1}^\infty$ be a sequence in $X(\mu)$ such that $\lim_{n\to\infty}\|f_n\|_{X(\mu)} = 0$. Then $\{f_n\}_{n=1}^\infty$ admits a subsequence converging μ-a.e. to 0.*

(iv) *$L^\infty(\mu) \subseteq X(\mu)$ with a continuous inclusion.*

(v) *The constant function χ_Ω is a weak order unit of $X(\mu)$.*

(vi) *There exists $r > 0$, depending on $\|\cdot\|_{X(\mu)}$, such that, if $\{f_n\}_{n=1}^\infty \subseteq X(\mu)$ is any sequence converging to a function $f \in X(\mu)$ relative to the quasi-norm $\|\cdot\|_{X(\mu)}$, then*

$$4^{-1/r}\limsup_{n\to\infty}\|f_n\|_{X(\mu)} \;\le\; \|f\|_{X(\mu)} \;\le\; 4^{1/r}\liminf_{n\to\infty}\|f_n\|_{X(\mu)}. \tag{2.6}$$

Proof. (i) Let $|||\cdot||| : X(\mu) \to [0,\infty)$ denote the F-norm defined by (2.4) with $Z := X(\mu)$ and an appropriate $r > 0$, so that (2.5) gives

$$4^{-1}\|f\|_{X(\mu)}^r \;\le\; |||f||| \;\le\; \|f\|_{X(\mu)}^r, \qquad f \in X(\mu). \tag{2.7}$$

Assume that the natural inclusion map $i : X(\mu) \to L^0(\mu)$ is not continuous. Then there exist a sequence $\{f_n\}_{n=1}^\infty$ in $X(\mu)$ and positive numbers ε, δ such that, with $A(n) := \{\omega \in \Omega : |f_n(\omega)| \ge \varepsilon\}$, we have

$$\mu\big(A(n)\big) \;\ge\; \delta \qquad \text{and} \qquad \|f_n\|_{X(\mu)} \;\le\; 2^{-n/r}\cdot\varepsilon \qquad \text{for all } \; n \in \mathbb{N}. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$|||\chi_{A(n)}||| \;\le\; \big\|\chi_{A(n)}\big\|_{X(\mu)}^r \;\le\; \big\|\varepsilon^{-1}f_n\big\|_{X(\mu)}^r \;=\; \varepsilon^{-r}\big\|f_n\big\|_{X(\mu)}^r \;\le\; 2^{-n} \tag{2.9}$$

for $n \in \mathbb{N}$ because $\chi_{A(n)} \le |\varepsilon^{-1}f_n|$. Let $A := \bigcap_{n=1}^\infty\bigcup_{k=n}^\infty A(k)$. Since

$$\mu\Big(\bigcup_{k=n}^\infty A(k)\Big) \ge \mu(A(n)) \ge \delta \quad \text{for all} \quad n \in \mathbb{N},$$

we have

$$\mu(A) \;=\; \lim_{n\to\infty}\mu\Big(\bigcup_{k=n}^\infty A(k)\Big) \;\ge\; \delta. \tag{2.10}$$

We now show that $\mu(A) = 0$. Let $B(k) := A(k) \cap A$ for $k \in \mathbb{N}$, so that $A = \bigcup_{k=j}^\infty B(k)$ for each $j \in \mathbb{N}$. Given $j, n \in \mathbb{N}$ with $n \ge j$ and defining the sets

$C(j,n) := \bigcup_{k=j}^{n} B(k) \subseteq \bigcup_{k=j}^{n} A(k)$, we have, via (F4), (2.7) and (2.9), that

$$\begin{aligned} \|\!|\chi_{C(j,n)}|\!\| \le \|\chi_{C(j,n)}\|_{X(\mu)}^r \;&\le\; \Big\| \sum_{k=j}^{n} \chi_{A(k)} \Big\|_{X(\mu)}^r \\ &\le 4 \Big|\!\Big|\!\Big| \sum_{k=j}^{n} \chi_{A(k)} \Big|\!\Big|\!\Big| \;\le\; 4 \sum_{k=j}^{n} |\!|\!|\chi_{A(k)}|\!|\!| \;\le\; 4 \sum_{k=j}^{n} 2^{-k} \;<\; 2^{-j+3}, \end{aligned} \tag{2.11}$$

because $\chi_{C(j,n)} \le \sum_{k=j}^{n} \chi_{B(k)} \le \sum_{k=j}^{n} \chi_{A(k)}$. For each $j \in \mathbb{N}$, select $n(j) \in \mathbb{N}$ with $n(j) \ge j$ such that $\mu(A \setminus C(j,n(j))) < 2^{-j}$, which is possible because $\{C(j,n)\}_{n=1}^{\infty}$ is increasing with union A. Letting $C := \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} \big(A \setminus C(j,n(j))\big)$ we have $\mu(C) = 0$, because

$$\mu(C) \;\le\; \sum_{j=k}^{\infty} \mu\big(A \setminus C(j,n(j))\big) \;\le\; \sum_{j=k}^{\infty} 2^{-j} \;=\; 2^{-k+1} \to 0$$

as $k \to \infty$. Given $k \in \mathbb{N}$, let $D(k) := \bigcap_{j=k}^{\infty} C(j,n(j))$. Then for every $j \ge k$ we have, via (2.7) and (2.11), that

$$|\!|\!|\chi_{D(k)}|\!|\!| \;\le\; \|\chi_{D(k)}\|_{X(\mu)}^r \;\le\; \|\chi_{C(j,n(j))}\|_{X(\mu)}^r \;\le\; 4\,|\!|\!|\chi_{C(j,n(j))}|\!|\!| \;\le\; 4(2^{-j+3}) \to 0$$

as $j \to \infty$. So, $|\!|\!|\chi_{D(k)}|\!|\!| = 0$, i.e., $\chi_{D(k)} = 0$ (μ-a.e.), and hence $\mu\big(D(k)\big) = 0$ for all $k \in \mathbb{N}$. Since $A = C \cup \big(\bigcup_{k=1}^{\infty} D(k)\big)$ and $\mu(C) = 0$, we have $\mu(A) = 0$. This contradicts (2.10). Therefore, the inclusion map i is continuous.

(ii) Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $X(\mu)$. By part (i), it is also Cauchy in the complete, metrizable topological vector space $L^0(\mu)$ and hence, converges in $L^0(\mu)$. That is, $\{f_n\}_{n=1}^{\infty}$ converges in measure, and therefore it admits a subsequence converging μ-a.e.

(iii) This is clear from (i).

(iv) Since $\chi_\Omega \in \operatorname{sim}\Sigma \subseteq X(\mu)$ and $|f| \le \|f\|_\infty \cdot \chi_\Omega$ for $f \in L^\infty(\mu)$, it follows from the ideal property of $X(\mu)$ in $L^0(\mu)$ that $L^\infty(\mu) \subseteq X(\mu)$. The continuity of the inclusion map follows from (Q2) and (2.3) which yield $\|f\|_{X(\mu)} \le \|\chi_\Omega\|_{X(\mu)} \cdot \|f\|_\infty$ for $f \in L^\infty(\mu)$.

(v) It is clear that $\big(f \wedge n\chi_\Omega\big) \uparrow f$ for every $f \in X(\mu)^+$. Then apply [108, Proposition 1.2.6(ii)] and the fact that $X(\mu) = X_{\mathbb{R}}(\mu) + iX_{\mathbb{R}}(\mu)$.

(vi) Recall the F-norm $|\!|\!| \cdot |\!|\!|$ and $r > 0$ as given in the proof of part (i). Since $|\!|\!| \cdot |\!|\!|$ satisfies the triangle inequality, it follows that $\lim_{n\to\infty} |\!|\!|f_n|\!|\!| = |\!|\!|f|\!|\!|$ and hence, also that $\lim_{n\to\infty} |\!|\!|f_n|\!|\!|^{1/r} = |\!|\!|f|\!|\!|^{1/r}$. This and (2.7) give

$$\limsup_{n\to\infty} \|f_n\|_{X(\mu)} \le 4^{1/r} \lim_{n\to\infty} |\!|\!|f_n|\!|\!|^{1/r} \;=\; 4^{1/r} |\!|\!|f|\!|\!|^{1/r} \;\le\; 4^{1/r} \|f\|_{X(\mu)}$$

and $$\|f\|_{X(\mu)} \le 4^{1/r} |\!|\!|f|\!|\!|^{1/r} \;=\; 4^{1/r} \lim_{n\to\infty} |\!|\!|f_n|\!|\!|^{1/r} \;\le\; 4^{1/r} \liminf_{n\to\infty} \|f_n\|_{X(\mu)}.$$

So, (2.6) is established. □

Remark 2.3. (i) Since $X(\mu)$ is an order ideal in $L^0(\mu)$, we see that $\operatorname{sim}\Sigma \subseteq X(\mu)$ if and only if $\chi_\Omega \in X(\mu)$.

(ii) Let $\rho : M^+(\mu) \to [0,\infty]$ be a *function norm* in the sense of general normed function spaces (also called normed Köthe function spaces), [164, Ch. 15], where $M^+(\mu)$ is the space of all (μ-equivalence classes of) $[0,\infty]$-valued, Σ-measurable functions on Ω. Then any $f \in M^+(\mu)$ satisfying $\rho(f) < \infty$ is necessarily finite μ-a.e., [164, Ch. 15, §63, Theorem 1], and hence, belongs to $L^0(\mu)$. Then, in our sense,

$$L_\rho := \{f \in L^0(\mu) : \rho(|f|) < \infty\} \tag{2.12}$$

is a normed function space equipped with the norm $f \mapsto \rho(|f|)$, for $f \in L_\rho$, and is continuously embedded into $L^0(\mu)$, [101, Theorem 6.9]. Conversely, given a normed function space $(X(\mu), \|\cdot\|_{X(\mu)})$ in our sense, define $\rho : M^+(\mu) \to [0,\infty]$ by

$$\rho(f) := \begin{cases} \|f\|_{X(\mu)} & \text{if} \quad f \in X(\mu)^+, \\ \infty & \text{if} \quad f \in M(\mu)^+ \setminus X(\mu)^+. \end{cases}$$

Then ρ is a function norm in the sense of [164, Ch. 15] and $L_\rho = (X(\mu), \|\cdot\|_{X(\mu)})$.

In the case of a quasi-normed function space $X(\mu)$ whose lattice quasi-norm $\|\cdot\|_{X(\mu)}$ fails to be a norm, there is a genuine difficulty in adapting the arguments from [101], because the equivalent F-norm $|\!|\!|\cdot|\!|\!|$ (see (2.4)) may not be compatible with the order (i.e., $|\!|\!|f|\!|\!| \le |\!|\!|g|\!|\!|$ may not hold for some $f, g \in X(\mu)$ satisfying $|f| \le |g|$). Therefore, we have adopted a different approach although the idea for our proof has been drawn from [101]. $\square$

An order ideal $X(\mu)$ of $L^0(\mu)$ is called a *quasi-Banach function space* (briefly q-B.f.s.) based on the measure space (Ω, Σ, μ) if $\operatorname{sim}\Sigma \subseteq X(\mu)$ and if $X(\mu)$ is equipped with a lattice quasi-norm $\|\cdot\|_{X(\mu)}$ for which it is complete. In other words, $(X(\mu), \|\cdot\|_{X(\mu)})$ is a complete quasi-normed function space such that $\operatorname{sim}\Sigma \subseteq X(\mu)$. If, in addition, $\|\cdot\|_{X(\mu)}$ happens to be a norm, then $X(\mu)$ is called a *Banach function space* (briefly B.f.s.). Well-known examples of q-B.f.s.' (which are non-normable) include $L^p([0,1])$ and (weighted) ℓ^p, for $0 < p < 1$, (see Examples 2.10 and 2.11 below, and [83, Ch. II]), certain Lorentz spaces, [30, p. 159], [87], Orlicz spaces, [30, p. 159], and certain mixed (p_1, p_2)-norm Lebesgue spaces on product spaces, [30, p. 159]. Our definition of a q-B.f.s. differs from that in [30, p. 155] which requires two additional properties. The requirement that $\operatorname{sim}\Sigma \subseteq X(\mu)$, crucial for the analysis that we develop in later chapters, is assumed in the definition of a B.f.s. or q-B.f.s. by some authors (e.g., [13, p. 2], [99, p. 28]) but not by all (e.g., [30, p. 155], [101, p. 42], [164, Ch. 15]). The closed ideal

$$X(\mu) := \big\{f \in L^\infty([0,1]) : f \text{ vanishes at } 0\big\}$$

of $L^\infty([0,1])$ is an example of a complete normed function space which fails to contain the simple functions; here "f vanishes at 0" means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|f\chi_{[0,\delta]}\|_\infty \le \varepsilon$.

Given a q-B.f.s. $X(\mu)$, let $X(\mu)_{\mathrm{b}} := \overline{\operatorname{sim}\Sigma}$ denote the closure of $\operatorname{sim}\Sigma$ in $X(\mu)$.

Lemma 2.4. *For any q-B.f.s. $X(\mu)$ the subspace $X(\mu)_{\mathrm{b}}$ is a closed order ideal in $X(\mu)$. In particular, $X(\mu)_{\mathrm{b}}$ is a q-B.f.s.*

Proof. It is clear that $X(\mu)_{\mathrm{b}}$ is complete and the restriction of $\|\cdot\|_{X(\mu)}$ to $X(\mu)_{\mathrm{b}}$ is a quasi-norm. It remains to verify that this restricted quasi-norm has the lattice property (2.3).

Given $f, g \in X(\mu)$, the inequality $\big||f| - |g|\big| \le |f - g|$ (which holds pointwise on Ω) together with the lattice property of $\|\cdot\|_{X(\mu)}$ imply that

$$h \in X(\mu)_{\mathrm{b}} \quad \Longrightarrow \quad |h| \in X(\mu)_{\mathrm{b}}. \tag{2.13}$$

An examination of the proof of Proposition 3.10 in Chapter 1 of [13] shows that it does *not* require the properties (P3) and (P5) listed on p. 2 of [13] (properties which we do not assume) and that the proof given there also applies in the setting of a q-B.f.s. This observation, together with an examination of that part of the proof of Theorem 3.11 in Chapter 1 of [13] establishing that $X(\mu)_{\mathrm{b}}$ is an ideal in $X(\mu)$ (which again does not use properties (P3) and (P5)), shows that in the setting of a q-B.f.s. we also have the property that $g \in X(\mu)_{\mathrm{b}}$ whenever $f \in X(\mu)_{\mathrm{b}}$ and $g \in L^0(\mu)$ are $\mathbb{R}$-valued functions satisfying $|g| \le |f|$ (μ-a.e.). The case for $\mathbb{C}$-valued f, g then follows from the $\mathbb{R}$-valued case, the property (2.13) and the inequalities $0 \le |\mathrm{Re}\,(g)| \le |g| \le |f|$ and $0 \le |\mathrm{Im}\,(g)| \le |g| \le |f|$. For more precise details we refer to [89]. □

Let $(X(\mu), \|\cdot\|_{X(\mu)})$ be a B.f.s. Recall that $X(\mu)$ is the complexification of the real B.f.s. $X_{\mathbb{R}}(\mu) := X(\mu) \cap L^0_{\mathbb{R}}(\mu)$ equipped with its given norm $\|\cdot\|_{X_{\mathbb{R}}(\mu)}$ and that $\|\cdot\|_{X_{\mathbb{R}}(\mu)}$ coincides with the restriction of $\|\cdot\|_{X(\mu)}$ to $X_{\mathbb{R}}(\mu)$, that is, $\|f\|_{X(\mu)} = \big\||f|\big\|_{X_{\mathbb{R}}(\mu)}$ for every $f \in X(\mu)$. Our setting is, of course, a special case of the complexification of a real Banach lattice. It is time to be more precise. A *real* Banach lattice is by definition a complete real normed lattice. Since we are mainly dealing with vector spaces over $\mathbb{C}$, vector spaces over $\mathbb{R}$ are called *real* vector spaces. Real Banach lattices are simply called Banach lattices in the literature and we refer to [2, 99, 108, 149, 165] for their properties. In order to give a formal definition of the complexification of a real Banach lattice $X_{\mathbb{R}}(\mu)$ according to [165, Exercise 100.15], let $(Z_{\mathbb{R}}, \|\cdot\|_{Z_{\mathbb{R}}})$ be a real Banach lattice. The complexification $Z := Z_{\mathbb{R}} + iZ_{\mathbb{R}}$ of the real vector lattice $Z_{\mathbb{R}}$ is a complex vector lattice with

$$|z| = \bigvee_{0 \le \theta < 2\pi} \Big|\big(\cos\theta\big)x + \big(\sin\theta\big)y\Big|, \qquad z = x + iy \in Z \quad \text{with} \quad x, y \in Z_{\mathbb{R}}; \tag{2.14}$$

[165, Ch. 14, §91], [109]. Since $Z_{\mathbb{R}}$ is uniformly complete, [165, Exercise 100.15], and Archimedean, [165, p. 282], the supremum in (2.14) necessarily exists, [165, Theorem 91.2]. Define a norm $\|\cdot\|_Z$ on Z by $\|z\|_Z := \big\||z|\big\|_{Z_{\mathbb{R}}}$ for $z \in Z$. Then the normed space $(Z, \|\cdot\|_Z)$ is complete and $\|\cdot\|_Z$ is a lattice norm, that is, $\|z\|_Z \le \|w\|_Z$ whenever $z, w \in Z$ with $|z| \le |w|$. We say that the pair $(Z, \|\cdot\|_Z)$ is a *complex* Banach lattice and is the complexification of the real Banach lattice

$(Z_{\mathbb{R}}, \|\cdot\|_{Z_{\mathbb{R}}})$. Unless we need an emphasis, complex Banach lattices are also simply called Banach lattices. We sometimes call $Z_{\mathbb{R}}$ the *real part* of the Banach lattice Z. From the monotonicity of the norm $\|\cdot\|_Z$, it follows that

$$\|x\|_{Z_{\mathbb{R}}} \leq \|x+iy\|_Z \quad \text{and} \quad \|y\|_{Z_{\mathbb{R}}} \leq \|x+iy\|_Z \qquad \text{whenever} \quad x, y \in Z_{\mathbb{R}}, \quad (2.15)$$

for which we require the inequalities $|x| \leq |x+iy|$ and $|y| \leq |x+iy|$ for $x, y \in Z_{\mathbb{R}}$, [165, Theorem 91.2].

The quasi-norm $\|\cdot\|_{X(\mu)}$ of a q-B.f.s. $X(\mu)$ is called *σ-order continuous* (briefly, σ-o.c.) if, for every decreasing sequence

$$\{f_n\}_{n=1}^{\infty} \quad \text{in} \quad X(\mu)^+ \quad \text{with} \quad \inf_{n\in\mathbb{N}}(f_n) = 0 \quad (\text{i.e., } f_n \downarrow 0),$$

we have $\lim_{n\to\infty} \|f_n\|_{X(\mu)} = 0$. We also say that $X(\mu)$ or, more explicitly, that $(X(\mu), \|\cdot\|_{X(\mu)})$ is a σ-o.c. q-B.f.s. The class of spaces which are σ-o.c. is rather large. Typical examples which fail to be σ-o.c. are $L^\infty([0,1])$, the Zygmund space $L_{\exp}$, the Lorentz spaces $L^{p,\infty}([0,1])$ for $1 < p < \infty$ and certain Marcinkiewicz spaces; for the definition of such spaces, see [13], [91], for example.

Remark 2.5. We define a q-B.f.s. $(X(\mu), \|\cdot\|_{X(\mu)})$ to be *order continuous* (briefly o.c.) if, for every downward directed net $\{f_\alpha\}_\alpha$ in $X(\mu)^+$ with $\inf_\alpha(f_\alpha) = 0$ (i.e., $f_\alpha \downarrow 0$), we have $\lim_\alpha \|f_\alpha\|_{X(\mu)} = 0$. The concepts of order continuity and σ-order continuity for general Banach lattices have long been investigated. In the case of a q-B.f.s. $X(\mu)$, these concepts coincide. Indeed, order continuity clearly implies σ-order continuity. Assume conversely that $X(\mu)$ is σ-order continuous. Let $f_\alpha \downarrow 0$ in $X(\mu)^+$. Since the real vector lattice $L^0_{\mathbb{R}}(\mu)$, which contains $X(\mu)^+$, is super Dedekind complete, [102, Example 23.3(iv)], we can select a decreasing sequence $\{f_{\alpha(n)}\}_{n=1}^\infty$ from the net $\{f_\alpha\}_\alpha$ such that $f_{\alpha(n)} \downarrow 0$. So

$$0 \;\leq\; \inf_\alpha \|f_\alpha\|_{X(\mu)} \;\leq\; \inf_{n\in\mathbb{N}} \|f_{\alpha(n)}\|_{X(\mu)} \;=\; 0.$$

In other words, $\lim_\alpha \|f_\alpha\|_{X(\mu)} = 0$, that is, $X(\mu)$ is order continuous. $\square$

Remark 2.6. Let $X(\mu)$ be a q-B.f.s. which is σ-o.c. Then $\operatorname{sim}\Sigma$ is *dense* in $X(\mu)$. To see this, suppose first that $f \in X(\mu)^+$. Choose Σ-simple functions $0 \leq s_n \uparrow f$ pointwise on Ω. Then $(f - s_n) \downarrow 0$ in the order of $X(\mu)$ from which it follows that $\lim_{n\to\infty} \|f - s_n\|_{X(\mu)} = 0$, that is, $s_n \to f$ in $X(\mu)$. The case for an arbitrary $f \in X(\mu)$ follows from the identity

$$f \;=\; \big((\mathrm{Re} f)^+ - (\mathrm{Re} f^-)\big) \;+\; i\big((\mathrm{Im} f)^+ - (\mathrm{Im} f)^-\big)$$

and the lattice property of $\|\cdot\|_{X(\mu)}$ (see (2.3)).

We point out that the same proof applies in a quasi-normed function space which is σ-o.c. and contains $\operatorname{sim}\Sigma$; the completeness of the space is not used.

The space $L^\infty(\mu)$ is not σ-o.c., yet $\operatorname{sim}\Sigma$ is dense in it. $\square$

Lemma 2.7. *Let $X(\mu)$ and $Y(\mu)$ be two q-B.f.s.' such that $X(\mu) \subseteq Y(\mu)$ as vector sublattices of $L^0(\mu)$. Then the natural inclusion map from $X(\mu)$ into $Y(\mu)$ is continuous.*

Proof. Take a sequence $\{f_n\}_{n=1}^{\infty} \subseteq X(\mu) \subseteq Y(\mu)$ such that $f_n \to f$ in $X(\mu)$ and $f_n \to g$ in $Y(\mu)$. By Proposition 2.2(iii), select a subsequence $\{f_{n(k)}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} f_{n(k)}(\omega) = f(\omega)$ for μ-a.e. $\omega \in \Omega$. Choose a subsequence of $\{f_{n(k)}\}_{k=1}^{\infty}$ which μ-a.e. converges to g, again via Proposition 2.2(iii). So, we have $f = g$ (μ-a.e.). The inclusion map is then closed and hence continuous via the Closed Graph Theorem, [88, §15.12. (3)], applied in the complete metrizable topological vector spaces $X(\mu)$ and $Y(\mu)$. □

The duality between a q-B.f.s. $X(\mu)$ and its (topological) dual space $X(\mu)^* = \mathcal{L}(X(\mu), \mathbb{C})$ is denoted by

$$\langle f, \xi \rangle := \xi(f), \qquad f \in X(\mu), \ \ \xi \in X(\mu)^*.$$

Then $X(\mu)^*$ is equipped with the dual norm

$$\| \cdot \|_{X(\mu)^*} : \xi \longmapsto \sup \Big\{ \big| \langle f, \xi \rangle \big| : f \in \mathbf{B}[X(\mu)] \Big\}, \qquad \xi \in X(\mu)^*. \tag{2.16}$$

Note that $\| \cdot \|_{X(\mu)^*}$ is indeed a norm for which $X(\mu)^*$ is complete because $\mathbb{C}$ is a Banach space and because of the inequality

$$|\langle f, \xi \rangle| \leq \|f\|_{X(\mu)} \|\xi\|_{X(\mu)^*}, \qquad f \in X(\mu), \ \ \xi \in X(\mu)^*. \tag{2.17}$$

Let $X_{\mathbb{R}}(\mu)^*$ denote the dual space of the real q-B.f.s. $X_{\mathbb{R}}(\mu)$, that is, $X_{\mathbb{R}}(\mu)^*$ is the real vector space consisting of all continuous $\mathbb{R}$-linear functionals from $X_{\mathbb{R}}(\mu)$ into $\mathbb{R}$. As in the case of $X(\mu)^*$, the real vector space $X_{\mathbb{R}}(\mu)^*$ is a real Banach space with respect to the corresponding dual norm $\| \cdot \|_{X_{\mathbb{R}}(\mu)^*}$. We define an order on $X_{\mathbb{R}}(\mu)^*$ by saying that a functional $\eta \in X_{\mathbb{R}}(\mu)^*$ is *positive* if $\langle f, \eta \rangle \geq 0$ for every $f \in X_{\mathbb{R}}(\mu)^+$.

Lemma 2.8. *Let $X(\mu)$ be a q-B.f.s. Then the following assertions hold.*

(i) *The dual space $(X_{\mathbb{R}}(\mu)^*, \| \cdot \|_{X_{\mathbb{R}}(\mu)^*})$ of the real q-B.f.s. $(X_{\mathbb{R}}(\mu), \| \cdot \|_{X_{\mathbb{R}}(\mu)})$ is a real Banach lattice with respect to the order defined just prior to the lemma.*

(ii) *The dual space $(X(\mu)^*, \ \| \cdot \|_{X(\mu)^*})$ of the q-B.f.s. $(X(\mu), \| \cdot \|_{X(\mu)})$ is the Banach lattice given by the complexification of the real Banach lattice $(X_{\mathbb{R}}(\mu)^*, \| \cdot \|_{X_{\mathbb{R}}(\mu)^*})$.*

Proof. (i) This is a well-known fact for the case when $X_{\mathbb{R}}(\mu)$ is a real Banach lattice. The proof of this standard result can be found in many places in the literature and we can adapt it to the case of a q-B.f.s. For example, a straightforward adaptation of the corresponding proof of [165, Theorem 85.6] establishes part (i).

(ii) This can be proved by adapting the argument in [165, pp. 323–324] which establishes our claim whenever $X(\mu)$ is a B.f.s. (indeed, even for Banach lattices). Such an adaptation requires us to establish the equality

$$\|\xi\|_{X(\mu)^*} \;=\; \sup\Big\{\big\langle f,\,|\xi|\big\rangle : f\in \mathbf{B}[X(\mu)]\cap X(\mu)^+\Big\}, \qquad \xi\in X(\mu)^*,$$

for the q-B.f.s. $X(\mu)$; this can be proved as in [165, pp. 323–324] because the triangle inequality is not used there. $\square$

If $X(\mu)$ is a B.f.s., then $X(\mu)^*$ is, of course, non-trivial and separates points of $X(\mu)$. This is not always the case for a q-B.f.s.; see Example 2.10 below. But, let us record a general fact first.

Lemma 2.9. *A q-B.f.s. $X(\mu)$ has trivial dual if and only if $\mathcal{L}\big(X(\mu),E\big)=\{0\}$ for every/some Banach space $E\neq\{0\}$.*

Proof. Suppose that $X(\mu)^*=\{0\}$ and E is any Banach space. Let $T\in\mathcal{L}(X(\mu),E)$. Then, for each $x^*\in E^*$ we have $x^*\circ T\in X(\mu)^*$ and hence, $x^*\circ T=0$. That is, for fixed $f\in X(\mu)$ we have $\langle T(f),\,x^*\rangle=0$ for all $x^*\in E^*$ and so $T(f)=0$. Accordingly, $T=0$.

Suppose that $E\neq\{0\}$ and $\mathcal{L}(X(\mu),E)=\{0\}$. Fix $x\in E\setminus\{0\}$. For each $\xi\in X(\mu)^*$, the map $f\mapsto\langle f,\xi\rangle x$ for $f\in X(\mu)$ belongs to $\mathcal{L}(X(\mu),E)$ and hence, is identically zero. Since $x\neq 0$, we conclude that $\langle f,\xi\rangle=0$ for all $f\in X(\mu)$, that is, $\xi=0$. $\square$

Example 2.10. If $\mu:\Sigma\to[0,\infty)$ is a non-atomic measure, then for $L^r(\mu)$ we have $L^r(\mu)^*=\{0\}$ whenever $0<r<1$, [83, Theorem 2.2]. According to Proposition 2.2(i) it follows that $L^0(\mu)^*\subseteq L^r(\mu)^*=\{0\}$, that is, $L^0(\mu)^*=\{0\}$; see also [83, Theorem 2.2]. In particular, $L^0(\mu)$ cannot be locally convex and hence, it is non-normable.

A direct consequence of the identity $L^r(\mu)^*=\{0\}$ is that $\mathcal{L}\big(L^r(\mu),E\big)=\{0\}$ for every/some Banach space E (see Lemma 2.9). $\square$

If μ is purely atomic, then every non-trivial q-B.f.s. over (Ω,Σ,μ) admits a non-trivial dual. We present such q-B.f.s.', which are order ideals of the vector lattice $\mathbb{C}^{\mathbb{N}}$ consisting of all $\mathbb{C}$-valued functions on $\mathbb{N}$. Here $\mathbb{C}^{\mathbb{N}}$ is endowed with the pointwise order in the sense that $f\in\mathbb{C}^{\mathbb{N}}$ is non-negative if and only if $f(n)\geq 0$ for each $n\in\mathbb{N}$.

Example 2.11. Take a function $\varphi\in\ell^1$ such that $\varphi(n)>0$ for every $n\in\mathbb{N}$. Define a *finite* measure $\mu:2^{\mathbb{N}}\to[0,\infty)$ by

$$\mu(A):=\sum_{n\in A}\varphi(n),\qquad A\in 2^{\mathbb{N}}. \tag{2.18}$$

Then

$$\mathbb{C}^{\mathbb{N}}\;=\;L^0(\mu)\qquad\text{and}\qquad \ell^\infty\;=\;L^\infty(\mu). \tag{2.19}$$

The first equality in (2.19) is meant both in the sense of vector lattices and also topologically, where $\mathbb{C}^{\mathbb{N}}$ is equipped with the topology of coordinatewise convergence (in which case it is a complete, metrizable, *locally convex* topological vector space). Indeed, it follows easily from Proposition 2.2(iii) and the fact that there are no non-trivial μ-null sets, that the identity map from $L^0(\mu)$ *onto* $\mathbb{C}^{\mathbb{N}}$ is closed and hence, continuous (by the Closed Graph Theorem, [88, §15.12.(3)]). Then the Open Mapping Theorem, [83, Corollary 1.5], ensures that the identity map is an isomorphism. Hence, $L^0(\mu)$ is precisely the locally convex space $\mathbb{C}^{\mathbb{N}}$ and, in particular, $L^0(\mu)^* = c_{00}(\mathbb{N}) = (\mathbb{C}^{\mathbb{N}})^*$. An immediate consequence of this is that every q-B.f.s $X(\mu) \neq \{0\}$ over the measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ has a non-trivial dual because $X(\mu)$ is continuously embedded into $L^0(\mu) = \mathbb{C}^{\mathbb{N}}$ (which has a non-trivial dual). Now, let us discuss some specific cases.

Let $0 < r < \infty$. To emphasize that μ is purely atomic, let us write

$$\ell^r(\mu) := L^r(\mu). \tag{2.20}$$

The order ideal $\ell^r(\mu)$ of $\mathbb{C}^{\mathbb{N}}$ is equipped with the quasi-norm

$$f \longmapsto \|f\|_{\ell^r(\mu)} := \left(\sum_{n=1}^{\infty} |f(n)|^r \varphi(n)\right)^{1/r} = \left(\int_{\mathbb{N}} |f|^r \, d\mu\right)^{1/r}, \qquad f \in \ell^r(\mu), \tag{2.21}$$

so that $\ell^r(\mu)$ is a q-B.f.s. based on the finite measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$. Of course, if $1 \leq r < \infty$, then $\ell^r(\mu)$ is a B.f.s. Since

$$\ell^r(\mu) = \varphi^{-1/r} \cdot \ell^r := \{\varphi^{-1/r} g : g \in \ell^r\}, \tag{2.22}$$

the q-B.f.s. $\ell^r(\mu)$ is order and isometrically isomorphic to ℓ^r via the canonical map $\Phi : \ell^r(\mu) \to \ell^r$ defined by

$$\Phi(f) := \varphi^{1/r} f, \qquad f \in \ell^r(\mu). \tag{2.23}$$

Now, let $0 < r < 1$. Since $(\ell^r)^* = \ell^\infty$, [83, Theorem 2.3], the dual space $\ell^r(\mu)^*$ of $\ell^r(\mu)$ is the weighted ℓ^∞-space

$$\varphi^{1/r} \cdot \ell^\infty := \{\varphi^{1/r} g : g \in \ell^\infty\}$$

with the duality

$$\langle f, \varphi^{1/r} g \rangle := \sum_{n=1}^{\infty} f(n) \cdot \big(\varphi^{1/r}(n) g(n)\big), \qquad f \in \ell^r(\mu), \quad g \in \ell^\infty, \tag{2.24}$$

induced by the canonical map Φ. So, $\ell^r(\mu)^*$ is non-trivial and separates points of $\ell^r(\mu)$.

Now, still for $0 < r < 1$, let us prove that $\ell^r(\mu)$ does not admit any equivalent norm. On the contrary, if there is an equivalent norm $\|\!|\cdot|\!\|_{\ell^r(\mu)}$ on $\ell^r(\mu)$, then there exist positive constants C_1 and C_2 such that

$$C_1 \|f\|_{\ell^r(\mu)} \leq \|\!|f|\!\|_{\ell^r(\mu)} \leq C_2 \|f\|_{\ell^r(\mu)}, \qquad f \in \ell^r(\mu).$$

With $f_n := \varphi^{-1/r}\chi_{\{n\}}$, for each $n \in \mathbb{N}$, it follows for every $k \in \mathbb{N}$ that

$$\begin{aligned} C_1 k^{1/r} = C_1 \Big\| \sum_{n=1}^{k} f_n \Big\|_{\ell^r(\mu)} &\le \Big\| \sum_{n=1}^{k} f_n \Big\|_{\ell^r(\mu)} \le \sum_{n=1}^{k} \| f_n \|_{\ell^r(\mu)} \\ &\le C_2 \sum_{n=1}^{k} \| f_n \|_{\ell^r(\mu)} = C_2 k, \end{aligned}$$

which is impossible. □

Let $X(\mu)$ be a general q-B.f.s. For each $f \in X(\mu)$, define

$$\|f\|_{\mathrm{b},X(\mu)} := \sup\Big\{ \big|\langle f, \xi\rangle\big| \;:\; \xi \in \mathbf{B}[X(\mu)^*] \Big\}. \tag{2.25}$$

This seminorm will play an important role in Chapter 6. A natural way to view it is via the canonical map J from $X(\mu)$ into its bidual $X(\mu)^{**} := (X(\mu)^*)^*$. Indeed, for each $f \in X(\mu)$ define $J(f)$ to be the linear functional $\xi \mapsto \langle f, \xi\rangle$ on $X(\mu)^*$. According to (2.17) we see that $J(f)$ is continuous with $\|J(f)\|_{X(\mu)^{**}} \le \|f\|_{X(\mu)}$. Moreover, $\|f\|_{\mathrm{b},X(\mu)} = \|J(f)\|_{X(\mu)^{**}}$; see (2.25). If $X(\mu)$ happens to be a B.f.s., then J is a linear isometry, and hence $\|\cdot\|_{\mathrm{b},X(\mu)} = \|\cdot\|_{X(\mu)}$. In the case when $X(\mu)^* = \{0\}$, then $J = 0$, and so $\|\cdot\|_{\mathrm{b},X(\mu)}$ is identically zero. To present basic properties of the seminorm $\|\cdot\|_{\mathrm{b},X(\mu)}$, we require some preliminary results. Recall that $X(\mu)$ is the complexification of its real part $X_{\mathbb{R}}(\mu) = X(\mu) \cap L^0_{\mathbb{R}}(\mu)$. As in the case of the dual $X_{\mathbb{R}}(\mu)^*$, the bidual $X(\mu)^{**} = X_{\mathbb{R}}(\mu)^{**} + iX_{\mathbb{R}}(\mu)^{**}$ is the complexification of $X_{\mathbb{R}}(\mu)^{**} := \mathcal{L}(X_{\mathbb{R}}(\mu)^*, \mathbb{R})$. Let $J_{\mathbb{R}} : X_{\mathbb{R}}(\mu) \to X_{\mathbb{R}}(\mu)^{**}$ denote the natural embedding. Then $J(f + ig) = J_{\mathbb{R}}(f) + iJ_{\mathbb{R}}(g)$ whenever $f, g \in X_{\mathbb{R}}(\mu)$. In other words, J is a unique $\mathbb{C}$-linear extension of $J_{\mathbb{R}}$ (see [149, p. 135]). Note that $J_{\mathbb{R}}$ is a *positive operator*.

Lemma 2.12. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. over (Ω, Σ, μ). Then*

$$J\big(|f|\big) \;=\; \big|J(f)\big|, \qquad f \in X(\mu).$$

Proof. First we assume that $f \in X_{\mathbb{R}}(\mu)$ and show that

$$J_{\mathbb{R}}\big(|f|\big) \;=\; \big|J_{\mathbb{R}}(f)\big|. \tag{2.26}$$

We shall verify this as in the proof of Theorem 5.4 in [2], although this theorem itself may not be applicable to our case because the topological bidual $X_{\mathbb{R}}(\mu)^{**}$ of $X_{\mathbb{R}}(\mu)$ may not coincide with the order bidual of $X_{\mathbb{R}}(\mu)$. Fix $\xi \in \big(X_{\mathbb{R}}(\mu)^*\big)^+$. By [2, Theorem 1.18], we have

$$J_{\mathbb{R}}\big(|f|\big)(\xi) \;=\; \big\langle |f|, \xi \big\rangle \;=\; \max\, \eta(f),$$

where the maximum is taken over all $\mathbb{R}$-linear functionals $\eta : X_{\mathbb{R}}(\mu) \to \mathbb{R}$ satisfying $-\xi \le \eta \le \xi$ (i.e., $\eta \in [-\xi, \xi]$). It is routine to check that such an η is continuous, that is, $\eta \in X_{\mathbb{R}}(\mu)^*$. Choose $\eta_0 \in [-\xi, \xi] \subseteq X_{\mathbb{R}}(\mu)^*$ such that $J_{\mathbb{R}}\big(|f|\big)(\xi) = \eta_0(f)$. Then

$$0 \leq J_{\mathbb{R}}\big(|f|\big)(\xi) \;=\; J_{\mathbb{R}}(f)(\eta_0) \;=\; \big|J_{\mathbb{R}}(f)(\eta_0)\big| \;\leq\; \big|J_{\mathbb{R}}(f)\big|(\xi),$$

which implies that $J_{\mathbb{R}}\big(|f|\big) \leq |J_{\mathbb{R}}(f)|$ because $\xi \in \big(X_{\mathbb{R}}(\mu)^*\big)^+$ is arbitrary. Since the reverse inequality $J_{\mathbb{R}}(|f|) \geq \big|J_{\mathbb{R}}(f)\big|$ is clear, we have established (2.26) for all $f \in X_{\mathbb{R}}(\mu)$. Therefore, $J_{\mathbb{R}}$ is a lattice homomorphism, that is, it preserves the order operations (see [2, Theorem 7.2]).

Now take a $\mathbb{C}$-valued function $f \in X(\mu)$. To prove the inequality $|J(f)| \geq J(|f|)$, let $\mathbf{F}$ be the collection of all finite subsets of the interval $[0, 2\pi)$ ordered by inclusion. Let $g := \mathrm{Re}(f)$ and $h := \mathrm{Im}(f)$. Then, by [149, p. 134], we have, in the complex vector lattice $X(\mu)$, that $|f| = \bigvee_{\theta \in [0,2\pi)} \big|(\cos\theta)g + (\sin\theta)h\big|$. The fact that $X(\mu)$ is o.c. (see Remark 2.5) yields

$$\begin{aligned} |f| &= \bigvee_{\Lambda \in \mathbf{F}} \Big(\bigvee_{\theta \in \Lambda} \big|(\cos\theta)g + (\sin\theta)h\big|\Big) \\ &= \lim_{\Lambda \in \mathbf{F}} \Big(\bigvee_{\theta \in \Lambda} \big|(\cos\theta)g + (\sin\theta)h)\big|\Big). \end{aligned}$$

Now, since $J_{\mathbb{R}}$ is a continuous lattice homomorphism and since

$$\big((\cos\theta)g + (\sin\theta)h\big) \in X_{\mathbb{R}}(\mu) \quad \text{for} \quad \theta \in [0, 2\pi),$$

we have that

$$\begin{aligned} J_{\mathbb{R}}(|f|) &= \lim_{\Lambda \in \mathbf{F}} J_{\mathbb{R}}\Big(\bigvee_{\theta \in \Lambda} \big|(\cos\theta)g \;+\; (\sin\theta)h\big|\Big) \\ &= \lim_{\Lambda \in \mathbf{F}} \bigvee_{\theta \in \Lambda} \Big(J_{\mathbb{R}}\big(\big|(\cos\theta)g \;+\; (\sin\theta)h\big|\big)\Big) \\ &= \lim_{\Lambda \in \mathbf{F}} \bigvee_{\theta \in \Lambda} \Big|(\cos\theta) J_{\mathbb{R}}(g) \;+\; (\sin\theta) J_{\mathbb{R}}(h)\Big|. \end{aligned}$$

On the other hand, from [149, Ch. IV, Theorem 1.8] applied to the continuous linear functional

$$J(f) \;=\; J_{\mathbb{R}}(g) \;+\; iJ_{\mathbb{R}}(h) \;\in X(\mu)^{**} = \mathcal{L}(X(\mu)^*, \mathbb{C}),$$

it follows that

$$|J(f)| = \bigvee_{\theta \in [0,2\pi)} \Big|(\cos\theta)J_{\mathbb{R}}(g) \;+\; (\sin\theta)J_{\mathbb{R}}(h)\Big|,$$

and hence,

$$\big|J(f)\big| \geq \lim_{\Lambda \in \mathbf{F}} \bigvee_{\theta \in \Lambda} \Big|(\cos\theta)J_{\mathbb{R}}(g) \;+\; (\sin\theta)J_{\mathbb{R}}(h)\Big| = J_{\mathbb{R}}\big(|f|\big) = J\big(|f|\big).$$

Conversely, $|J(f)| \leq |J|(|f|) = J(|f|)$ (see [149, Ch. IV, Definition 1.7]). So, the lemma has been established. $\square$

Some basic properties of the seminorm $\|\cdot\|_{\mathrm{b},X(\mu)}$ are now presented.

Proposition 2.13. *Let $X(\mu)$ be a q-B.f.s. based on (Ω,Σ,μ). The following statements hold for the seminorm $\|\cdot\|_{\mathrm{b},X(\mu)}: X(\mu)\to[0,\infty)$ defined by (2.25).*

(i) *We have*

$$0 \le \|f\|_{\mathrm{b},X(\mu)} = \big\|J(f)\big\|_{X(\mu)^{**}} \le \|f\|_{X(\mu)}, \qquad f\in X(\mu). \tag{2.27}$$

(ii) *Let $X(\mu)$ be either a B.f.s. or a σ-order continuous q-B.f.s. Then the seminorm $\|\cdot\|_{\mathrm{b},X(\mu)}$ is a lattice seminorm, namely $\|f\|_{\mathrm{b},X(\mu)} \le \|g\|_{\mathrm{b},X(\mu)}$ whenever $f,g\in X(\mu)$ and $|f|\le|g|$.*

(iii) *The seminorm $\|\cdot\|_{\mathrm{b},X(\mu)}$ is identically zero on $X(\mu)$ if and only if $X(\mu)^*=\{0\}$.*

(iv) *The space $(X(\mu),\|\cdot\|_{X(\mu)})$ is a B.f.s. if and only if $\|f\|_{\mathrm{b},X(\mu)}=\|f\|_{X(\mu)}$ for all $f\in X(\mu)$.*

(v) *The following conditions are equivalent.*
  - (a) *$\|\cdot\|_{\mathrm{b},X(\mu)}$ is a norm on $X(\mu)$.*
  - (b) *The linear map $J: X(\mu)\to X(\mu)^{**}$ is injective.*
  - (c) *The closed unit ball $\mathbf{B}[X(\mu)^*]$ separates points of $X(\mu)$.*

(vi) *Let $X(\mu)$ be either a B.f.s. or a σ-order continuous q-B.f.s. Then the following conditions are equivalent.*
  - (a) *$(X(\mu),\|\cdot\|_{\mathrm{b},X(\mu)})$ is a B.f.s.*
  - (b) *$\|\cdot\|_{X(\mu)}$ and $\|\cdot\|_{\mathrm{b},X(\mu)}$ are equivalent on $X(\mu)$.*
  - (c) *$X(\mu)$ admits a lattice norm equivalent to $\|\cdot\|_{X(\mu)}$.*

(vii) *Let $X(\mu)$ be either a B.f.s. or a σ-order continuous q-B.f.s. and $f\in X(\mu)^+$. Then there exists $\xi\in\mathbf{B}[X(\mu)^*]\cap\big(X(\mu)^*\big)^+$ such that $\langle f,\xi\rangle=\|f\|_{\mathrm{b},X(\mu)}$.*

Proof. Statement (i) has already been verified and (iii) is clear from (2.25).

(ii) If $X(\mu)$ is a B.f.s., then $\|\cdot\|_{b,X(\mu)}=\|\cdot\|_{X(\mu)}$ is a lattice norm.

So, assume that $X(\mu)$ is a σ-order continuous q-B.f.s. Let $f,g\in X(\mu)$ with $|f|\le|g|$. Since $\|\cdot\|_{X(\mu)^{**}}$ is a lattice norm, Lemma 2.12 yields

$$\|f\|_{\mathrm{b},X(\mu)}=\big\|J(f)\big\|_{X(\mu)^{**}}=\big\|\,|J(f)|\,\big\|_{X(\mu)^{**}}=\big\|J(|f|)\big\|_{X(\mu)^{**}}.$$

Similarly, $\|g\|_{\mathrm{b},X(\mu)}\le\big\|J(|g|)\big\|_{X(\mu)^{**}}$. Since $J(|f|)\le J(|g|)$, it follows that

$$\|f\|_{\mathrm{b},X(\mu)}\le\|g\|_{\mathrm{b},X(\mu)},$$

which verifies part (ii).

(iv) The "only if" portion has already been established. The "if" portion follows because $\|\cdot\|_{X(\mu)}=\|\cdot\|_{\mathrm{b},X(\mu)}$ implies that $\|\cdot\|_{\mathrm{b},X(\mu)}$ and hence, also $\|\cdot\|_{X(\mu)}$, is a norm, which is necessarily a lattice norm by (ii).

(v) Since we already know that $\|\cdot\|_{\mathrm{b},X(\mu)}$ is a seminorm, (a) $\Leftrightarrow$ (b) follows from (2.27) and (a) $\Leftrightarrow$ (c) follows from the respective definition.

(vi) (a) $\Rightarrow$ (b). By (2.27), the topology defined by the quasi-norm $\|\cdot\|_{X(\mu)}$ on $X(\mu)$ is stronger than that defined by $\|\cdot\|_{\mathrm{b},X(\mu)}$. Condition (a) enables us to apply the Closed Graph Theorem, [88, §15, 12(3)], to deduce (b).

(b) $\Rightarrow$ (c). Since $\|\cdot\|_{\mathrm{b},X(\mu)}$ is a lattice seminorm (by (ii)), this is clear.

(c) $\Rightarrow$ (a). Let $|||\cdot|||_{X(\mu)}$ be a lattice norm which is equivalent to $\|\cdot\|_{X(\mu)}$ on $X(\mu)$. Select constants $C_1, C_2 > 0$ such that

$$C_1|||f|||_{X(\mu)} \;\le\; \|f\|_{X(\mu)} \;\le\; C_2|||f|||_{X(\mu)}, \qquad f \in X(\mu). \tag{2.28}$$

We shall show that

$$C_1|||f|||_{X(\mu)} \;\le\; \|f\|_{\mathrm{b},X(\mu)} \;\le\; C_2|||f|||_{X(\mu)}, \qquad f \in X(\mu). \tag{2.29}$$

To prove the first inequality, let $|||\cdot|||_{X(\mu)^*}$ denote the dual norm on the space $X(\mu)^* = (X(\mu), |||\cdot|||_{X(\mu)})^*$. Given $\xi \in X(\mu)^*$, it follows from (2.28) that

$$|\langle f,\xi\rangle| \le |||f|||_{X(\mu)}\cdot|||\xi|||_{X(\mu)^*} \le C_1^{-1}\|f\|_{X(\mu)}\cdot|||\xi|||_{X(\mu)^*} = \big(C_1^{-1}|||\xi|||_{X(\mu)^*}\big)\,\|f\|_{X(\mu)}$$

for every $f \in X(\mu)$, which implies that

$$\|\xi\|_{X(\mu)^*} \le C_1^{-1}|||\xi|||_{X(\mu)^*} \tag{2.30}$$

because

$$\|\xi\|_{X(\mu)^*} = \inf\big\{C > 0 : |\langle f,\xi\rangle| \le C\|f\|_{X(\mu)} \ \text{ for all } \ f \in X(\mu)\big\},$$

which follows routinely from (2.16). Now, fix $f \in X(\mu)$. The Hahn-Banach Theorem applied in the Banach space $(X(\mu), |||\cdot|||_{X(\mu)})$ guarantees that there exists $\xi_0 \in X(\mu)^*$ with $|||\xi_0|||_{X(\mu)^*} \le 1$ for which $|||f|||_{X(\mu)} = |\langle f,\xi_0\rangle|$. Therefore, the definition of $\|f\|_{\mathrm{b},X(\mu)}$ (see (2.25)) yields that

$$C_1|||f|||_{X(\mu)} = |\langle f,\, C_1\xi_0\rangle| \le \|f\|_{\mathrm{b},X(\mu)}$$

because $\|C_1\xi_0\|_{X(\mu)^*} \le |||\xi_0|||_{X(\mu)^*} \le 1$ via (2.30) (with $\xi := \xi_0$). So, the first inequality in (2.29) is established.

On the other hand, from (2.27) and (2.28), we have

$$\|f\|_{\mathrm{b},X(\mu)} \le \|f\|_{X(\mu)} \le C_2|||f|||_{X(\mu)}, \qquad f \in X(\mu),$$

so that the second inequality in (2.29) also holds.

Since $(X(\mu), |||\cdot|||_{X(\mu)})$ is a B.f.s. and $\|\cdot\|_{\mathrm{b},X(\mu)}$ is a lattice seminorm (by (ii)), it follows that $(X(\mu), \|\cdot\|_{\mathrm{b},X(\mu)})$ is a B.f.s.

(vii) Recall that $X(\mu)^*$ is a *Banach space* (relative to (2.16)), possibly equaling $\{0\}$. The weak* closed subset

$$\mathbf{B}^+[X(\mu)^*] := \mathbf{B}[X(\mu)^*] \cap (X(\mu)^*)^+$$

of the weak* compact set $\mathbf{B}[X(\mu)^*]$ (see [142, Theorem 3.15]) is also weak* compact. Moreover, $\|f\|_{\mathrm{b},X(\mu)} = \sup_{\xi\in\mathbf{B}^+[X(\mu)^*]}\langle f,\xi\rangle$. Indeed, (2.27) implies that

$$\begin{aligned}\|f\|_{\mathrm{b},X(\mu)} &= \|J(f)\|_{X(\mu)^{**}} = \sup_{\xi\in\mathbf{B}[X(\mu)^*]} \big|\langle \xi, J(f)\rangle\big| = \sup_{\xi\in\mathbf{B}[X(\mu)^*]} \big|\langle f,\xi\rangle\big| \\ &\le \sup_{\xi\in\mathbf{B}[X(\mu)^*]} \big\langle |f|,|\xi|\big\rangle = \sup_{\eta\in\mathbf{B}^+[X(\mu)^*]} \big\langle |f|,\eta\big\rangle \\ &\le \|\,|f|\,\|_{\mathrm{b},X(\mu)} \;=\; \|f\|_{\mathrm{b},X(\mu)},\end{aligned}$$

where the last equality follows from part (ii). The scalar function $\xi\mapsto\langle f,\xi\rangle$, for $\xi\in\mathbf{B}^+[X(\mu)^*]$, which is continuous on $\mathbf{B}^+[X(\mu)^*]$ equipped with the relative weak* topology, attains its maximum at some point of $\mathbf{B}^+[X(\mu)^*]$. □

Example 2.14. Let the notation be as in Example 2.11 with $0<r<1$. Then, given $f\in\ell^r(\mu)$ we have

$$\|f\|_{\mathrm{b},\ell^r(\mu)} \;=\; \|f\varphi^{1/r}\|_{\ell^1}.$$

Indeed, for every $h\in\ell^r(\mu)^* = \varphi^{1/r}\cdot\ell^\infty$ we have $\|h\|_{\ell^r(\mu)^*} = \|\varphi^{-1/r}\,h\|_{\ell^\infty}$. So, (2.24) gives

$$\begin{aligned}\|f\|_{\mathrm{b},\ell^r(\mu)} &= \sup\big\{\,\big|\langle f,h\rangle\big| : h\in\mathbf{B}[\ell^r(\mu)^*]\,\big\} \\ &= \sup\Big\{\,\Big|\sum_{n=1}^{\infty}\big(f\varphi^{1/r}\big)(n)\cdot(\varphi^{-1/r}\,h)(n)\Big| : h\in\mathbf{B}[\ell^r(\mu)^*]\Big\} \\ &= \sup\Big\{\,\Big|\sum_{n=1}^{\infty}\big(f\varphi^{1/r}\big)(n)\cdot g(n)\,\Big| \;:\; g\in\mathbf{B}[\ell^\infty]\Big\} \;=\; \|f\varphi^{1/r}\|_{\ell^1}.\end{aligned}$$

Note that $\|\cdot\|_{\mathrm{b},\ell^r(\mu)}$ is a lattice *norm* on $\ell^r(\mu)$, but surely not equivalent to $\|\cdot\|_{\ell^r(\mu)}$. □

Let $X(\mu)$ be a q-B.f.s. over (Ω,Σ,μ). Given a non-μ-null set $A\in\Sigma$, let μ_A denote the restriction of μ to $\Sigma\cap A$. Let

$$X(\mu_A) := \{f|_A : f\in X(\mu)\},$$

where $f|_A$ denotes the restriction of each function $f\in X(\mu)$ to A. The functional $\|\cdot\|_{X(\mu_A)} : X(\mu_A)\to[0,\infty)$, defined for each $f|_A$ by

$$\|f|_A\|_{X(\mu_A)} := \|\widetilde{f}\chi_A\|_{X(\mu)}, \tag{2.31}$$

for all $\widetilde{f}\in X(\mu)$ satisfying $f=\tilde{f}$ on A, is clearly a well-defined lattice quasinorm in $X(\mu_A)$. Given $g\in X(\mu_A)$, let the element $i_A(g)\in X(\mu)$ be defined by $i_A(g)(\omega):=g(\omega)$ for every $\omega\in A$ and by $g(\omega):=0$ for every $\omega\in\Omega\setminus A$. Then the so-defined linear map $i_A : X(\mu_A)\to X(\mu)$ is positive and an isometry onto its range because

$$\|g\|_{X(\mu_A)} \;=\; \|i_A(g)\cdot\chi_A\|_{X(\mu)} \;=\; \|i_A(g)\|_{X(\mu)}, \qquad g\in X(\mu_A). \tag{2.32}$$

Lemma 2.15. *Let $X(\mu)$ be a q-B.f.s. over (Ω, Σ, μ). Let $A \in \Sigma$ be a non-μ-null set.*

(i) *The function space $X(\mu_A)$ equipped with the quasi-norm $\|\cdot\|_{X(\mu_A)}$ defined by (2.31) is a q-B.f.s. over the finite measure space $(A, \Sigma \cap A, \mu_A)$. Moreover, $X(\mu_A)$ is σ-o.c. whenever $X(\mu)$ is σ-o.c.*

(ii) *The dual operator $i_A^* : X(\mu)^* \to X(\mu_A)^*$ of $i_A : X(\mu_A) \to X(\mu)$ satisfies*

$$i_A^*\big(\mathbf{B}[X(\mu)^*]\big) = \mathbf{B}[X(\mu_A)^*].$$

(iii) *We have*

$$\|g\|_{\mathrm{b},X(\mu_A)} = \|i_A(g)\|_{\mathrm{b},X(\mu)}, \qquad g \in X(\mu_A). \tag{2.33}$$

Proof. (i) The first statement follows from the fact that i_A is a linear isometry from $X(\mu_A)$ onto the closed sublattice $\chi_A \cdot X(\mu) := \{\chi_A f : f \in X(\mu)\}$ of $X(\mu)$ which preserves the order operations.

Suppose now that $X(\mu)$ is σ-o.c. Let $f_n \downarrow 0$ with $\{f_n\}_{n=1}^\infty \subseteq X(\mu_A)^+$. Then $\{i_A(f_n)\}_{n=1}^\infty \subseteq X(\mu)^+$ with $i_A(f_n) \downarrow 0$. Hence, $\|i_A(f_n)\|_{X(\mu)} \downarrow 0$ and, by (2.32), also $\|f_n\|_{X(\mu_A)} \downarrow 0$. Accordingly, $X(\mu_A)$ is also σ-o.c.

(ii) Let $\xi \in \mathbf{B}[X(\mu)^*]$. Then (2.32) gives

$$\big|\langle g, i_A^*(\xi)\rangle\big| = \big|\langle i_A(g), \xi\rangle\big| \le \big\|i_A(g)\big\|_{X(\mu)} = \|g\|_{X(\mu_A)}, \qquad g \in X(\mu_A),$$

which implies that $\|i_A^*(\xi)\|_{X(\mu_A)^*} \le 1$, that is, $i_A^*(\xi) \in \mathbf{B}[X(\mu_A)^*]$. So,

$$i_A^*\big(\mathbf{B}[X(\mu)^*]\big) \subseteq \mathbf{B}[X(\mu_A)^*].$$

Conversely, let $\eta \in \mathbf{B}[X(\mu_A)^*]$. Define a linear functional $\widetilde{\eta} : X(\mu) \to \mathbb{C}$ by $\widetilde{\eta}(f) := \langle f|_A, \eta\rangle$ for every $f \in X(\mu)$. Then $\widetilde{\eta} \in X(\mu)^*$ and satisfies $\|\widetilde{\eta}\|_{X(\mu)^*} \le 1$ because

$$\big|\widetilde{\eta}(f)\big| = \big|\langle f|_A, \eta\rangle\big| \le \|f|_A\|_{X(\mu_A)} = \|f\chi_A\|_{X(\mu)} \le \|f\|_{X(\mu)}, \qquad f \in X(\mu).$$

Moreover, given $g \in X(\mu_A)$, we have

$$\big\langle g, i_A^*(\widetilde{\eta})\big\rangle = \big\langle i_A(g), \widetilde{\eta}\big\rangle = \big\langle i_A(g)|_A, \eta\big\rangle = \langle g, \eta\rangle,$$

which implies that $\eta = i_A^*(\widetilde{\eta}) \in i_A^*\big(\mathbf{B}[X(\mu)^*]\big)$. So, we have established part (ii).

(iii) This is a straightforward application of part (ii) and (2.25), applied to both $X(\mu)$ and $X(\mu_A)$. $\square$

Given a q-B.f.s. $X(\mu)$ and a non-μ-null set $A \in \Sigma$, the linear operator $i_A : X(\mu_A) \to X(\mu)$ is an isometry via (2.32) and its range is the closed sublattice $\chi_A \cdot X(\mu)$ of $X(\mu)$. So we can have the identification

$$X(\mu_A) = \chi_A \cdot X(\mu) \tag{2.34}$$

via i_A. With this identification we may rephrase (2.32) as

$$\|f\chi_A\|_{X(\mu_A)} = \|f\chi_A\|_{X(\mu)}, \qquad f \in X(\mu). \tag{2.35}$$

Moreover, we may then rewrite (2.33) in Lemma 2.15 above as

$$\big\|f\chi_A\big\|_{\mathrm{b},X(\mu_A)} = \big\|f\chi_A\big\|_{\mathrm{b},X(\mu)}, \qquad f \in X(\mu). \tag{2.36}$$

Let us return to a general q-B.f.s. $X(\mu)$. Given $g \in L^0(\mu)$, let

$$g \cdot X(\mu) := \{gf : f \in X(\mu)\}.$$

The *Köthe dual* of $X(\mu)$, also called the *associate space* of $X(\mu)$, [102, 164], is the order ideal of $L^0(\mu)$ defined by

$$X(\mu)' := \{g \in L^0(\mu) : g \cdot X(\mu) \subseteq L^1(\mu)\}.$$

Since both $X(\mu)$ and $L^1(\mu)$ are order ideals with $\chi_\Omega \in X(\mu)$, it follows that $X(\mu)' \subseteq L^1(\mu)$. Given $g \in X(\mu)'$, the linear functional

$$\xi_g : f \mapsto \int_\Omega gf\, d\mu, \qquad f \in X(\mu), \tag{2.37}$$

is necessarily continuous, that is, $\xi_g \in X(\mu)^*$. Indeed, the multiplication operator $M_g : f \mapsto gf$, for $f \in X(\mu)$, is a closed operator (cf. Proposition 2.2(iii)) from $X(\mu)$ into $L^1(\mu)$ and hence, is continuous via the Closed Graph Theorem, [88, §15.12.(3)]. So, ξ_g is continuous because it is the composition of M_g with the continuous linear functional $h \mapsto \int_\Omega h\, d\mu$ on $L^1(\mu)$.

Next, observe that the linear map

$$g \mapsto \xi_g, \qquad g \in X(\mu)', \tag{2.38}$$

from $X(\mu)'$ into $X(\mu)^*$ is injective. Indeed, suppose that $\xi_g = 0$. Since $\operatorname{sim}\Sigma \subseteq X(\mu)$, we have $\xi_g(\chi_A) = \int_A g\, d\mu = 0$ for all $A \in \Sigma$. But, $g \in L^1(\mu)$ and so $g = 0$. Hence, we can define a norm on $X(\mu)'$ for which the canonical map (2.38) becomes an isometry, namely

$$\|g\|_{X(\mu)'} := \|\xi_g\|_{X(\mu)^*}, \qquad g \in X(\mu)'.$$

Clearly, $\|\cdot\|_{X(\mu)'}$ is a lattice norm. Accordingly, $(X(\mu)', \|\cdot\|_{X(\mu)'})$ is a lattice normed function space over (Ω, Σ, μ).

Proposition 2.16. *Let $(X(\mu), \|\cdot\|_{X(\mu)})$ be a q-B.f.s. over (Ω, Σ, μ).*

(i) *The Köthe dual $(X(\mu)', \|\cdot\|_{X(\mu)'})$ of $X(\mu)$ is a complete, normed function space. Consequently, if $X(\mu)'$ contains all Σ-simple functions, then $X(\mu)'$ is a B.f.s.*

(ii) *If, in addition, $X(\mu)$ is σ-o.c., then the linear isometry* (2.38) *is surjective.*

(iii) *Assume that $X(\mu)$ is a B.f.s. Then the linear isometry* (2.38) *is surjective if and only if $X(\mu)$ has σ-o.c. norm.*

To prove this we shall use the following Vitali type convergence theorem as given in [97, Lemma 2.3].

Lemma 2.17. *Let $\lambda : \Sigma \to \mathbb{C}$ be a complex measure. Then a Σ-measurable function $f : \Omega \to \mathbb{C}$ is λ-integrable (i.e., $\int_\Omega |f|\, d|\lambda| < \infty$, where $|\lambda|$ is the variation measure of λ) if and only if there exist $f_n \in L^1(\lambda)$, for $n \in \mathbb{N}$, such that $f_n \to f$ pointwise as $n \to \infty$ and, for each $A \in \Sigma$, the complex sequence $\{\int_A f_n\, d\lambda\}_{n=1}^\infty$ converges. In this case we have*

$$\int_A f\, d\mu = \lim_{n\to\infty} \int_A f_n\, d\lambda, \qquad A \in \Sigma.$$

Proof of Proposition 2.16. Observe first that

$$\int_\Omega |g|\, d\mu \ \le\ \|\chi_\Omega\|_{X(\mu)}\, \|g\|_{X(\mu)'}, \qquad g \in X(\mu)'. \tag{2.39}$$

Let $\{g_n\}_{n=1}^\infty$ be a Cauchy sequence in $X(\mu)'$. By (2.39) we have that

$$\int_\Omega |g_n - g_k|\, d\mu \ \le\ \|\chi_\Omega\|_{X(\mu)}\, \|g_n - g_k\|_{X(\mu)'}, \qquad n, k \in \mathbb{N},$$

and so the sequence $\{g_n\}_{n=1}^\infty$ is Cauchy in $L^1(\mu)$. Hence, there is a subsequence $\{g_{n(j)}\}_{j=1}^\infty$ converging μ-a.e. to a function $g \in L^1(\mu)$.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$\|g_{n(j)} - g_{n(k)}\|_{X(\mu)'} \ <\ \varepsilon, \qquad j, k \ge N. \tag{2.40}$$

Fix $f \in X(\mu)$. Then $g_{n(j)} f \to gf$ pointwise μ-a.e. as $j \to \infty$ and, given $A \in \Sigma$, the scalar sequence $\big\{ \int_A (g_{n(j)} f)\, d\mu \big\}_{j=1}^\infty$ converges due to the inequalities

$$\Big| \int_A \big(g_{n(j)} f - g_{n(k)} f\big)\, d\mu \Big| \ \le\ \int_A \big|g_{n(j)} f - g_{n(k)} f\big|\, d\mu \ \le\ \|f\|_{X(\mu)} \cdot \varepsilon, \qquad j, k \in \mathbb{N}, \tag{2.41}$$

which follows from (2.40). By Lemma 2.17 we can conclude that $gf \in L^1(\mu)$, Moreover, Fatou's Lemma and (2.41) give, for all $j \ge N$ and $f \in \mathbf{B}[X(\mu)]$, that

$$\begin{aligned} \Big| \int_\Omega (g_{n(j)} - g) f\, d\mu \Big| \ &\le\ \int_\Omega |g_{n(j)} - g| \cdot |f|\, d\mu \\ &= \int_\Omega \lim_{k\to\infty} |g_{n(j)} - g_{n(k)}| \cdot |f|\, d\mu \ \le\ \liminf_{k\to\infty} \int_\Omega |g_{n(j)} - g_{n(k)}| \cdot |f|\, d\mu \ \le\ \varepsilon. \end{aligned}$$

Therefore, g is the limit of the subsequence $\{g_{n(j)}\}_{j=1}^\infty$ and hence, also of $\{g_n\}_{n=1}^\infty$, in the norm of $X(\mu)'$. So, (i) is proved.

To prove (ii), let $\eta \in X(\mu)^*$. Define a set function $\lambda_\eta : A \mapsto \langle \chi_A, \eta\rangle$, for $A \in \Sigma$. Then λ_η is a complex measure because, for every sequence $\{A(n)\}_{n=1}^\infty \subseteq \Sigma$ decreasing to the empty set, we have $|\langle \chi_{A(n)}, \eta\rangle| \le \|\chi_{A(n)}\|_{X(\mu)} \|\eta\|_{X(\mu)^*} \to 0$ as $n \to \infty$ because $\|\cdot\|_{X(\mu)}$ is σ-o.c. If $A \in \Sigma$ is μ-null (i.e., $\mu(A) = 0$), then $\chi_A = 0$ in $X(\mu) \subseteq L^0(\mu)$, and hence, $\lambda_\eta(A) = \langle \chi_A, \eta\rangle = 0$. So, λ_η is absolutely continuous with respect to μ; see [42, Ch. I, Theorem 2.1]. Let $g \in L^1(\mu)$ denote the Radon–Nikodým derivative $d\lambda_\eta/d\mu$, that is, $\lambda_\eta(A) = \int_A g\, d\mu$ for $A \in \Sigma$. We claim that $g \in X(\mu)'$ and $\int_\Omega fg\, d\mu = \langle f, \eta\rangle$ for every $f \in X(\mu)$. In fact, fix $f \in X(\mu)$. Since $X(\mu)$ is σ-o.c., we can choose a sequence $\{s_n\}_{n=1}^\infty$ from $\operatorname{sim}\Sigma$ such that $|s_n| \le |f|$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = f$ both pointwise and in the quasi-norm $\|\cdot\|_{X(\mu)}$; see Remark 2.6. So, $s_n g \to fg$ pointwise as $n \to \infty$, and the limit

$$\langle f\chi_A, \eta\rangle \;=\; \lim_{n\to\infty} \langle s_n\, \chi_A, \eta\rangle \;=\; \lim_{n\to\infty} \int_A s_n\, d\lambda_\eta \;=\; \lim_{n\to\infty} \int_A s_n g\, d\mu$$

exists for all $A \in \Sigma$. Then Lemma 2.17 applies to conclude that $fg \in L^1(\mu)$ and $\int_\Omega fg\, d\mu = \langle f, \eta\rangle$. Accordingly, $g \in X(\mu)'$ (because $f \in X(\mu)$ is arbitrary) and $\eta = \xi_g$ (see (2.37)). This establishes (ii).

(iii) This is well known; see for example [164, §72, Theorem 5] where the setting is B.f.s.' over $\mathbb{R}$ and $\mathbb{C}$. □

Remark 2.18. (i) We can regard $X(\mu)'$ as a subspace of $X(\mu)^*$ via the linear isometry (2.38); that is, we identify each $g \in X(\mu)'$ with $\xi_g \in X(\mu)^*$. With this identification, part (i) of Proposition 2.16 means that the Köthe dual $X(\mu)'$ is a closed subspace of the (topological) dual $X(\mu)^*$, and part (ii) means that $X(\mu)' = X(\mu)^*$.

(ii) If $X(\mu)$ is a σ-order continuous B.f.s. of the form L_ρ for some function norm $\rho : M^+(\mu) \to [0, \infty]$, then $(L_\rho)' = (L_\rho)^*$; see [164, Ch. 15, Theorem 15].

(iii) Part (iii) of Proposition 2.16 is well known for *real* B.f.s., [108, Theorem 2.4.2]. This result may fail in a q-B.f.s.; see Example 2.19 below. □

When Ω happens to be a topological Hausdorff space, we denote by $\mathcal{B}(\Omega)$ the *Borel σ-algebra* of Ω, that is, the σ-algebra generated by all the open subsets of Ω.

Example 2.19. Let $\mu : \mathcal{B}([0,1]) \to [0,1]$ denote the Lebesgue measure. Define $A(n) := \big[(n+1)^{-1}, n^{-1}\big)$ for $n \in \mathbb{N}$. Let $X(\mu)$ denote the order ideal in $L^0(\mu)$ consisting of all $f \in L^0(\mu)$ such that

$$\|f\|_{X(\mu)} := \sup_{n\in\mathbb{N}} \|f\chi_{A(n)}\|_{L^{1/2}(\mu)} \;<\; \infty. \tag{2.42}$$

The functional defined on $X(\mu)$ by (2.42) is a quasi-norm satisfying

$$\|f + g\|_{X(\mu)} \;\le\; 2\Big(\|f\|_{X(\mu)} + \|g\|_{X(\mu)}\Big), \qquad f, g \in X(\mu).$$

It is routine to check that $X(\mu)$ is complete relative to $\|\cdot\|_{X(\mu)}$. Since $\operatorname{sim}\Sigma \subseteq X(\mu)$, we conclude that $(X(\mu), \|\cdot\|_{X(\mu)})$ is a q-B.f.s. based on $([0,1], \mathcal{B}([0,1]), \mu)$. Moreover, $L^{1/2}(\mu) \subseteq X(\mu)$ and the natural inclusion map is continuous because $\|f\|_{X(\mu)} \leq \|f\|_{L^{1/2}(\mu)}$ for all $f \in L^{1/2}(\mu)$. Therefore,

$$X(\mu)^* \subseteq L^{1/2}(\mu)^* = \{0\};$$

see Example 2.10. Hence, $X(\mu)' = X(\mu)^* = \{0\}$. However, $X(\mu)$ is not σ-o.c. In fact, for each $k \in \mathbb{N}$, define

$$f_k := \sum_{n=k}^{\infty} \|\chi_{A(n)}\|_{L^{1/2}(\mu)}^{-1} \cdot \chi_{A(n)}.$$

Then $\|f_k\|_{X(\mu)} = 1$ for all $k \in \mathbb{N}$ whereas $f_k \downarrow 0$. So, the quasi-norm $\|\cdot\|_{X(\mu)}$ is not σ-o.c. $\square$

2.2 The p-th power of a quasi-Banach function space

The following inequalities for $a, b \in [0,\infty)$ will be useful in subsequent arguments:

$$(a+b)^r \leq a^r + b^r \qquad \text{for} \quad 0 < r \leq 1, \tag{2.43}$$

$$a^r + b^r \leq (a+b)^r \leq 2^{r-1}(a^r + b^r) \qquad \text{for} \quad r \geq 1, \qquad \text{and} \tag{2.44}$$

$$|a^r - b^r| \leq r \cdot |a^{r-1} + b^{r-1}| \cdot |a-b| \qquad \text{for} \quad r \geq 1. \tag{2.45}$$

Given $0 < p < \infty$, we define the *p-th power* $X(\mu)_{[p]} \subseteq L^0(\mu)$ of a q-B.f.s. $(X(\mu), \|\cdot\|_{X(\mu)})$, according to [61, Definition 1.9] and [30, p. 156]; see also [104]. Namely,

$$X(\mu)_{[p]} := \left\{ f \in L^0(\mu) : |f|^{1/p} \in X(\mu) \right\} \tag{2.46}$$

and

$$\|f\|_{X(\mu)_{[p]}} := \big\| |f|^{1/p} \big\|_{X(\mu)}^p, \qquad f \in X(\mu)_{[p]}. \tag{2.47}$$

The term "p-th power" is derived from the fact that

$$|f| \in X(\mu) \Longleftrightarrow |f|^p \in X(\mu)_{[p]} \qquad \text{provided} \quad f \in L^0(\mu), \tag{2.48}$$

which is clear from the definition. It follows from (2.43) and (2.44) that $X(\mu)_{[p]}$ is an order ideal of the vector lattice $L^0(\mu)$, with $\operatorname{sim}\Sigma \subseteq X(\mu)_{[p]}$, and that (2.47) defines a quasi-norm $\|\cdot\|_{X(\mu)_{[p]}}$; see Lemma 2.21 below. Of course, $X(\mu)_{[1]} = X(\mu)$. Even when $X(\mu)$ is a B.f.s., the space $X(\mu)_{[p]}$ may only be a quasi-normed function space (for some p). For instance, if $X(\mu) := L^1([0,1])$ and $1 < p < \infty$, then $X(\mu)_{[p]} = L^{1/p}([0,1])$ with $0 < (1/p) < 1$ and so is a non-normable q-B.f.s., whereas for $0 < p \leq 1$ we see that $X(\mu)_{[p]} = L^{1/p}([0,1])$ with $1 \leq (1/p) < \infty$ is actually a B.f.s. The proof of the following lemma is routine.

Lemma 2.20. *Let $X(\mu)$ be a q-B.f.s. based on (Ω, Σ, μ).*

(i) *For all $0 < p, r < \infty$ we have $\big(X(\mu)_{[p]}\big)_{[r]} = X(\mu)_{[pr]}$.*

(ii) *Given a q-B.f.s. $Y(\mu)$ over (Ω, Σ, μ), we have $X(\mu) \subseteq Y(\mu)$ if and only if $X(\mu)_{[p]} \subseteq Y(\mu)_{[p]}$ for some/every $p \in (0, \infty)$.*

Given q-B.f.s.' $X(\mu)$ and $Y(\mu)$ over (Ω, Σ, μ), let us adopt the convention

$$X(\mu) \cdot Y(\mu) := \big\{ fg : f \in X(\mu) \text{ and } g \in Y(\mu) \big\}.$$

The following result collects together some useful facts concerning $X(\mu)_{[p]}$.

Lemma 2.21. *Let $(X(\mu), \|\cdot\|_{X(\mu)})$ be a q-B.f.s. based on (Ω, Σ, μ) and $K \geq 1$ be a constant satisfying*

$$\|f_1 + f_2\|_{X(\mu)} \;\leq\; K\Big(\|f_1\|_{X(\mu)} \;+\; \|f_2\|_{X(\mu)}\Big), \qquad f_1, f_2 \in X(\mu). \tag{2.49}$$

(i) *Let $q, r, s > 0$ be numbers such that $q = r + s$. If $f \in X(\mu)_{[r]}$ and $g \in X(\mu)_{[s]}$, then $fg \in X(\mu)_{[q]}$ and*

$$\|fg\|_{X(\mu)_{[q]}} \leq K^q \|f\|_{X(\mu)_{[r]}} \|g\|_{X(\mu)_{[s]}}. \tag{2.50}$$

Conversely, given $h \in X(\mu)_{[q]}$, there exist functions $f \in X(\mu)_{[r]}$ and $g \in X(\mu)_{[s]}$ such that $h = fg$. Consequently,

$$X(\mu)_{[q]} = X(\mu)_{[r]} \cdot X(\mu)_{[s]}. \tag{2.51}$$

(ii) *Let $0 < p < \infty$. The p-th power $X(\mu)_{[p]}$ of $X(\mu)$ is an ideal of the vector lattice $L^0(\mu)$ with $\operatorname{sim}\Sigma \subseteq X(\mu)_{[p]}$. Moreover, the function $\|\cdot\|_{X(\mu)_{[p]}}$ defined by (2.47) is a lattice quasi-norm on $X(\mu)_{[p]}$ satisfying, for all $f, g \in X(\mu)_{[p]}$,*

$$\|f+g\|_{X(\mu)_{[p]}} \;\leq\; K^2\Big(\|f\|_{X(\mu)_{[p]}} + \|g\|_{X(\mu)_{[p]}}\Big) \qquad \text{when} \quad 0 < p < 1, \tag{2.52}$$

and

$$\|f + g\|_{X(\mu)_{[p]}} \;\leq\; 2^{p-1}K^p\Big(\|f\|_{X(\mu)_{[p]}} \;+\; \|g\|_{X(\mu)_{[p]}}\Big) \qquad \text{when} \quad 1 \leq p < \infty. \tag{2.53}$$

(iii) *If the quasi-norm of $X(\mu)$ is σ-o.c., then the quasi-norm of $X(\mu)_{[p]}$ is also σ-o.c. for every $0 < p < \infty$. In particular, $\operatorname{sim}\Sigma$ is dense in $X(\mu)_{[p]}$ for every $0 < p < \infty$.*

(iv) *If $0 < p \leq q < \infty$, then $X(\mu)_{[p]} \subseteq X(\mu)_{[q]}$ and the natural inclusion map is continuous. Indeed,*

$$\|f\|_{X(\mu)_{[q]}} \;\leq\; K^q \,\|\chi_\Omega\|_{X(\mu)}^{q-p} \,\|f\|_{X(\mu)_{[p]}}, \qquad f \in X(\mu)_{[p]}.$$

In particular, $X(\mu) \subseteq X(\mu)_{[p]}$ for $1 \leq p < \infty$ and $X(\mu)_{[p]} \subseteq X(\mu)$ for $0 < p \leq 1$.

Proof. (i) We adapt the standard proof of Hölder's inequality as done in [61, Proposition 1.10]. In order to establish (2.50) it suffices to assume that $\|f\|_{X(\mu)_{[r]}} = 1 = \|g\|_{X(\mu)_{[s]}}$; see (Q2). Let $F := |f|^{1/q} \in X(\mu)_{[r/q]}$ and $G := |g|^{1/q} \in X(\mu)_{[s/q]}$. Then $FG \in X(\mu)$ because $(r/q) + (s/q) = 1$ yields

$$FG = \left(F^{q/r}\right)^{r/q}\left(G^{q/s}\right)^{s/q} \leq \frac{r}{q}F^{q/r} + \frac{s}{q}G^{q/s}$$

with both $F^{q/r}, G^{q/s} \in X(\mu)$. This inequality and (2.49) give

$$\|FG\|_{X(\mu)} \leq K\left(\frac{r}{q}\left\|F^{q/r}\right\|_{X(\mu)} + \frac{s}{q}\left\|G^{q/s}\right\|_{X(\mu)}\right) = K$$

because $\|F^{q/r}\|_{X(\mu)} = \|\,|f|^{1/r}\|_{X(\mu)} = 1$ and $\|G^{q/s}\|_{X(\mu)} = \|\,|g|^{1/s}\|_{X(\mu)} = 1$. Observing that $fg \in X(\mu)_{[q]}$, we have, via $|fg|^{1/q} = FG \in X(\mu)$, that

$$\left\|fg\right\|_{X(\mu)_{[q]}} = \left\|FG\right\|_{X(\mu)}^{q} \leq K^q,$$

which establishes (2.50).

Concerning the converse, let $h \in X(\mu)_{[q]}$. Define a function $\operatorname{sgn} h : \Omega \to \mathbb{C}$ by $(\operatorname{sgn} h)(\omega) := h(\omega)/|h(\omega)|$ for every $\omega \in \Omega$ with the understanding that $0/0 = 0$. Then the function $f := (\operatorname{sgn} h) \cdot |h|^{r/q}$ belongs to $X(\mu)_{[r]}$ because $|f|^{1/r} = |h|^{1/q} \in X(\mu)$. Similarly, the function $g := |h|^{s/q}$ belongs to $X(\mu)_{[s]}$ because $|g|^{1/s} = |h|^{1/q} \in X(\mu)$. Therefore, since $1 = (r/q) + (s/q)$, we have that

$$h = (\operatorname{sgn} h) \cdot |h| = (\operatorname{sgn} h) \cdot |h|^{r/q} \cdot |h|^{s/r} = fg \in X(\mu)_{[r]} \cdot X(\mu)_{[s]}.$$

So, (2.51) holds.

(ii) Clearly $X(\mu)_{[p]}$ is closed under scalar multiplication. Let $f, g \in X(\mu)_{[p]}$. Then by applying (2.43) and (2.44) we have

$$|f+g|^{1/p} \leq 2^{(1/p)-1}\left(|f|^{1/p} + |g|^{1/p}\right) \quad \text{and} \quad |f+g|^{1/p} \leq |f|^{1/p} + |g|^{1/p}$$

when $0 < p < 1$ and $1 \leq p < \infty$, respectively. In both cases, $f + g \in X(\mu)_{[p]}$, that is, $X(\mu)_{[p]}$ is a linear subspace of $L^0(\mu)$. So, from the definition of $X(\mu)_{[p]}$, it is clear that $X(\mu)$ is an ideal of $L^0(\mu)$ with $\operatorname{sim}\Sigma \subseteq X(\mu)$.

A straightforward adaptation of the proof of Proposition 1.11 in [61] establishes (2.52) and (2.53) as follows. When $0 < p < 1$, we have, from (2.49) and part (i) with $r := p$, $s := (1-p)$ and $q := 1$, that

$$\begin{aligned}\left\|\,|f+g|^{1/p}\,\right\|_{X(\mu)} &= \left\|\,|f+g| \cdot |f+g|^{(1/p)-1}\,\right\|_{X(\mu)} \\ &\leq K\Big(\left\|\,|f| \cdot |f+g|^{(1/p)-1}\,\right\|_{X(\mu)} + \left\|\,|g| \cdot |f+g|^{(1/p)-1}\,\right\|_{X(\mu)}\Big) \\ &\leq K^2\Big(\|f\|_{X(\mu)_{[p]}} + \|g\|_{X(\mu)_{[p]}}\Big) \cdot \left\|\,|f+g|^{(1/p)-1}\,\right\|_{X(\mu)_{[1-p]}}.\end{aligned}$$

The fact that $\left\|\,|f+g|^{(1/p)-1}\,\right\|_{X(\mu)_{[1-p]}} = \left\|\,|f+g|^{1/p}\,\right\|_{X(\mu)}^{1-p}$ gives (2.52).

Next, let $1 \le p < \infty$. Then (2.43), (2.44) and (2.49) yield

$$\begin{aligned}\|f+g\|_{X(\mu)_{[p]}} &= \big\|\, |f+g|^{1/p} \,\big\|^p_{X(\mu)} \le \big\|\, |f|^{1/p} + |g|^{1/p} \,\big\|^p_{X(\mu)} \\ &\le K^p \big(\big\|\, |f|^{1/p} \,\big\|_{X(\mu)} + \big\|\, |g|^{1/p} \,\big\|_{X(\mu)} \big)^p \\ &\le 2^{p-1} K^p \Big(\big\|\, |f|^{1/p} \big\|^p_{X(\mu)} + \big\|\, |g|^{1/p} \,\big\|^p_{X(\mu)} \Big),\end{aligned}$$

that is, (2.53) holds.

(iii) Fix $0 < p < \infty$. Suppose the sequence $f_n \downarrow 0$ in $\big(X(\mu)_{[p]}\big)^+$. Then also $f_n^{1/p} \downarrow 0$ in $X(\mu)^+$ and hence, $\|f_n^{1/p}\|_{X(\mu)} \to 0$. Consequently, $\|f_n\|_{X(\mu)_{[p]}} := \|f_n^{1/p}\|^p_{X(\mu)} \to 0$, showing that $\|\cdot\|_{X(\mu)_{[p]}}$ is σ-o.c. Since $\operatorname{sim}\Sigma \subseteq X(\mu)_{[p]}$, the density of $\operatorname{sim}\Sigma$ follows from Remark 2.6.

(iv) Let $f \in X(\mu)_{[p]}$. Since $\chi_\Omega \in X(\mu)_{[q-p]}$, we have from (i) that the function $f = f\chi_\Omega \in X(\mu)_{[q]}$ and

$$\|f\|_{X(\mu)_{[q]}} = \|f\chi_\Omega\|_{X(\mu)_{[q]}} \le K^q \|\chi_\Omega\|_{X(\mu)_{[q-p]}} \|f\|_{X(\mu)_{[p]}}.$$

Since $\big\|\chi_\Omega\big\|_{X(\mu)_{[q-p]}} = \big\|\chi_\Omega\big\|^{q-p}_{X(\mu)}$, statement (iv) holds. □

Given $0 < p < \infty$ and a q-B.f.s. $X(\mu)$, by applying the fact that $X(\mu)_{[p]}$ is a vector space (cf. Lemma 2.21(ii)), it follows from (2.46) and (2.48) that

$$\Big(\sum_{j=1}^n \big|f_j\big|^p\Big)^{1/p} \in X(\mu), \qquad f_1, \dots, f_n \in X(\mu), \quad n \in \mathbb{N}. \tag{2.54}$$

We now establish the important fact that the p-th power of $X(\mu)$ is *always* complete. This is stated in [61, Proposition 1.11], without proof, for the special case when $X(\mu)$ is a B.f.s.

Proposition 2.22. *Let $X(\mu)$ be any q-B.f.s. based on (Ω, Σ, μ). Then its p-th power $X(\mu)_{[p]}$ is complete for every $0 < p < \infty$. In particular, $X(\mu)_{[p]}$ is again a q-B.f.s.*

Proof. Thanks to Lemma 2.21, the only remaining thing to be proved is the completeness of $X(\mu)_{[p]}$. To this end let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $X(\mu)_{[p]}$. Due to the ideal property, both sequences $\{\mathrm{Re} f_n\}_{n=1}^\infty$ and $\{\mathrm{Im} f_n\}_{n=1}^\infty$ are also Cauchy in $X(\mu)_{[p]}$. The ideal property and the equality $|a-b| = |a^+ - b^+| + |a^- - b^-|$ for real numbers a, b then ensures that the sequences $\big\{(\mathrm{Re} f_n)^+\big\}_{n=1}^\infty$, $\big\{(\mathrm{Re} f_n)^-\big\}_{n=1}^\infty$, $\big\{(\mathrm{Im} f_n)^+\big\}_{n=1}^\infty$ and $\big\{(\mathrm{Im} f_n)^-\big\}_{n=1}^\infty$ are all Cauchy in $X(\mu)_{[p]}$. This allows us to assume that $f_n \in X(\mu)^+_{[p]}$, for all $n \in \mathbb{N}$, without loss of generality.

Assume that $0 < p < 1$. Fix $n, k \in \mathbb{N}$. By (2.45) with $r := 1/p$ we have

$$\big|f_n^{1/p} - f_k^{1/p}\big| \le \frac{1}{p} \, \big|f_n^{(1/p)-1} \; + \; f_k^{(1/p)-1}\big| \cdot \big|f_n - f_k\big|. \tag{2.55}$$

The function $\left|f_n^{(1/p)-1} + f_k^{(1/p)-1}\right|$ belongs to $X(\mu)_{[1-p]}$. So (2.55) and Lemma 2.21(i) with $q := 1$, $r := (1-p)$ and $s := p$ ensure that

$$\left\|f_n^{1/p} - f_k^{1/p}\right\|_{X(\mu)} \leq \frac{K}{p}\left\|f_n^{(1/p)-1} \ + \ f_k^{(1/p)-1}\right\|_{X(\mu)_{[1-p]}} \left\|f_n - f_k\right\|_{X(\mu)_{[p]}}. \quad (2.56)$$

Since $(1/p) - 1 = (1-p)/p$, the inequality (2.52) implies that

$$\begin{aligned}\left\|f_n^{(1/p)-1} + f_k^{(1/p)-1}\right\|_{X(\mu)_{[1-p]}} &\leq K^2\Big(\ \left\|f_n^{(1-p)/p}\right\|_{X(\mu)_{[1-p]}} \ + \ \left\|f_k^{(1-p)/p}\right\|_{X(\mu)_{[1-p]}}\Big) \\ = K^2\Big(\ \left\|f_n^{1/p}\right\|_{X(\mu)}^{1-p} \ + \ \left\|f_k^{1/p}\right\|_{X(\mu)}^{1-p}\Big) &\leq 2K^2 \sup_{j\in\mathbb{N}} \left\|f_j\right\|_{X(\mu)_{[p]}}^{(1-p)/p}. \quad (2.57)\end{aligned}$$

This inequality and (2.56) show that $\{f_n^{1/p}\}_{n=1}^\infty$ is Cauchy in the q-B.f.s. $X(\mu)$ and hence, has a limit $g \in X(\mu)$. Proposition 2.2(ii) implies that $g \geq 0$ (μ-a.e.). We now use the inequality $|a-b|^{1/p} \leq |a^{1/p} - b^{1/p}|$ for positive numbers a and b, which, for instance, follows from (2.44). This and the ideal property of $X(\mu)$ give

$$\left\|f_n - g^p\right\|_{X(\mu)_{[p]}} = \left\|\,|f_n - g^p|^{1/p}\right\|_{X(\mu)}^p \leq \left\|f_n^{1/p} - g\right\|_{X(\mu)}^p \to 0$$

as $n \to \infty$. Since $g^p \in X(\mu)_{[p]}$, this implies the completeness of $X(\mu)_{[p]}$.

Now assume that $1 \leq p < \infty$. We can again suppose that each $f_n \in X(\mu)_{[p]}^+$ for $n \in \mathbb{N}$. The inequality $|a^{1/p} - b^{1/p}| \leq |a-b|^{1/p}$ for non-negative numbers a and b (this can be derived from (2.43)) gives

$$\left\|f_n^{1/p} - f_k^{1/p}\right\|_{X(\mu)} = \left\|\,|f_n^{1/p} - f_k^{1/p}|\,\right\|_{X(\mu)} \leq \left\|f_n - f_k\right\|_{X(\mu)_{[p]}}^{1/p}, \qquad n,k \in \mathbb{N}. \quad (2.58)$$

Therefore, $\{f_n^{1/p}\}_{n=1}^\infty$ is Cauchy and has a limit $h \in X(\mu)$. Again by Proposition 2.2(ii) we have $h \geq 0$. Clearly $h^p \in X(\mu)_{[p]}$ and $h^{p-1} \in X(\mu)_{[p-1]}$. Fix $n \in \mathbb{N}$. By (2.45) with $r := p$, we have

$$|f_n - h^p| \leq p\left|f_n^{(p-1)/p} + h^{p-1}\right| \cdot \left|f_n^{1/p} - h\right|. \quad (2.59)$$

Apply Lemma 2.21(i) with $q := p$, $r := (p-1)$ and $s := 1$ to the right-hand side of (2.59) to obtain

$$\begin{aligned}&p\left\|(f_n^{(p-1)/p} + h^{p-1})(f_n^{1/p} - h)\right\|_{X(\mu)_{[p]}} \\ &\qquad \leq p\,K^p \left\|f_n^{(p-1)/p} + h^{p-1}\right\|_{X(\mu)_{[p-1]}} \left\|f_n^{1/p} - h\right\|_{X(\mu)}.\end{aligned} \quad (2.60)$$

Moreover, Lemma 2.21(ii) gives, with $\alpha_p := \max\left\{2^{(p-2)}K^{(p-1)},\, K^2\right\}$, that

$$\begin{aligned}\left\|f_n^{(p-1)/p} + h^{p-1}\right\|_{X(\mu)_{[p-1]}} &\leq \alpha_p\,\Big(\ \left\|f_n^{(p-1)/p}\right\|_{X(\mu)_{[p-1]}} \ + \ \left\|h^{p-1}\right\|_{X(\mu)_{[p-1]}}\Big) \\ &= \alpha_p\,\Big(\ \left\|f_n^{1/p}\right\|_{X(\mu)}^{p-1} \ + \ \left\|h\right\|_{X(\mu)}^{p-1}\Big).\end{aligned}$$

This inequality, together with (2.59) and (2.60), yield

$$\|f_n - h^p\|_{X(\mu)_{[p]}} \le pK^p\alpha_p\Big[\sup_{n\in\mathbb{N}}\Big(\|f_n^{1/p}\|_{X(\mu)}^{p-1} + \|h\|_{X(\mu)}^{p-1}\Big)\Big] \cdot \|f_n^{1/p} - h\|_{X(\mu)}$$

for $n \in \mathbb{N}$. That is, $f_n \to h^p$ in $X(\mu)_{[p]}$ as $n \to \infty$, and hence $X(\mu)_{[p]}$ is complete. □

It is of interest to determine when the p-th power of a q-B.f.s. $(X(\mu), \|\cdot\|_{X(\mu)})$ is normable.

Let $0 < q < \infty$. The q-B.f.s. $X(\mu)$ is called *q-convex* if there is a constant $c > 0$ such that

$$\Big\|\Big(\sum_{j=1}^{n}|f_j|^q\Big)^{1/q}\Big\|_{X(\mu)} \le c\Big(\sum_{j=1}^{n}\|f_j\|_{X(\mu)}^q\Big)^{1/q} \tag{2.61}$$

for all $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X(\mu)$; see [30, p. 156] for the case of a real q-B.f.s. The smallest constant satisfying (2.61) for all such $n \in \mathbb{N}$ and f_j's ($j = 1, \dots, n$) is called the *q-convexity constant* of $X(\mu)$ and is denoted by $\mathbf{M}^{(q)}[X(\mu)]$. Letting $n := 1$ in (2.61) yields

$$\mathbf{M}^{(q)}[X(\mu)] \ge 1. \tag{2.62}$$

Proposition 2.23. *Let $X(\mu)$ be a q-B.f.s. with quasi-norm $\|\cdot\|_{X(\mu)}$.*

(i) *Let $0 < p \le 1$. If $\|\cdot\|_{X(\mu)}$ is a norm, then $\|\cdot\|_{X(\mu)_{[p]}}$ is a norm and hence, $\big(X(\mu)_{[p]}, \|\cdot\|_{X(\mu)_{[p]}}\big)$ is a B.f.s.*

(ii) *Let $0 < p < \infty$. Then the q-B.f.s. $X(\mu)$ is p-convex if and only if its p-th power $X(\mu)_{[p]}$ admits a lattice norm equivalent to $\|\cdot\|_{X(\mu)_{[p]}}$. Moreover, it is possible to select an equivalent lattice norm $\eta_{[p]}$ on $X(\mu)_{[p]}$ satisfying*

$$\eta_{[p]}(f) \le \|f\|_{X(\mu)_{[p]}} \le \big(\mathbf{M}^{(p)}[X(\mu)]\big)^p \cdot \eta_{[p]}(f), \qquad f \in X(\mu)_{[p]}. \tag{2.63}$$

(iii) *Assume that $X(\mu)$ is p-convex for some $0 < p < \infty$. Then the lattice quasi-norm $\|\cdot\|_{X(\mu)_{[p]}}$ is a norm if and only if $\mathbf{M}^{(p)}[X(\mu)] = 1$.*

(iv) *If $X(\mu)$ is p-convex for some $1 \le p < \infty$, then $X(\mu)$ admits a lattice norm equivalent to $\|\cdot\|_{X(\mu)}$.*

Proof. (i) The fact that $\|\cdot\|_{X(\mu)_{[p]}}$ is a norm is known, [61, Proposition 1.11]. For the completeness of $X(\mu)_{[p]}$ see Proposition 2.22.

(ii) Suppose that $X(\mu)$ is p-convex. Define

$$\eta_{[p]}(f) := \inf\Big\{\sum_{j=1}^{n}\|f_j\|_{X(\mu)_{[p]}} : |f| \le \sum_{j=1}^{n}|f_j|,\ f_j \in X(\mu)_{[p]},\ j = 1, \dots, n,\ n \in \mathbb{N}\Big\} \tag{2.64}$$

for every $f \in X(\mu)_{[p]}$. That $\eta_{[p]} : X(\mu) \to [0, \infty)$ is a lattice norm equivalent to the quasi-norm $\|\cdot\|_{X(\mu)_{[p]}}$ is stated in [30, Lemma 3] without a proof and with extra properties required of the quasi-norm. So, we provide a proof for our more general setting.

From the definition of $\eta_{[p]}$ it follows, for $\alpha \in \mathbb{C}$ and $f, g \in X(\mu)_{[p]}$, that $\eta_{[p]}(\alpha f) = |\alpha|\eta_{[p]}(f)$ and $\eta_{[p]}(f+g) \leq \eta_{[p]}(f) + \eta_{[p]}(g)$. It is easy to verify that $\eta_{[p]}(f) \leq \eta_{[p]}(g)$ whenever $f, g \in X(\mu)_{[p]}$ with $|f| \leq |g|$. We now claim that

$$\eta_{[p]}(f) \;\leq\; \|f\|_{X(\mu)_{[p]}} \;\leq\; \big(\mathbf{M}^{(p)}[X(\mu)]\big)^p \cdot \eta_{[p]}(f), \qquad f \in X(\mu)_{[p]}. \tag{2.65}$$

The first inequality in (2.65) is clear from (2.64). To prove the second inequality, fix $f \in X(\mu)_{[p]}$. Given $\varepsilon > 0$ select $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in X(\mu)_{[p]}$ such that $|f| \leq \sum_{j=1}^n |f_j|$ and

$$\sum_{j=1}^n \big\|f_j\big\|_{X(\mu)_{[p]}} \;\leq\; \eta_{[p]}(f) \;+\; \varepsilon. \tag{2.66}$$

The p-convexity of $X(\mu)$ gives

$$\begin{aligned}
\|f\|_{X(\mu)_{[p]}} \;&=\; \|\,|f|^{1/p}\,\|^p_{X(\mu)} \;\leq\; \Big\|\Big(\sum_{j=1}^n |f_j|\Big)^{1/p}\Big\|^p_{X(\mu)} \\
&=\; \Big\|\Big(\sum_{j=1}^n \big(|f_j|^{1/p}\big)^p\Big)^{1/p}\Big\|^p_{X(\mu)} \;\leq\; \big(\mathbf{M}^{(p)}[X(\mu)]\big)^p \Big(\sum_{j=1}^n \|\,|f_j|^{1/p}\,\|^p_{X(\mu)}\Big) \\
&=\; \big(\mathbf{M}^{(p)}[X(\mu)]\big)^p \Big(\sum_{j=1}^n \|f_j\|_{X(\mu)_{[p]}}\Big) \;\leq\; \big(\mathbf{M}^{(p)}[X(\mu)]\big)^p \big(\eta_{[p]}(f) \;+\; \varepsilon\big).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it is clear that the second inequality in (2.65) holds. It follows from (Q1) and (2.65) that $\eta_{[p]}(f) = 0$ if and only if $f = 0$. Therefore, $\eta_{[p]}$ is a lattice norm on $X(\mu)_{[p]}$ equivalent to $\|\cdot\|_{X(\mu)_{[p]}}$.

Assume now that $X(\mu)_{[p]}$ admits an equivalent lattice norm ρ, that is, there exist $C_1, C_2 > 0$ such that

$$C_1\|f\|_{X(\mu)_{[p]}} \leq \rho(f) \leq C_2\|f\|_{X(\mu)_{[p]}}, \qquad f \in X(\mu)_{[p]}. \tag{2.67}$$

Fix $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in X(\mu)$. From (2.67) and the triangle inequality for the norm ρ, it follows that

$$\begin{aligned}
&\Big\|\Big(\sum_{j=1}^n |f_j|^p\Big)^{1/p}\Big\|_{X(\mu)} = \Big\|\sum_{j=1}^n |f_j|^p\Big\|^{1/p}_{X(\mu)_{[p]}} \\
&\leq \Big(C_1^{-1}\rho\Big(\sum_{j=1}^n |f_j|^p\Big)\Big)^{1/p} \leq \Big(C_1^{-1}\sum_{j=1}^n \rho\big(|f_j|^p\big)\Big)^{1/p} \\
&\leq \Big(C_1^{-1}C_2\sum_{j=1}^n \big\|\,|f_j|^p\big\|_{X(\mu)_{[p]}}\Big)^{1/p} = \big(C_1^{-1}C_2\big)^{1/p}\Big(\sum_{j=1}^n \big\|f_j\big\|^p_{X(\mu)}\Big)^{1/p}.
\end{aligned}$$

So, $X(\mu)$ is p-convex and we have established part (ii).

(iii) Assume that $\mathbf{M}^{(p)}[X(\mu)] = 1$. Then the lattice norm $\eta_{[p]}$ defined by (2.64) equals $\|\cdot\|_{X(\mu)_{[p]}}$ because $\eta_{[p]}$ satisfies (2.63) as verified in the proof of part (ii). Consequently, $\|\cdot\|_{X(\mu)_{[p]}}$ is a norm.

Conversely, assume that $\|\cdot\|_{X(\mu)_{[p]}}$ is a lattice norm and take arbitrary $f_1, \ldots, f_n \in X(\mu)$ with $n \in \mathbb{N}$. Then, the triangle inequality of $\|\cdot\|_{X(\mu)_{[p]}}$ implies that

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{n} |f_j|^p\Big)^{1/p}\Big\|_{X(\mu)} &= \Big\|\sum_{j=1}^{n} |f_j|^p\Big\|_{X(\mu)_{[p]}}^{1/p} \\ &\le \Big(\sum_{j=1}^{n} \big\| |f_j|^p \big\|_{X(\mu)_{[p]}}\Big)^{1/p} = \Big(\sum_{j=1}^{n} \|f_j\|_{X(\mu)}^p\Big)^{1/p}.\end{aligned}$$

Thus, $\mathbf{M}^{(p)}[X(\mu)] \le 1$, which together with (2.62) imply that $\mathbf{M}^{(p)}[X(\mu)] = 1$.

(iv) By part (ii), the q-B.f.s. $X(\mu)_{[p]}$ admits an equivalent lattice norm $\eta_{[p]}$ given by (2.64) and satisfying (2.65). Define

$$\eta_{X(\mu)}(f) := \Big(\eta_{[p]}\big(|f|^p\big)\Big)^{1/p}, \qquad f \in X(\mu).$$

For $f \in X(\mu)$, note that $\Big(\eta_{[p]}(|f|^p)\Big)^{1/p}$ is precisely $\|f\|_{(\widetilde{X}(\mu)_{[p]})_{[1/p]}}$, where $\widetilde{X}(\mu)_{[p]}$ denotes the B.f.s. $(X(\mu)_{[p]},\, \eta_{[p]})$, after observing (by Lemma 2.20) that $(X(\mu)_{[p]})_{[1/p]} = X(\mu)$ as vector lattices. By part (i), with $1/p \in (0,1]$ in place of p and $\widetilde{X}(\mu)_{[p]}$ in place of $X(\mu)$, we conclude that $\eta_{X(\mu)}$ is a lattice norm on $X(\mu)$. Given $g \in X(\mu)$, substituting $f := |g|^p \in X(\mu)_{[p]}$ into (2.65) yields

$$\eta_{X(\mu)}(g) \ \le\ \|g\|_{X(\mu)} \ \le\ \mathbf{M}^{(p)}[X(\mu)] \cdot \eta_{X(\mu)}(g).$$

This shows that $\eta_{X(\mu)}$ and $\|\cdot\|_{X(\mu)}$ are equivalent. □

A natural question is whether or not the p-th power $X(\mu)_{[p]}$ of a q-B.f.s. $X(\mu)$ is different from the original space $X(\mu)$. If $X(\mu)$ is the B.f.s. $L^\infty(\mu)$, then $X(\mu)_{[p]} = X(\mu)$ for all $p \in (0,\infty)$. The converse also turns out to be valid. To verify this we need the following Lemma 2.24. Given $g \in L^\infty(\mu)$ and any q-B.f.s. $X(\mu)$, let $M_g : X(\mu) \to X(\mu)$ denote the multiplication operator defined by

$$M_g(f) := gf, \qquad f \in X(\mu). \tag{2.68}$$

Since $|g| \le \big\|g\big\|_\infty \chi_\Omega$ and $\|\cdot\|_{X(\mu)}$ is a lattice quasi-norm, it follows that

$$\|M_g(f)\|_{X(\mu)} \le \|g\|_\infty \|f\|_{X(\mu)} \quad \text{for all} \quad f \in X(\mu),$$

that is, M_g is continuous.

Lemma 2.24. *Let $X(\mu)$ be a q-B.f.s. based on a positive, finite measure space (Ω, Σ, μ) and suppose that $T \in \mathcal{L}\big(X(\mu)\big)$ satisfies $T \circ M_{\chi_A} = M_{\chi_A} \circ T$ for every $A \in \Sigma$. Then the function $T(\chi_\Omega) \in L^\infty(\mu)$.*

Proof. Let $\varphi := T(\chi_\Omega)$. We show that

$$|\varphi| \leq \|T\| \chi_\Omega. \tag{2.69}$$

Suppose that $\mu\big(\{\omega \in \Omega : |\varphi(\omega)| > \|T\|\}\big) > 0$. Then there exists $\varepsilon > 0$ such that the set $A := \big\{\omega \in \Omega : |\varphi(\omega)| > \varepsilon + \|T\|\big\}$ satisfies $\mu(A) > 0$. Observe that

$$\big|\varphi\chi_A\big| = \big|(M_{\chi_A} \circ T)(\chi_\Omega)\big| = \big|(T \circ M_{\chi_A})(\chi_\Omega)\big| = \big|T(\chi_A)\big|$$

and, by definition of A, that $\big|\varphi\chi_A\big| > \big(\varepsilon + \|T\|\big)\chi_A$. It follows that $\big|T(\chi_A)\big| > \big(\varepsilon + \|T\|\big)\chi_A$ and hence, by the lattice property of $\|\cdot\|_{X(\mu)}$, that

$$\big\|T(\chi_A)\big\|_{X(\mu)} = \big\|\,\big|T(\chi_A)\big|\,\big\|_{X(\mu)} \geq \big(\varepsilon + \|T\|\big)\|\chi_A\|_{X(\mu)}.$$

On the other hand, by the definition of $\|T\|$, it follows that

$$\big\|T(\chi_A)\big\|_{X(\mu)} \leq \|T\| \cdot \|\chi_A\|_{X(\mu)}.$$

Since $\|\chi_A\|_{X(\mu)} > 0$, the previous two inequalities are contradictory to one another. Accordingly, $\big\{\omega \in \Omega : |\varphi(\omega)| > \|T\|\big\}$ is a μ-null set, from which (2.69) follows immediately. In particular, $\varphi \in L^\infty(\mu)$. $\square$

Remark 2.25. Let $T \in \mathcal{L}(X(\mu))$ satisfy the hypothesis of Lemma 2.24. Then $T \circ M_s = M_s \circ T$ for every $s \in \operatorname{sim}\Sigma$ and, with φ denoting $T(\chi_\Omega) \in L^\infty(\mu)$, we have $T(s) = \varphi s = M_\varphi(s)$ for $s \in \operatorname{sim}\Sigma$. From this formula, the continuity of both T and M_φ on $X(\mu)$, and the fact that $\operatorname{sim}\Sigma$ is dense in $X(\mu)_\mathrm{b}$, it follows that

$$T(f) = \varphi f, \qquad f \in X(\mu)_\mathrm{b}. \tag{2.70}$$

In particular, if $X(\mu) = X(\mu)_\mathrm{b}$ (e.g., if $\|\cdot\|_{X(\mu)}$ is σ-o.c.), then $T = M_\varphi$ on $X(\mu)$. However, if $\operatorname{sim}\Sigma$ is not dense in $X(\mu)$, then it is not possible to conclude, via topological means, that (2.70) holds for arbitrary $f \in X(\mu)$. However, using the Dedekind completeness of $L^\infty_\mathbb{R}(\mu)$ it can be shown via an order argument that (2.70) does hold, in general, for all $f \in X_\mathbb{R}(\mu)$ and $T \in \mathcal{L}(X_\mathbb{R}(\mu))$ satisfying the hypothesis of Lemma 2.24; see [38, Proposition 2.2] for the B.f.s. setting (and more general measures μ) and [89, Ch. 3] for the q-B.f.s. setting (both over $\mathbb{R}$). It would be interesting to know if these arguments extend to spaces over $\mathbb{C}$. $\square$

Proposition 2.26. *Let $X(\mu)$ be a q-B.f.s. based on (Ω, Σ, μ). Then the following statements are equivalent.*

(i) $X(\mu) = L^\infty(\mu)$.

(ii) $X(\mu)_{[p]} = X(\mu)$ *for every* $p \in (0, \infty)$.

(iii) *There exists* $p \in (0,1) \cup (1,\infty)$ *such that* $X(\mu)_{[p]} = X(\mu)$.

(iv) *The q-B.f.s.* $X(\mu)$ *is an algebra of functions with respect to* (μ*-a.e.*) *pointwise multiplication.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear.

(iii) $\Rightarrow$ (iv). Mathematical induction and Lemma 2.20(i) yield

$$X(\mu)_{[p^n]} = X(\mu), \qquad n \in \mathbb{N}. \tag{2.71}$$

Assume first that $0 < p < 1$. Then $p^N \leq 1/2 < \infty$ for some $N \in \mathbb{N}$. Then, Lemma 2.21(iv) and (2.71) with $n := N$ give

$$X(\mu)_{[1/2]} = X(\mu) \tag{2.72}$$

because $X(\mu) = X(\mu)_{[p^N]} \subseteq X(\mu)_{[1/2]} \subseteq X(\mu)_{[1]} = X(\mu)$. Next assume that $1 < p < \infty$. Then $X(\mu) = (X(\mu)_{[p]})_{[1/p]} = X(\mu)_{[1/p]}$, via Lemma 2.20(i) and our hypothesis that $X(\mu)_{[p]} = X(\mu)$. Using the fact that $(1/p)^n \leq 1/2$ for a large enough $n \in \mathbb{N}$, we can again deduce (2.72).

Let $f \in X(\mu)$. Then, $f \in X(\mu)_{[1/2]}$ via (2.72). So, $|f|^2 \in X(\mu)$ by definition. Hence, $|f^2| = |f|^2 \in X(\mu)$ and, since $X(\mu)$ is a lattice, also f^2 belongs to $X(\mu)$. Since f is arbitrary, the general identity $4gh = (g+h)^2 - (g-h)^2$ for $g, h \in X(\mu)$ yields statement (iv).

(iv) $\Rightarrow$ (i). Let $g \in X(\mu)$ be arbitrary, but fixed. Since $X(\mu)$ is an algebra, we have $gf \in X(\mu)$ for every $f \in X(\mu)$. An application of Proposition 2.2(iii) and the Closed Graph Theorem, [88, §15.12.(3)], implies that the operator T defined (on all of $X(\mu)$) by $f \mapsto gf$ is continuous. Since T clearly satisfies $T \circ M_{\chi_A} = M_{\chi_A} \circ T$, for every $A \in \Sigma$, it follows from Lemma 2.24 that $g = T(\chi_\Omega) \in L^\infty(\mu)$. □

Let $X(\mu)$ and $Y(\mu)$ be q-B.f.s.' based on (Ω, Σ, μ). Define an order ideal of $L^0(\mu)$ by

$$\mathcal{M}\big(X(\mu), Y(\mu)\big) := \big\{g \in L^0(\mu) \,:\, g \cdot X(\mu) \subseteq Y(\mu)\big\}. \tag{2.73}$$

Two special cases have already been considered. Indeed, if $Y(\mu) = L^1(\mu)$, then it follows from the definition of the Köthe dual $X(\mu)'$ that

$$\mathcal{M}\big(X(\mu), L^1(\mu)\big) = X(\mu)'. \tag{2.74}$$

Moreover, if $X(\mu) = Y(\mu)$, then

$$\mathcal{M}\big(X(\mu), X(\mu)\big) = L^\infty(\mu). \tag{2.75}$$

Indeed, the inclusion $L^\infty(\mu) \subseteq \mathcal{M}\big(X(\mu), X(\mu)\big)$ is clear.

Conversely, if $g \in \mathcal{M}\big(X(\mu), X(\mu)\big)$ then the usual "closed graph argument" shows that the linear map $T : X(\mu) \to X(\mu)$ defined by $f \mapsto gf$ is continuous. Then Lemma 2.24 implies that $g \in L^\infty(\mu)$.

We now investigate the general case from the viewpoint of p-th powers and Köthe duals; see also [104]. Each $g \in \mathcal{M}\big(X(\mu), Y(\mu)\big)$ corresponds to the multiplication operator $M_g : X(\mu) \to Y(\mu)$ defined by

$$M_g(f) := gf, \qquad\qquad f \in X(\mu). \tag{2.76}$$

This M_g is continuous via the Closed Graph Theorem, [88, §15, 12.(3)]. So, there is a one-to-one correspondence between $\mathcal{M}\big(X(\mu), Y(\mu)\big)$ and the subspace of $\mathcal{L}(X(\mu), Y(\mu))$ which consists of all multiplication operators.

The following useful fact will be needed later.

Proposition 2.27. *Let $X(\mu)$ and $Y(\mu)$ be q-B.f.s.' over (Ω, Σ, μ) with $Y(\mu)$ being σ-o.c. Suppose that $g \in \mathcal{M}\big(X(\mu), Y(\mu)\big)$ satisfies $g \neq 0$ (μ-a.e.). Then the range $\mathcal{R}(M_g)$, of M_g, is dense in $Y(\mu)$.*

Proof. According to Remark 2.6 it suffices to show that $\chi_A \in Y(\mu)$ belongs to the closure $\overline{\mathcal{R}(M_g)}$ of $\mathcal{R}(M_g)$ (in $Y(\mu)$) for each $A \in \Sigma$ with $\mu(A) > 0$. Observe that

$$A(n) := \big\{\omega \in A : |g(\omega)| \geq n^{-1}\big\} \uparrow \big\{w \in A : g(w) \neq 0\big\},$$

that is, $\chi_{A(n)} \uparrow \chi_A$. Since $\chi_A \in Y(\mu)$ and $Y(\mu)$ is σ-o.c., it follows that $\chi_{A(n)} \to \chi_A$ in the topology of $Y(\mu)$. Moreover, $g^{-1}\chi_{A(n)} \in L^\infty(\mu) \subseteq X(\mu)$ for all $n \in \mathbb{N}$ and so $M_g(g^{-1}\chi_{A(n)}) = \chi_{A(n)} \in \mathcal{R}(M_g)$ for all $n \in \mathbb{N}$. Accordingly, $\chi_A \in \overline{\mathcal{R}(M_g)}$, as required. $\square$

In Proposition 2.27, the condition that $Y(\mu)$ is σ-o.c. is necessary.

Example 2.28. Let $X(\mu) = Y(\mu) = L^\infty([0,1])$ with μ denoting Lebesgue measure on $\Omega := [0,1]$, in which case $Y(\mu)$ is *not* σ-o.c. Define $g \in \mathcal{M}\big(X(\mu), Y(\mu)\big)$ by $g(\omega) := \omega$ for $\omega \in \Omega$. To show that $\overline{\mathcal{R}(M_g)} \neq Y(\mu)$, consider the constant function $\mathbf{1} = \chi_\Omega \in Y(\mu)$. If $\mathbf{1} \in \overline{\mathcal{R}(M_g)}$, then there exists a function $f \in X(\mu)$ such that $\|\mathbf{1} - gf\|_{L^\infty([0,1])} < 1/2$ and hence, also $\|\mathbf{1} - g\mathrm{Re}(f)\|_{L^\infty([0,1])} < 1/2$. So, we may assume that f is $\mathbb{R}$-valued. It then follows from the inequalities

$$-1/2 \leq \omega f(\omega) - 1 \leq 1/2 \qquad \mu\text{-a.e. } \omega \in \Omega$$

that actually $f \geq 0$. So, f satisfies

$$0 \leq \omega f(\omega) \ \leq \ \omega \|f\|_{L^\infty([0,1])}, \qquad \mu\text{-a.e. } \omega \in \Omega,$$

from which it follows that

$$\|\mathbf{1} - gf\|_{L^\infty([0,1])} \geq \ \sup\big\{1 - \omega\|f\|_{L^\infty([0,1])} \ : \ 0 \leq \omega \leq (1 \wedge \|f\|^{-1}_{L^\infty([0,1])})\big\} = 1.$$

This contradicts $\|\mathbf{1} - gf\|_{L^\infty([0,1])} < 1/2$ and so no such f can exist, that is, $\mathbf{1} \notin \overline{\mathcal{R}(M_g)}$. $\square$

Given $0 < p < \infty$, define the p-th power of the order ideal $\mathcal{M}\big(X(\mu), Y(\mu)\big)$ by

$$\mathcal{M}\big(X(\mu), Y(\mu)\big)_{[p]} := \Big\{ g \in L^0(\mu) \ : \ |g|^{1/p} \in \mathcal{M}\big(X(\mu), Y(\mu)\big) \Big\}$$

as for the case of the p-th power of a q-B.f.s. Then we have

$$\mathcal{M}\big(X(\mu),\, Y(\mu)\big)_{[p]} = \mathcal{M}\big(X(\mu)_{[p]},\, Y(\mu)_{[p]}\,\big). \tag{2.77}$$

Given $g \in L^0(\mu)^+$, we have from (2.77) that $g^{1/p} \in \mathcal{M}\big(X(\mu), Y(\mu)\big)$ if and only if $g \in \mathcal{M}\big(X(\mu)_{[p]}, Y(\mu)_{[p]}\,\big)$. Let $\mu_g : \Sigma \to [0,\infty]$ denote the indefinite integral $A \mapsto \int_A g\, d\mu$, for $A \in \Sigma$. The following result will be useful in Chapter 6. Its proof will be omitted as it is a direct application of the definitions, of (2.77) with $Y(\mu) := L^p(\mu)$ and of (2.74) with $X(\mu)_{[p]}$ in place of $X(\mu)$.

Proposition 2.29. *Let $X(\mu)$ be a q-B.f.s. over (Ω,Σ,μ) and let $0 < p < \infty$. The following statements for a function $g \in L^1(\mu)^+$ are equivalent.*

(i) $\int_\Omega |f|^p g\, d\mu \; = \; \int_\Omega \big|f g^{1/p}\big|^p\, d\mu \; < \; \infty$ *for all $f \in X(\mu)$.*

(ii) $g^{1/p} \in \mathcal{M}\big(X(\mu),\, L^p(\mu)\big)$.

(iii) $g^{r/p} \in \mathcal{M}\big(X(\mu)_{[r]},\, L^{p/r}(\mu)\big)$ *for some/every $0 < r < \infty$.*

(iv) $g \in (X(\mu)_{[p]})'$.

Given $1 \le w \le \infty$, we have from Hölder's inequality that

$$\mathcal{M}\big(L^w(\mu),\, L^1(\mu)\big) = L^w(\mu)' = L^{w'}(\mu), \tag{2.78}$$

where $(1/w)+(1/w') = 1$ (with the understanding that $1/\infty = 0$). Let us generalize this in Example 2.30(i) below.

Example 2.30. (i) Let $0 < p, q, r \le \infty$ satisfy

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}. \tag{2.79}$$

We claim that

$$\mathcal{M}\big(L^p(\mu),\, L^r(\mu)\big) = L^q(\mu). \tag{2.80}$$

In fact, if any one of p, q, r is infinity, then (2.80) is clear. So assume that $0 < p, q, r < \infty$. Then (2.79) gives $(p/r)^{-1} + (q/r)^{-1} = 1$. Since $1 < (p/r) < \infty$, we have from (2.78) with $w := (p/r)$ that $\mathcal{M}\big(L^{p/r}(\mu), L^1(\mu)\big) = L^{q/r}(\mu)$. This and (2.77) imply that

$$\begin{aligned} \mathcal{M}\big(L^p(\mu),\, L^r(\mu)\big) \; &= \; \mathcal{M}\big(L^{p/r}(\mu)_{[1/r]},\, L^1(\mu)_{[1/r]}\,\big) \\ = \; \mathcal{M}\big(L^{p/r}(\mu),\, L^1(\mu)\,\big)_{[1/r]} \; &= \; L^{q/r}(\mu)_{[1/r]} \; = \; L^q(\mu). \end{aligned}$$

(ii) Let $0 < r < 1 \le p < \infty$. Choose $0 < q < \infty$ to satisfy (2.79), in which case (2.80) holds. Hence, $g \cdot L^p(\mu) = M_g\big(L^p(\mu)\big) \subseteq L^r(\mu)$ for all $g \in L^q(\mu)$. In particular, the choice $g := \chi_\Omega$ gives that $L^p(\mu) \subseteq L^r(\mu)$, that is, the B.f.s. $L^p(\mu)$ is contained in the q-B.f.s. $L^r(\mu)$.

On the other hand, the space $\mathcal{M}(L^r(\mu),\ L^p(\mu))$ is not as rich as

$$\mathcal{M}(L^p(\mu), L^r(\mu)) = L^q(\mu).$$

To see this, assume first that μ is non-atomic. Then, $\mathcal{M}(L^r(\mu), L^p(\mu)) = \{0\}$. Indeed, let $h \in \mathcal{M}(L^r(\mu), L^p(\mu))$. To show that $h = 0$, we may assume that $h \geq 0$. Apply Proposition 2.29 with $g := h^p$ and $X(\mu) := L^r(\mu)$ to obtain that $h^p \in \big(L^r(\mu)_{[p]}\big)' = L^{r/p}(\mu)'$. But, $L^{r/p}(\mu)' = L^{r/p}(\mu)^* = \{0\}$ because $(r/p) < 1$ (see Example 2.10 and Remark 2.18(i)). So, $\mathcal{M}(L^r(\mu), L^p(\mu)) = \{0\}$.

Suppose now that μ is purely atomic. To make the presentation simpler, assume that $\mu : 2^{\mathbb{N}} \to [0, \infty)$ is a finite measure, with $\mu(\{n\}) > 0$ for every $n \in \mathbb{N}$ and that $p = 1$. Then, $\mathcal{M}(\ell^r(\mu), \ell^1(\mu)) \neq \{0\}$ because $\mathcal{M}(\ell^r(\mu), \ell^1(\mu)) \ni \chi_A$ for every non-empty finite set $A \subseteq \mathbb{N}$. Of course, $\mathcal{M}(\ell^r(\mu), \ell^1(\mu)) \not\ni \chi_\Omega$ because $\ell^r(\mu) \nsubseteq \ell^1(\mu)$. However, for a suitable choice of the measure μ and the weight function ψ on $\mathbb{N}$, we can have the inclusion

$$\ell^r(\psi\, d\mu) \subseteq \ell^1(\mu); \tag{2.81}$$

in other words, the q-B.f.s. $\ell^r(\psi\, d\mu)$ is contained in the B.f.s. $\ell^1(\mu)$. We shall deal with similar cases in Chapter 6. Here, $\psi\, d\mu$ denotes the scalar measure defined by

$$A \longmapsto \sum_{n\in A} \psi(n)\, \mu(\{n\}), \qquad A \in 2^{\mathbb{N}}. \tag{2.82}$$

Let us present an explicit μ and ψ which satisfy (2.81). Choose $\varphi \in \ell^r \subsetneq \ell^1$ such that $\varphi(n) > 0$ for all $n \in \mathbb{N}$. Define μ by $\mu(A) := \sum_{n\in A} \varphi(n)$ for $A \in 2^{\mathbb{N}}$, and let $\psi(n) := \big(\mu(\{n\})\big)^{r-1}$ for $n \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} |f(n)|\, \mu(\{n\}) &= \sum_{n=1}^{\infty} \Big[\big(|f(n)|\, \mu(\{n\})\big)^r\Big]^{1/r} \leq \Big(\sum_{n=1}^{\infty} \big(|f(n)|\, \mu(\{n\})\big)^r\Big)^{1/r} \\ &= \Big(\sum_{n=1}^{\infty} |f(n)|^r \psi(n)\, \mu(\{n\})\Big)^{1/r} = \|f\|_{\ell^r(\psi\, d\mu)} < \infty\end{aligned}$$

for all $f \in \ell^r(\psi\ d\mu)$, which establishes (2.81). We have used the fact that the inequality $\|g\|_{\ell^{1/r}} \leq \|g\|_{\ell^1}$ holds for every $g \in \ell^1$. $\square$

2.3 Completeness criteria

Proposition 2.22 shows that the spaces $X(\mu)_{[p]}$, for $0 < p < \infty$, have the important property that they are complete whenever the q-B.f.s. $X(\mu)$ is complete. We now present a criterion which ensures the completeness of $X(\mu)$ itself; see Proposition 2.35 below. The criterion applies to an extensive class of spaces and is well known for normed function spaces; we are unaware of any such results for lattice quasi-norms.

The proof proceeds via several lemmata, and relies on the existence of an F-norm $|||\cdot|||$ on $X(\mu)$ which generates the same topology as the original topology and satisfies various useful inequalities.

Lemma 2.31. *Suppose that $(X(\mu), \|\cdot\|)$ is a lattice quasi-normed function space. Let $|||\cdot||| : X(\mu) \to [0,\infty)$ be the* F*-norm defined via* (2.4) *with $Z := X(\mu)$ and for an appropriate $r > 0$, so that* (2.5) *gives*

$$\frac{1}{4}\|f\|^r \le |||f||| \le \|f\|^r, \qquad f \in X(\mu). \tag{2.83}$$

(i) *The inequality $|||f||| \le 4\,|||g|||$ holds whenever $f, g \in X(\mu)$ satisfy $|f| \le |g|$.*

(ii) *For all $f \in X(\mu)$ we have*

$$\frac{1}{4}\,|||\,|f|\,||| \le |||f||| \le 4\,|||\,|f|\,|||. \tag{2.84}$$

(iii) *For each $\alpha \in \mathbb{C}$ and $f \in X(\mu)$ we have*

$$\frac{|\alpha|^r}{4}\,|||f||| \le |||\alpha f||| \le 4\,|\alpha|^r\,|||\,|f|\,|||. \tag{2.85}$$

(iv) *For each $\beta > 0$ and $f \in X(\mu)$ we have*

$$\frac{\beta}{4}\,|||f||| \le |||\beta^{1/r} f||| \le 4\beta\,|||f|||. \tag{2.86}$$

Proof. (i) According to (2.83), if $f, g \in X(\mu)$ satisfy $|f| \le |g|$, then the lattice property (2.3) implies that

$$|||f||| \le \|f\|^r \le \|g\|^r \le 4|||g|||,$$

which is precisely the inequality in part (i).

(ii) Since $|f| \le |f|$, part (i) yields both $|||f||| \le 4|||\,|f|\,|||$ and $4^{-1}|||\,|f|\,||| \le |||f|||$. These two inequalities together form (2.84).

(iii) Using (Q2) and (2.83) we have

$$|||\alpha f||| \le \|\alpha f\|^r = |\alpha|^r \|f\|^r \le 4|\alpha|^r |||f|||$$

and also

$$|||\alpha f||| \ge \frac{1}{4}\,\|\alpha f\|^r = \frac{|\alpha|^r}{4}\,\|f\|^r \ge \frac{|\alpha|^r}{4}\,|||f|||.$$

These two inequalities give (2.85).

(iv) Substitute $\alpha := \beta^{1/r}$ into (2.85) yields (2.86). $\square$

Let $X(\mu)$ be an order ideal in $L^0(\mu)$ and $\eta : X(\mu) \to [0,\infty)$ be either a quasi-norm or an F-norm. We say that $(X(\mu),\eta)$ has the *weak Fatou property* if, whenever $\{u_n\}_{n=1}^\infty \subseteq X(\mu)^+$ with $u_n \uparrow$ and $\sup_{n\in\mathbb{N}} \eta(u_n) < \infty$, there exists $u \in X(\mu)^+$ such that $u_n \uparrow u$ (μ-a.e.). Since $u \in X(\mu)^+ \subseteq L^0(\mu)$, we note that $u = \sup_{n\in\mathbb{N}} u_n$ is finite-valued μ-a.e.

Remark 2.32. (a) Let $X(\mu)$ be an order ideal in $L^0(\mu)$. Given any lattice quasi-norm $\|\cdot\|$ on $X(\mu)$, let $|\!|\!|\cdot|\!|\!|$ be the F-norm defined by (2.4) with $Z := X(\mu)$ and for an appropriate $r > 0$. In view of (2.83), we see that $(X(\mu),\|\cdot\|)$ has the weak Fatou property if and only if $(X(\mu),|\!|\!|\cdot|\!|\!|)$ has the weak Fatou property.

(b) Let $\rho : M^+(\mu) \to [0,\infty]$ be a function norm; see Remark 2.3. With the definitions

$$X(\mu) := \{f \in L^0(\mu) : \rho(|f|) < \infty\} = L_\rho$$

and $\eta(f) := \rho(|f|)$ for $f \in X(\mu)$, it follows that the normed function space $(X(\mu),\eta)$ has the weak Fatou property, as defined above, precisely when it possesses this property in the classical sense, [164, p. 446]. $\square$

Let $|\!|\!|\cdot|\!|\!|$ be any F-norm defined on an order ideal $X(\mu)$ of $L^0(\mu)$. We say that $(X(\mu),|\!|\!|\cdot|\!|\!|)$ has the *Riesz–Fischer property* if, whenever $\{u_n\}_{n=1}^\infty \subseteq X(\mu)^+$ satisfies $\sum_{n=1}^\infty |\!|\!|u_n|\!|\!| < \infty$, there exists $u \in X(\mu)^+$ such that $u = \sum_{n=1}^\infty u_n$ pointwise μ-a.e. on Ω. Since $u \in X(\mu)^+ \subseteq L^0(\mu)$, we note that $\sum_{n=1}^\infty u_n$ is finite-valued μ-a.e. In the case when $|\!|\!|\cdot|\!|\!|$ is a lattice norm, it is known that the normed function space $(X(\mu),|\!|\!|\cdot|\!|\!|)$ is complete if and only if it has the Riesz–Fischer property (see, for example, [164, Ch. 15, Theorem 2]).

Lemma 2.33. *Let $X(\mu)$ be an order ideal in $L^0(\mu)$ and $|\!|\!|\cdot|\!|\!|$ be an* F*-norm on $X(\mu)$ with the weak Fatou property. Then $(X(\mu),|\!|\!|\cdot|\!|\!|)$ has the Riesz–Fischer property.*

Proof. Let $\{u_n\}_{n=1}^\infty \subseteq X(\mu)^+$ satisfy $\sum_{n=1}^\infty |\!|\!|u_n|\!|\!| < \infty$. Define $s_n := \sum_{j=1}^n u_j$, for $n \in \mathbb{N}$, in which case $s_n \uparrow$ in $X(\mu)^+$. By the triangle inequality (F4) for F-norms we have

$$|\!|\!|s_n|\!|\!| \le \sum_{j=1}^n |\!|\!|u_j|\!|\!| \le \sum_{j=1}^\infty |\!|\!|u_j|\!|\!| < \infty, \qquad n \in \mathbb{N}.$$

Then the weak Fatou property ensures the existence of $u \in X(\mu)^+$ such that $s_n \uparrow u$ (i.e., $u = \sum_{n=1}^\infty u_n$) pointwise μ-a.e. on Ω. $\square$

The converse of the previous lemma is not valid, in general; for instance, see Example 3.34. The following result provides an appropriate substitute (for a particular F-norm) for the infinite triangle inequality in lattice normed function spaces possessing the Riesz–Fischer property.

Lemma 2.34. *Let $X(\mu)$ be an order ideal in $L^0(\mu)$ and $\|\cdot\|$ be a lattice quasi-norm on $X(\mu)$. Let $|\!|\!|\cdot|\!|\!|$ be the* F*-norm on $X(\mu)$ defined via* (2.4) *with $Z := X(\mu)$ and* *for an appropriate* $r > 0$*. If* $\big(X(\mu), \|\!|\cdot|\!\|\big)$ *has the Riesz–Fischer property, then*

$$\Big\|\!\Big| \sum_{n=1}^{\infty} u_n \Big|\!\Big\| \le 16 \sum_{n=1}^{\infty} \|\!| u_n |\!\|$$

for every sequence $\{u_n\}_{n=1}^{\infty} \subseteq X(\mu)^+$ *whose pointwise* μ*-a.e. sum* $\sum_{n=1}^{\infty} u_n$ *belongs to* $X(\mu)^+$*.*

Proof. Assume the contrary, that is, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subseteq X(\mu)^+$ whose pointwise μ-a.e. sum $\sum_{n=1}^{\infty} u_n$ belongs to $X(\mu)^+$ but

$$\Big\|\!\Big| \sum_{n=1}^{\infty} u_n \Big|\!\Big\| > 16 \sum_{n=1}^{\infty} \|\!| u_n |\!\|.$$

Note that $\sum_{n=1}^{\infty} \|\!| u_n |\!\| < \infty$ since $\sum_{n=1}^{\infty} u_n \in X(\mu)^+$ implies that $\|\!| \sum_{n=1}^{\infty} u_n |\!\| < \infty$. Choose $\varepsilon > 0$ such that

$$\Big\|\!\Big| \sum_{n=1}^{\infty} u_n \Big|\!\Big\| > \varepsilon + 16 \sum_{n=1}^{\infty} \|\!| u_n |\!\|. \tag{2.87}$$

For $k \in \mathbb{N}$, set $\beta_k := 4k/\varepsilon$ and then multiply (2.87) by β_k to get

$$\beta_k \Big\|\!\Big| \sum_{n=1}^{\infty} u_n \Big|\!\Big\| > 4k + 16\,\beta_k \sum_{n=1}^{\infty} \|\!| u_n |\!\|. \tag{2.88}$$

By (2.86) we conclude that

$$\beta_k \Big\|\!\Big| \sum_{n=1}^{\infty} u_n \Big|\!\Big\| \le 4 \Big\|\!\Big| \sum_{n=1}^{\infty} \beta_k^{1/r} u_n \Big|\!\Big\| \tag{2.89}$$

and also that

$$4k + 16\,\beta_k \sum_{n=1}^{\infty} \|\!| u_n |\!\| \ge 4k + 4 \sum_{n=1}^{\infty} \|\!| \beta_k^{1/r} u_n |\!\|, \tag{2.90}$$

for all $k \in \mathbb{N}$. Combining (2.88), (2.89) and (2.90) yields

$$4 \Big\|\!\Big| \sum_{n=1}^{\infty} \beta_k^{1/r} u_n \Big|\!\Big\| > 4k + 4 \sum_{n=1}^{\infty} \|\!| \beta_k^{1/r} u_n |\!\|, \qquad k \in \mathbb{N}.$$

That is, with $v_n^{(k)} := \beta_k^{1/r} u_n \in X(\mu)^+$ we have $\sum_{n=1}^{\infty} v_n^{(k)} = \beta_k^{1/r} \sum_{n=1}^{\infty} u_n \in X(\mu)^+$ and

$$\Big\|\!\Big| \sum_{n=1}^{\infty} v_n^{(k)} \Big|\!\Big\| > k + \sum_{n=1}^{\infty} \|\!| v_n^{(k)} |\!\|, \qquad k \in \mathbb{N}. \tag{2.91}$$

For each $k \in \mathbb{N}$, subtract the inequality $\left\| \sum_{n=1}^{j-1} v_n^{(k)} \right\| \leq \sum_{n=1}^{j-1} \left\| v_n^{(k)} \right\|$, valid for all $j \geq 2$, from (2.91) to conclude that

$$\left\| \sum_{n=j}^{\infty} v_n^{(k)} \right\| \geq k \ + \sum_{n=j}^{\infty} \left\| v_n^{(k)} \right\|, \qquad j \geq 2. \tag{2.92}$$

For each $k \in \mathbb{N}$, choose $j_k \geq 2$ such that $\sum_{n=j_k}^{\infty} \left\| v_n^{(k)} \right\| < 1/k^2$. Also, with $j := j_k$ in (2.92), we have $\left\| \sum_{n=j_k}^{\infty} v_n^{(k)} \right\| > k$. By rearranging indices we see that there is a sequence $\{w_{n,k}\}_{n,k=1}^{\infty}$ in $X(\mu)^+$ whose μ-a.e. pointwise sums $\sum_{n=1}^{\infty} w_{n,k} \in X(\mu)^+$, for $k \in \mathbb{N}$, satisfy

$$\sum_{n=1}^{\infty} \| w_{n,k} \| < 1/k^2 \qquad \text{and} \qquad \left\| \sum_{n=1}^{\infty} w_{n,k} \right\| > k. \tag{2.93}$$

But, $\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \| w_{n,k} \| < \sum_{k=1}^{\infty} 1/k^2 < \infty$ and so, by the Riesz–Fischer property, there exists $w \in X(\mu)^+$ such that $w = \sum_{k,n=1}^{\infty} w_{n,k}$ pointwise μ-a.e. In particular, $\| w \| < \infty$. On the other hand, since $w \geq \sum_{n=1}^{\infty} w_{n,k}$ for each $k \in \mathbb{N}$, we have, by (2.93) and Lemma 2.31(i), that

$$\| w \| \geq \frac{1}{4} \left\| \sum_{n=1}^{\infty} w_{n,k} \right\| > \frac{k}{4}, \qquad k \in \mathbb{N}.$$

This contradicts $\| w \| < \infty$. □

Finally, the promised completeness criterion for $X(\mu)$.

Proposition 2.35. *Let $X(\mu)$ be an order ideal in $L^0(\mu)$ and $\|\cdot\|$ be a lattice quasi-norm on $X(\mu)$ such that $(X(\mu), \|\cdot\|)$ has the weak Fatou property. Then $X(\mu)$ is complete relative to $\|\cdot\|$. If, in addition, $X(\mu) \supseteq \operatorname{sim}\Sigma$, then $(X(\mu), \|\cdot\|)$ is a q-B.f.s.*

Proof. Let $\|\cdot\|$ be the F-norm on $X(\mu)$ defined via (2.4) with $Z := X(\mu)$ and an appropriate $r > 0$. According to Remark 2.32(a), the F-norm $\|\cdot\|$ also has the weak Fatou property and hence, by Lemma 2.33, has the Riesz–Fischer property.

Let $\{f_n\}_{n=1}^{\infty}$ be any Cauchy sequence in $(X(\mu), \|\cdot\|)$. Choose a subsequence $\{f_{n(k)}\}_{k=1}^{\infty}$ satisfying

$$\| f_{n(k+1)} - f_{n(k)} \| < 1/2^k, \qquad k \in \mathbb{N},$$

which is possible by (2.83). Define $u_k := |f_{n(k+1)} - f_{n(k)}|$ for $k \in \mathbb{N}$. Then (2.84) yields

$$\sum_{k=1}^{\infty} \| u_k \| \leq 4 \sum_{k=1}^{\infty} \| f_{n(k+1)} - f_{n(k)} \| < \infty.$$

By the Riesz–Fischer property, there is $u \in X(\mu)^+$ such that $u = \sum_{k=1}^{\infty} u_k$ (μ-a.e.). Set

$$f(\omega) := f_{n(1)}(\omega) \ + \ \sum_{k=1}^{\infty} \Big(f_{n(k+1)}(\omega) - f_{n(k)}(\omega) \Big)$$

for all $\omega \in \Omega$ satisfying $u(\omega) = \sum_{k=1}^{\infty} u_k(\omega)$. Now, fix $j \in \mathbb{N}$. Then we have

$$f(\omega) - f_{n(j+1)}(\omega) = \sum_{k=j+1}^{\infty} \Big(f_{n(k+1)}(\omega) - f_{n(k)}(\omega) \Big)$$

for μ-a.e. $\omega \in \Omega$. So, by parts (i) and (ii) of Lemma 2.31 we have

$$|\!|\!| f - f_{n(j+1)} |\!|\!| \le 4 \Big|\!\Big|\!\Big| \Big| \sum_{k=j+1}^{\infty} (f_{n(k+1)} - f_{n(k)}) \Big| \Big|\!\Big|\!\Big| \le 16 \Big\| \sum_{k=j+1}^{\infty} \big| f_{n(k+1)} - f_{n(k)} \big| \Big\|.$$

Now apply (2.84) and Lemma 2.34 to conclude that

$$\begin{aligned} |\!|\!| f - f_{n(j+1)} |\!|\!| &\le 16^2 \sum_{k=j+1}^{\infty} \big\| \, | f_{n(k+1)} - f_{n(k)} | \, \big\| \\ &\le 4 \cdot 16^2 \sum_{k=j+1}^{\infty} |\!|\!| f_{n(k+1)} - f_{n(k)} |\!|\!| \le 4^5 \sum_{k=j+1}^{\infty} 2^{-k}. \end{aligned}$$

Accordingly, $f_{n(j+1)} \to f$ in $(X(\mu), |\!|\!| \cdot |\!|\!|)$ as $j \to \infty$. Since $\{f_n\}_{n=1}^{\infty}$ is $|\!|\!| \cdot |\!|\!|$-Cauchy, we conclude that $f_k \to f$ as $k \to \infty$ (with respect to $|\!|\!| \cdot |\!|\!|$) because

$$|\!|\!| f_k - f |\!|\!| \le |\!|\!| f_k - f_{n(j+1)} |\!|\!| \ + \ |\!|\!| f_{n(j+1)} - f |\!|\!|$$

for all $k, j \in \mathbb{N}$. By (2.83), also $f_k \to f$ in $(X(\mu), \| \cdot \|)$ as $k \to \infty$. $\square$

2.4 Completely continuous operators

A continuous linear operator between Banach spaces is called *completely continuous* if it maps every weakly convergent sequence to a norm convergent sequence. This is equivalent to the condition that the operator maps every weakly compact set to a relatively compact set, thanks to the Eberlein-Smulian Theorem. Completely continuous operators are often called *Dunford-Pettis operators*. Compact operators are always completely continuous. The converse is not valid, in general, unless the domain of the operator is reflexive. A well-known example is the natural inclusion map from ℓ^1 into ℓ^2 which is completely continuous but not compact; see Example 2.44 below.

The following conditions for a continuous linear operator T from the B.f.s. $L^1(\mu)$ into a Banach space E are equivalent:

(a) T is completely continuous.

(b) The set $\{T(\chi_A) : A \in \Sigma\}$ is relatively compact in E.

This fact is, for example, an immediate consequence of [67, Fact 2.9]. If we replace $L^1(\mu)$ with a general B.f.s., then condition (b) is necessary for complete continuity but, in general, is no longer sufficient as can be seen from the following example.

Example 2.36. Let the measure μ and the notation be as in Example 2.11 and $1 < r < \infty$. Then the canonical isometry $\Phi : \ell^r(\mu) \to \ell^r$ defined by (2.23) is surely not completely continuous because its image $\mathbf{B}[\ell^r]$, of the weakly compact subset $\mathbf{B}[\ell^r(\mu)]$ of $\ell^r(\mu)$, is non-compact. On the other hand, the bounded subset $W := \{\Phi(\chi_A) : A \in 2^{\mathbb{N}}\} = \{\varphi^{1/r}\chi_A : A \in 2^{\mathbb{N}}\}$ of ℓ^r is compact. In fact, since $\|\varphi^{1/r}\chi_A\|_{\ell^r} \le \|\varphi^{1/r}\|_{\ell^r}$ for every $A \in 2^{\mathbb{N}}$, we see that

$$\lim_{n\to\infty}\sum_{j=n}^{\infty}\left|\varphi^{1/r}(j)\chi_A(j)\right|^r = 0$$

uniformly with respect to $A \in 2^{\mathbb{N}}$, and hence, it follows from [46, Ch. IV, Exercise 13.3] that W is relatively compact. Clearly W is closed, and therefore, is compact. $\square$

We proceed to identify those continuous linear operators T defined on a general q-B.f.s. which satisfy condition (b) above; see Proposition 2.41 below. Our proof will follow the lines of [42, p. 93].

We say that a subset K of a q-B.f.s. $X(\mu)$ is *uniformly μ-absolutely continuous* if $\sup_{f\in K}\|f\chi_A\|_{X(\mu)} \to 0 \quad \text{as} \quad \mu(A) \to 0$. More precisely, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sup_{f\in K}\|f\chi_A\|_{X(\mu)} < \varepsilon$ for all $A \in \Sigma$ satisfying $\mu(A) < \delta$. This property is called "uniformly equi-integrable" in [20, p. 6]. For $X(\mu) = L^1(\mu)$, uniform μ-absolute continuity is precisely uniform μ-integrability in the sense of [42, p. 74]. If $X(\mu)$ is σ-o.c., then its subset $\{\chi_A : A \in \Sigma\}$ is always uniformly μ-absolutely continuous as will be shown in Lemma 2.37 below. Let us adopt the following notation (which is standard in the case of real Banach lattices):

$$[-g, g] := \{f \in X(\mu) : |f| \le g\}, \qquad g \in X(\mu)^+.$$

Lemma 2.37. *Let $X(\mu)$ be a q-B.f.s. with σ-o.c. quasi-norm.*

(i) *For each $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \varepsilon$ whenever $A \in \Sigma$ satisfies $\|\chi_A\|_{X(\mu)} < \delta$.*

(ii) *For each $g \in X(\mu)$ we have $\lim_{\mu(A)\to 0}\|g\chi_A\|_{X(\mu)} = 0$.*

(iii) *The following conditions for a subset $K \subseteq X(\mu)$ are equivalent.*

  (a) *The set K is bounded and uniformly μ-absolutely continuous.*

  (b) *For every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that*

$$K \subseteq [-N\chi_\Omega, N\chi_\Omega] + \varepsilon\mathbf{B}[X(\mu)]. \tag{2.94}$$

  (c) *For every $\varepsilon > 0$, there is $g \in X(\mu)^+$ such that*

$$K \subseteq [-g, g] + \varepsilon\mathbf{B}[X(\mu)]. \tag{2.95}$$

Proof. (i) Assume that the claim is false. Then we can select $\varepsilon > 0$ and a sequence $\{A(j)\}_{j=1}^{\infty} \subseteq \Sigma$ such that

$$\|\chi_{A(j)}\|_{X(\mu)} < j^{-1} \quad \text{and} \quad \mu\big(A(j)\big) > \varepsilon \qquad \text{for every} \quad j \in \mathbb{N}. \tag{2.96}$$

The natural injection from $X(\mu)$ into $L^0(\mu)$ is continuous by Proposition 2.2(i). From this and the first inequality in (2.96), it follows that the sequence $\{\chi_{A(j)}\}_{j=1}^{\infty}$ converges to zero in measure. Therefore,

$$\lim_{j\to\infty} \mu\big(\{\omega \in \Omega : |\chi_{A(j)}(\omega)| > 2^{-1}\}\big) = 0. \tag{2.97}$$

However, $A(j) = \{\omega \in \Omega : |\chi_{A(j)}(\omega)| > 2^{-1}\}$ for every $j \in \mathbb{N}$. So, (2.97) means that $\lim_{j\to\infty} \mu(A(j)) = 0$, which contradicts the second inequality in (2.96).

(ii) Assume that the claim is false. Then there exist $g \in X(\mu)$, $\varepsilon > 0$ and a sequence $\{B(j)\}_{j=1}^{\infty} \subseteq \Sigma$ such that

$$\mu\big(B(j)\big) < 2^{-j} \quad \text{and} \quad \|g\chi_{B(j)}\|_{X(\mu)} > \varepsilon \qquad \text{for} \quad j \in \mathbb{N}. \tag{2.98}$$

With $C(n) := \bigcup_{j=n}^{\infty} B(j)$ for $n \in \mathbb{N}$, let $C := \bigcap_{n=1}^{\infty} C(n)$. Then $C(n) \downarrow C$ and so

$$\mu(C) = \lim_{n\to\infty} \mu\big(C(n)\big) \le \lim_{n\to\infty} \sum_{j=n}^{\infty} \mu\big(B(j)\big) \le \lim_{n\to\infty} 2^{-n+1} = 0,$$

that is, $\chi_C = 0$ (μ-a.e.). Hence, $|g|\chi_{C(n)} \downarrow 0$ (μ-a.e.) in the σ-o.c. space $X(\mu)$ and therefore $\|g\chi_{C(n)}\|_{X(\mu)} \to 0$ as $n \to \infty$. This is impossible because (2.98) yields $\|g\chi_{C(n)}\|_{X(\mu)} \ge \|g\chi_{B(n)}\|_{X(\mu)} > \varepsilon$ for every $n \in \mathbb{N}$.

(iii) (a) $\Rightarrow$ (b). Fix $\varepsilon > 0$. Let $B(f, n) := \{\omega \in \Omega : |f(\omega)| \ge n\}$ for every $f \in K$ and $n \in \mathbb{N}$, and let $c := \sup_{f\in K} \|f\|_{X(\mu)} < \infty$. Since $n\chi_{B(f,n)} \le |f\chi_{B(f,n)}| \le |f|$, we have that

$$\|\chi_{B(f,n)}\|_{X(\mu)} \le cn^{-1}, \qquad f \in K, \quad n \in \mathbb{N}.$$

By uniform μ-absolute continuity of K, we can select $\delta_1 > 0$ satisfying

$$\sup_{f\in K} \|f\chi_A\|_{X(\mu)} < \varepsilon \tag{2.99}$$

whenever $A \in \Sigma$ satisfies $\mu(A) < \delta_1$. By part (i) there exists $\delta_2 > 0$ such that $\mu(A) < \delta_1$ whenever $A \in \Sigma$ and $\|\chi_A\|_{X(\mu)} < \delta_2$. Choose $N \in \mathbb{N}$ for which $cN^{-1} < \delta_2$, so that $\|\chi_{B(f,N)}\|_{X(\mu)} \le cN^{-1} < \delta_2$. Then, for every $f \in K$, we have $\mu\big(B(f, N)\big) < \delta_1$. Consequently, with $A := B(f, N)$, (2.99) gives

$$\sup_{f\in K} \big\|f\chi_{B(f,N)}\big\|_{X(\mu)} < \varepsilon. \tag{2.100}$$

Thus, for every $f \in K$, we have that

$$f = f\chi_{\Omega\setminus B(f,N)} + f\chi_{B(f,N)} \in [-N\chi_\Omega, N\chi_\Omega] + \varepsilon\mathbf{B}[X(\mu)].$$

That is, (b) holds.

(b) $\Rightarrow$ (c). Clear.

(c) $\Rightarrow$ (a). Let $\varepsilon > 0$. Choose $g \in X(\mu)^+$ satisfying (2.95). Apply part (ii) to select $\delta > 0$ such that $\|g\chi_A\|_{X(\mu)} < \varepsilon$ whenever $A \in \Sigma$ satisfies $\mu(A) < \delta$. Fix $f \in K$. By (2.95) there exists $h \in \mathbf{B}[X(\mu)]$ such that $|f - \varepsilon h| \leq g$. Consequently, if $A \in \Sigma$ satisfies $\mu(A) < \delta$, then $\|f\chi_A\|_{X(\mu)} \leq \|g\chi_A\|_{X(\mu)} + \varepsilon\|h\chi_A\|_{X(\mu)} \leq 2\varepsilon$. Since $f \in K$ is arbitrary, this shows that (a) holds. □

Remark 2.38. (a) Note that part (i) of Lemma 2.37 does not require $X(\mu)$ to be σ-o.c.; see its proof.

(b) By part (iii) of Lemma 2.37, if $X(\mu)$ is a real B.f.s., then the bounded, uniformly μ-absolutely continuous subsets are exactly the L-weakly compact subsets of $X(\mu)$. For this, see the definition of L-weakly compact sets in a real Banach lattice and its equivalent formulation in [108, Ch. 3, §3.6]. Moreover, L-weakly compact sets are necessarily relatively weakly compact in real Banach lattices, [108, Proposition 3.6.5]. Instead of extending this to the case of complex Banach lattices via the usual complexification, we prefer to give an elementary and direct proof applicable to B.f.s.' with σ-o.c. norm over $\mathbb{R}$ or $\mathbb{C}$; see the following result. □

Proposition 2.39. *Let $X(\mu)$ be a B.f.s. with σ-o.c. norm.*

(i) *The order interval $[-\chi_\Omega, \chi_\Omega]$ is weakly compact in $X(\mu)$.*

(ii) *Every bounded, uniformly μ-absolutely continuous subset of $X(\mu)$ is relatively weakly compact.*

Proof. (i) Let $J : L^\infty(\mu) \to X(\mu)$ be the natural embedding; it will be shown to be weakly compact. We claim that its dual operator $J^* : X(\mu)^* \to L^\infty(\mu)^*$ satisfies $J^*\big(X(\mu)^*\big) \subseteq L^1(\mu)$ with $L^1(\mu)$ considered as a closed subspace of $L^\infty(\mu)^*$ in a natural way. In fact, let $g \in X(\mu)^* = X(\mu)' \subseteq L^1(\mu)$ (see Remark 2.18(i)). For every $f \in L^\infty(\mu)$ we have

$$\big\langle f, J^*(g)\big\rangle = \big\langle J(f), g\big\rangle = \int_\Omega J(f)g\,d\mu = \int_\Omega fg\,d\mu,$$

which implies that $J^*(g) = g \in L^1(\mu)$. This establishes the claim.

The following fact will be needed.

Fact A: *Let Y and Z be Banach spaces. If the dual operator $T^* : Z^* \to Y^{**}$ of an operator $T \in \mathcal{L}(Y^*, Z)$ satisfies $T^*(Z^*) \subseteq Y$ as subspaces of Y^{**}, then T is weakly compact.*

Let us verify this. The set $T^*(\mathbf{B}[Z^*])$ is $\sigma(Y^{**},Y^*)$-relatively compact by Alaoglu's Theorem, [46, Ch. V, Theorem 4.2], because $T^*(\mathbf{B}[Z^*]) \subseteq \|T\| \cdot \mathbf{B}[Y^{**}]$. By assumption $T^*(\mathbf{B}[Z^*]) \subseteq Y$ and hence, $T^*(\mathbf{B}[Z^*])$ is $\sigma(Y,Y^*)$-relatively compact, that is, $T^* : Z^* \to Y$ is weakly compact. Since the inclusion of Y into Y^{**} is weakly continuous, it follows that $T^*(\mathbf{B}(Z^*))$ is $\sigma(Y^{**},Y^{***})$-relatively compact in Y^{**}. Now Gantmacher's Theorem, [46, Ch. VI, Theorem 4.8], proves Fact A.

By applying Fact A with $T := J$, $Y := L^1(\mu)$ and $Z := X(\mu)$, we see that J is weakly compact, and so $[-\chi_\Omega, \chi_\Omega] = J\big(\mathbf{B}[L^\infty(\mu)]\big)$ is relatively weakly compact. As $[-\chi_\Omega, \chi_\Omega]$ is clearly a closed convex subset of $X(\mu)$, statement (i) holds.

(ii) Let W be a bounded, uniformly μ-absolutely continuous subset of $X(\mu)$. Lemma 2.37 implies that, given $\varepsilon > 0$, there is $N \in \mathbb{N}$ for which

$$W \;\subseteq\; [-N\chi_\Omega, N\chi_\Omega] \;+\; \varepsilon \mathbf{B}[X(\mu)].$$

Since $[-N\chi_\Omega, N\chi_\Omega] = N \cdot [-\chi_\Omega, \chi_\Omega]$ is weakly compact by (i), we conclude that W is relatively weakly compact via [2, Theorem 10.17]; this reference is for spaces over $\mathbb{R}$ but, an examination of its proof shows that it is also true for spaces over $\mathbb{C}$. $\square$

The converse of Fact A in the above proof is false, i.e., there are weakly compact operators $T \in \mathcal{L}(Y^*, Z)$ which do not satisfy $T^*(Z^*) \subseteq Y$. For instance, consider the case of $Y := c_0$, $Z = \mathbb{C}$ and the operator $T : \ell^1 \to \mathbb{C}$ given by $T(\varphi) := \sum_{n=1}^\infty \varphi(n)$, for $\varphi \in \ell^1$. Then $T^* : \mathbb{C} \to \ell^\infty$ is given by $T^*(\alpha) = (\alpha, \alpha, \alpha, \dots) = \alpha \mathbf{1}$ for $\alpha \in \mathbb{C} = \mathbb{C}^*$. In fact, for $\varphi \in Y^*$ and $\alpha \in \mathbb{C}$, we have

$$\big\langle \varphi,\ T^*(\alpha) \big\rangle = \big\langle T(\varphi),\ \alpha \big\rangle = \Big\langle \sum_{n=1}^\infty \varphi(n),\ \alpha \Big\rangle = \alpha \sum_{n=1}^\infty \varphi(n) = \langle \varphi, \alpha \mathbf{1} \rangle.$$

Thus, $T^*(Z^*) \;=\; T^*(\mathbb{C}) \not\subseteq c_0 = Y$ whereas the operator T is even compact.

Remark 2.40. (i) According to Dunford's Theorem, [42, Ch. II,Theorem 2.15], a bounded subset of $L^1(\mu)$ is uniformly μ-absolutely continuous if and only if it is relatively weakly compact. In the case of real B.f.s.' there is a class of B.f.s.' having this property. A real Banach function space $X(\mu)$ is said to have the *positive Schur property* if every positive weakly null sequence is norm-convergent. It is known that every weakly compact subset of a real B.f.s. $X(\mu)$ is uniformly μ-absolutely continuous if and only if $X(\mu)$ has the positive Schur property; see [144, Theorem 1.16] and [145, Theorem 1].

(ii) Weak compactness of the order intervals characterizes order continuity in the case of general real Banach lattices. Indeed, a real Banach lattice Z is order continuous if and only if, for every pair $\{y, z\}$ in Z with $y \leq z$, the order interval $[y, z] = \{x \in Z : y \leq x \leq z\}$ is weakly compact, [99, Theorem 1.b.16]. $\square$

Proposition 2.41. *Let $X(\mu)$ be a q-B.f.s. with σ-o.c. quasi-norm and E be a Banach space. For each $T \in \mathcal{L}(X(\mu), E)$ the following assertions are equivalent.*

(i) *T maps every bounded, uniformly μ-absolutely continuous subset of $X(\mu)$ to a relatively compact subset of E.*

(ii) *The set $\{T(\chi_A) : A \in \Sigma\}$ is relatively compact in E.*

(iii) *The restriction of T to the Banach space $L^\infty(\mu) \subseteq X(\mu)$ is a compact operator from $L^\infty(\mu)$ into E.*

Proof. (i) $\Rightarrow$ (ii). This implication is a consequence of Lemma 2.37(ii) from which it follows (with $g := \chi_\Omega$) that the bounded subset $\{\chi_A : A \in \Sigma\}$ of $X(\mu)$ is uniformly μ-absolutely continuous.

(ii) $\Rightarrow$ (iii). By considering pointwise limits of functions from $\operatorname{sim}\Sigma$ it is easy to check that $\mathbf{B}[L^\infty(\mu)] \subseteq 4\,\overline{\operatorname{bco}}\{\chi_A : A \in \Sigma\}$, where $\overline{\operatorname{bco}}$ stands for the closed, balanced, convex hull. So $T\big(\mathbf{B}[L^\infty(\mu)]\big) \subseteq 4\,\overline{\operatorname{bco}}\{T(\chi_A) : A \in \Sigma\}$.

(iii) $\Rightarrow$ (i). Take a bounded, uniformly μ-absolutely continuous subset K of $X(\mu)$. Let $\varepsilon > 0$ and select $N \in \mathbb{N}$ satisfying (2.94), so that

$$T(K) \;\subseteq\; T\big(N\mathbf{B}[L^\infty(\mu)]\big) \;+\; \varepsilon T\big(\mathbf{B}[X(\mu)]\big). \tag{2.101}$$

For every $a > 0$ and vector $x \in E$, let $U_a(x) := \{y \in E : \|x - y\|_E < a\}$. By (iii), the set $T\big(N\mathbf{B}[L^\infty(\mu)]\big)$ is relatively compact and hence, totally bounded in E. Thus, there exist $k \in \mathbb{N}$ and $x_1, \dots, x_k \in E$ such that

$$T\big(N\mathbf{B}[L^\infty(\mu)]\big) \;\subseteq\; \bigcup_{j=1}^{k} U_\varepsilon(x_j). \tag{2.102}$$

It follows from (2.101) and (2.102) that $T(K) \subseteq \bigcup_{j=1}^{k} U_{\varepsilon(1+\|T\|)}(x_j)$. So, $T(K)$ is totally bounded as ε is arbitrary. In other words, $T(K)$ is relatively compact in the Banach space E, and hence, (i) holds. $\square$

The above proposition, of course, applies to the case when $X(\mu) := L^1(\mu)$. Let us formally record this.

Corollary 2.42. *Let μ be a positive, finite measure, E be a Banach space and $T \in \mathcal{L}(L^1(\mu), E)$. Then the following statements are equivalent.*

(i) *T is completely continuous.*

(ii) *The subset $\{T(\chi_A) : A \in \Sigma\}$ of E is relatively compact.*

(iii) *The restriction of T to the Banach space $L^\infty(\mu) \subseteq L^1(\mu)$ is a compact operator from $L^\infty(\mu)$ into E.*

Proof. All we require is Dunford's Theorem; see Remark 2.40(i). $\square$

The above characterization can be extended to the setting of $L^1(\lambda)$ for any $[0,\infty]$-valued measure λ; see Corollary 2.43 below. Although $L^1(\lambda)$ is not a B.f.s. in our sense, such a characterization seems to be interesting and, moreover, it will be needed later when λ is the variation of a vector measure (see Chapter 3).

Corollary 2.43. *Let E be a Banach space, $\lambda : \Sigma \to [0, \infty]$ be a measure and $T \in \mathcal{L}(L^1(\lambda), E)$. Then T is completely continuous if and only if, for every $A \in \Sigma$ with $\lambda(A) < \infty$, the set $\{T(\chi_B) : B \in \Sigma \cap A\}$ is relatively compact in E.*

Proof. Suppose that T is completely continuous. Let $A \in \Sigma$ with $\lambda(A) < \infty$. The restriction λ_A of λ to the measurable space $(A, \Sigma \cap A)$ is a finite measure and the restriction T_A of T to $L^1(\lambda_A)$ is also completely continuous. Hence, the set $\{T(\chi_B) : B \in \Sigma \cap A\} = \{T_A(\chi_B) : B \in \Sigma \cap A\}$ is relatively compact in E.

For the converse, first observe that T is completely continuous if and only if, for each $B \in \Sigma$ with σ-finite λ-measure, the restriction of T to $L^1(\lambda_B)$ is completely continuous.

This follows from the definition of completely continuous operators because, for any weakly convergent sequence $\{f_n\}_{n=1}^\infty$ in $L^1(\lambda)$, there exists a set $B \in \Sigma$ with σ-finite λ-measure off which each f_n, for $n \in \mathbb{N}$, vanishes (λ-a.e.). So, we may assume that λ is σ-finite. Hence, we can take an infinite Σ-partition $\{A(n)\}_{n=1}^\infty$ of Ω such that $0 < \lambda\big(A(n)\big) < \infty$ for every $n \in \mathbb{N}$. Define a Σ-measurable function $g : \Omega \to (0, \infty)$ by

$$g := \sum_{n=1}^{\infty} \Big[2^n \lambda(A(n))\Big]^{-1} \chi_{A(n)}.$$

Let λ_g denote the indefinite integral of g with respect to λ, i.e., $\lambda_g(A) = \int_A g \, d\lambda$ for $A \in \Sigma$. Since $g \in L^1(\lambda)$, the measure λ_g is finite on Σ. The map $f \mapsto gf$ from $L^1(\lambda_g)$ onto $L^1(\lambda)$ is a Banach space isometry. Hence, $T : L^1(\lambda) \to E$ is completely continuous if and only if the E-valued operator $\widetilde{T} : f \mapsto T(gf)$, for $f \in L^1(\lambda_g)$, is completely continuous. To show that $\widetilde{T}$ is completely continuous we will apply Corollary 2.42.

For each $n \in \mathbb{N}$, let $W_n := \big\{[2^n \lambda(A(n))]^{-1} T(\chi_B) : B \in \Sigma \cap A(n)\big\}$. Given any $B(n) \in \Sigma \cap A(n)$, for $n \in \mathbb{N}$, the sequence $\big\{[2^n \lambda(A(n))]^{-1} T(\chi_{B(n)})\big\}_{n=1}^\infty$ is absolutely summable in E since

$$\begin{aligned} \sum_{n=1}^{\infty} \Big\| [2^n \lambda(A(n))]^{-1} T\big(\chi_{B(n)}\big) \Big\|_E &= \sum_{n=1}^{\infty} \Big\| T\big(g\chi_{B(n)}\big) \Big\|_E \\ &\leq \|T\| \sum_{n=1}^{\infty} \big\| g\chi_{B(n)} \big\|_{L^1(\lambda)} \leq \|T\| \cdot \|g\|_{L^1(\lambda)} < \infty. \end{aligned}$$

Consequently, $\big\{[2^n \lambda(A(n))]^{-1} T(\chi_{B(n)})\big\}_{n=1}^\infty$ is also unconditionally summable in the Banach space E. This enables us to define the *sum of* $\{W_n\}_{n=1}^\infty$ by

$$\sum_{n=1}^{\infty} W_n := \Big\{ \sum_{n=1}^{\infty} x_n : x_n \in W_n \ \text{ for } \ n \in \mathbb{N} \Big\} \subseteq E;$$

see [86, p. 3]. By assumption, each W_n is relatively compact in E, for $n \in \mathbb{N}$, and hence, the sum $\sum_{n=1}^\infty W_n$ is also relatively compact (this follows from [86, Ch. I, Lemma 1.3]). Since $g \in L^1(\lambda)$, the Dominated Convergence Theorem implies that $g\chi_B = \sum_{n=1}^{\infty} g\chi_{B\cap A(n)}$ converges absolutely in $L^1(\lambda)$ and hence, that

$$T(g\chi_B) = T\Big(\sum_{n=1}^{\infty} g\chi_{B\cap A(n)}\Big) = \sum_{n=1}^{\infty} T\big(g\chi_{B\cap A(n)}\big)$$

for every $B \in \Sigma$. It follows that

$$\big\{\widetilde{T}(\chi_B) : B \in \Sigma\big\} = \big\{T(g\chi_B) : B \in \Sigma\big\} = \sum_{n=1}^{\infty} W_n.$$

So, the operator $\widetilde{T} : L^1(\lambda_g) \to E$ is completely continuous via Corollary 2.42. Therefore, so is $T : L^1(\lambda) \to E$. $\square$

By Corollary 2.43, if the subset $\{T(\chi_A) : A \in \Sigma \text{ and } \lambda(A) < \infty\}$ of E is relatively compact, then $T : L^1(\lambda) \to E$ is completely continuous. The converse is not true, in general, as seen from the following well-known example.

Example 2.44. Let $\lambda : 2^{\mathbb{N}} \to [0, \infty]$ denote counting measure, so that $L^1(\lambda) = \ell^1$. Let $T : \ell^1 \to \ell^2$ denote the canonical inclusion map. By the Schur property of ℓ^1, the map T is completely continuous. But, the set $\{T(\chi_A) : A \in \Sigma,\ \lambda(A) < \infty\}$ contains all unit basis vectors of ℓ^2 and so cannot be relatively compact. $\square$

2.5 Convexity and concavity properties of linear operators

The aim of this section is to give formal definitions of q-convexity and q-concavity properties of linear operators between quasi-Banach lattices, and to collect together some basic facts on such operators.

Let Z be either a Banach lattice or a q-B.f.s. Given $0 < q < \infty$, defining q-convex and q-concave operators requires us to consider elements of the form $\big(\sum_{j=1}^{n} |z_j|^q\big)^{1/q} \in Z$ for any $z_1, \ldots, z_n \in Z$ with $n \in \mathbb{N}$. For Z a q-B.f.s. this poses no problem because elements of Z are $\mathbb{C}$-valued functions. On the other hand, the case when Z is a complex Banach lattice requires the Krivine calculus applied to its real part $Z_{\mathbb{R}}$. Given $n \in \mathbb{N}$, let $\mathcal{H}_n$ denote the space of all $\mathbb{R}$-valued continuous functions f on $\mathbb{R}^n$ which are homogeneous of degree 1, that is, $f(\lambda x) = \lambda f(x)$ for all $x \in \mathbb{R}^n$ and $\lambda \geq 0$. Then $\mathcal{H}_n$ is a real vector lattice with respect to the pointwise order. The Krivine calculus which we use is the following one, [99, Theorem 1.d.1]; see also [41, Ch. 16] and [92].

Lemma 2.45. *Let $Z_{\mathbb{R}}$ be a real Banach lattice, $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in Z_{\mathbb{R}}$. Then there is a unique linear map $\tau_n : \mathcal{H}_n \to Z_{\mathbb{R}}$ such that:*

(i) *For each $j = 1, \dots, n$, the map τ_n sends the canonical projection on $\mathbb{R}^n$ given by $(t_1, \dots, t_n) \mapsto t_j$ to the vector $x_j \in Z_{\mathbb{R}}$, and*

(ii) *τ_n preserves the lattice operations.*

Note that the linear map τ_n in Lemma 2.45 depends on both $n \in \mathbb{N}$ and $x_1, \dots, x_n$, whereas the notation does not show the dependence on the latter.

Now, fix $0 < q < \infty$ and let $n \in \mathbb{N}$ and $z_1, \dots, z_n$ be vectors in a complex Banach lattice. Since $|z_1|, \dots, |z_n|$ belong to the real part $Z_{\mathbb{R}}$ of Z, we can apply Lemma 2.45 to $x_j := |z_j|$, for $j = 1, \dots, n$, to define the element $\big(\sum_{i=1}^n |z_j|^q\big)^{1/q} \in Z_{\mathbb{R}} \subseteq Z$ as follows. The function $f : (t_1, \dots, t_n) \mapsto \big(\sum_{j=1}^n |t_j|^q\big)^{1/q}$ on $\mathbb{R}^n$ belongs to $\mathcal{H}_n$. So Lemma 2.45 allows us to define

$$\Big(\sum_{j=1}^n |z_j|^q\Big)^{1/q} := \tau_n(f) \in Z_{\mathbb{R}} \subseteq Z.$$

We do not consider the class of all (complex) quasi-Banach lattices here because it seems that the Krivine calculus does not extend to it, while such a class would be an ideal one for unifying Banach lattices and q-B.f.s.' instead of saying at each stage that Z is either a Banach lattice or a q-B.f.s. However, for our purposes this suffices.

Definition 2.46. Let W be a quasi-Banach space and let Z be either a Banach lattice or a q-B.f.s. Let $0 < q < \infty$.

(i) A linear operator $T : W \to Z$ is said to be *q-convex* if there exists a constant $c > 0$ such that

$$\Big\| \Big(\sum_{j=1}^n |T(w_j)|^q\Big)^{1/q} \Big\|_Z \le c \Big(\sum_{j=1}^n \|w_j\|_W^q\Big)^{1/q}, \qquad w_1, \dots, w_n \in W, \quad n \in \mathbb{N}. \tag{2.103}$$

The smallest constant c satisfying (2.103) is called the *q-convexity constant* of T and is denoted by $\mathbf{M}^{(q)}[T]$.

(ii) A linear operator $S : Z \to W$ is said to be *q-concave* if there exists a constant $c > 0$ such that

$$\Big(\sum_{j=1}^n \|S(z_j)\|_W^q\Big)^{1/q} \le c \Big\| \Big(\sum_{j=1}^n |z_j|^q\Big)^{1/q} \Big\|_Z, \qquad z_1, \dots, z_n \in Z, \quad n \in \mathbb{N}. \tag{2.104}$$

The smallest constant $c > 0$ satisfying (2.104) is called the *q-concavity constant* of S and is denoted by $\mathbf{M}_{(q)}[S]$.

Letting $n := 1$ in (2.103) and (2.104) shows that such operators T and S are necessarily continuous and that

$$\|T\| \le \mathbf{M}^{(q)}[T] < \infty \qquad \text{and} \qquad \|S\| \le \mathbf{M}_{(q)}[S] < \infty, \tag{2.105}$$

respectively.

Let $0 < q < \infty$ and Z be either a Banach lattice or a q-B.f.s. We say that Z is *q-convex* (resp. *q-concave*) if the identity operator id_{Z} on Z is q-convex (resp. q-concave). In this case we write

$$\mathbf{M}^{(q)}[Z] := \mathbf{M}^{(q)}[\mathrm{id}_{\mathrm{Z}}] \qquad \Big(\text{resp.} \qquad \mathbf{M}_{(q)}[Z] := \mathbf{M}_{(q)}[\mathrm{id}_{\mathrm{Z}}]\ \Big) \tag{2.106}$$

and call $\mathbf{M}^{(q)}[Z]$ (resp. $\mathbf{M}_{(q)}[Z]$) the *q-convexity constant* (resp. the *q-concavity constant*) of Z. Since $\|\mathrm{id}_Z\| = 1$, it follows from (2.105), with $T := \mathrm{id}_Z$ and $S := \mathrm{id}_Z$, that

$$\mathbf{M}^{(q)}[Z] \geq 1 \qquad \text{and} \qquad \mathbf{M}_{(q)}[Z] \geq 1, \tag{2.107}$$

respectively.

Note that we have already defined q-convexity of a q-B.f.s. immediately prior to Proposition 2.23.

Notation 2.47. Under the assumption of Definition 2.46, let $\mathcal{K}^{(q)}(W, Z)$ (resp. $\mathcal{K}_{(q)}(Z, W)$) denote the space of all q-convex (resp. q-concave) operators from W into Z (resp. Z into W).

Remark 2.48. Consider the case when $W_{\mathbb{R}}$ is a real quasi-Banach space and $Z_{\mathbb{R}}$ is either a real Banach lattice or a real q-B.f.s. Let $0 < q < \infty$.

(i) The q-convexity (resp. q-concavity) of a real linear operator $T_{\mathbb{R}} : W_{\mathbb{R}} \to Z_{\mathbb{R}}$ (resp. $S_{\mathbb{R}} : Z_{\mathbb{R}} \to W_{\mathbb{R}}$) can be defined as in Definition 2.46(i) by replacing W by $W_{\mathbb{R}}$, Z by $Z_{\mathbb{R}}$ and T by $T_{\mathbb{R}}$ (resp. S by $S_{\mathbb{R}}$).

(ii) The q-convexity and q-concavity of $Z_{\mathbb{R}}$ can also be defined similar to the complex case. Then, of course, the q-convexity and q-concavity constants are defined by

$$\mathbf{M}^{(q)}[Z_{\mathbb{R}}] := \mathbf{M}^{(q)}[\mathrm{id}_{\mathrm{Z}_{\mathbb{R}}}] \qquad \text{and} \quad \mathbf{M}_{(q)}[Z_{\mathbb{R}}] := \mathbf{M}_{(q)}[\mathrm{id}_{\mathrm{Z}_{\mathbb{R}}}]$$

via the identity $\mathrm{id}_{\mathrm{Z}_{\mathbb{R}}}$ on $Z_{\mathbb{R}}$, respectively.

(iii) These definitions in parts (i) and (ii) coincide with those in [99, Definition 1.d.3] when $W_{\mathbb{R}}$ is a real quasi-Banach space, $Z_{\mathbb{R}}$ is a real Banach lattice and $1 \leq q < \infty$. □

When Z is either a complex Banach lattice or a q-B.f.s, convexity and concavity properties of Z are equivalent to those of its real part $Z_{\mathbb{R}}$. The precise statement is given by the following result.

Lemma 2.49. *Let $0 < q < \infty$ and Z be either a complex Banach lattice or a q-B.f.s.*

(i) *Z is q-convex if and only if its real part $Z_{\mathbb{R}}$ is q-convex, in which case $\mathbf{M}^{(q)}[Z_{\mathbb{R}}] = \mathbf{M}^{(q)}[Z]$.*

(ii) *Z is q-concave if and only if its real part $Z_{\mathbb{R}}$ is q-concave, in which case $\mathbf{M}_{(q)}[Z_{\mathbb{R}}] = \mathbf{M}_{(q)}[Z]$.*

Proof. (i) Assume that Z is q-convex. Fix $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in Z_{\mathbb{R}} \subseteq Z$. Then

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{n}|x_j|^q\Big)^{1/q}\Big\|_{Z_{\mathbb{R}}} &= \Big\|\Big(\sum_{j=1}^{n}|x_j|^q\Big)^{1/q}\Big\|_{Z} \\ &\leq \Big(\mathbf{M}^{(q)}[Z]\Big)\Big(\sum_{j=1}^{n}\|x_j\|_Z^q\Big)^{1/q} = \Big(\mathbf{M}^{(q)}[Z]\Big)\Big(\sum_{j=1}^{n}\|x_j\|_{Z_{\mathbb{R}}}^q\Big)^{1/q},\end{aligned}$$

which implies that $Z_{\mathbb{R}}$ is also q-convex and $\mathbf{M}^{(q)}[Z_{\mathbb{R}}] \leq \mathbf{M}^{(q)}[Z]$.

Suppose now that $Z_{\mathbb{R}}$ is q-convex. Fix $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in Z$. Since $|z_1|, \ldots, |z_n| \in Z_{\mathbb{R}}$, it follows that

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}\Big\|_{Z} &= \Big\|\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}\Big\|_{Z_{\mathbb{R}}} \\ &\leq \Big(\mathbf{M}^{(q)}[Z_{\mathbb{R}}]\Big)\Big\|\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}\Big\|_{Z_{\mathbb{R}}} = \Big(\mathbf{M}^{(q)}[Z_{\mathbb{R}}]\Big)\Big\|\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}\Big\|_{Z}.\end{aligned}$$

This shows that Z is also q-convex and $\mathbf{M}^{(q)}[Z] \leq \mathbf{M}^{(q)}[Z_{\mathbb{R}}]$.

(ii) Analogous to part (i). □

Every Banach-lattice-valued linear map defined on a quasi-Banach space is 1-convex.

Lemma 2.50. *Let W be a quasi-Banach space and E be a Banach lattice. Then $\mathcal{L}(W,E) = \mathcal{K}^{(1)}(W,E)$. Moreover, $\|T\| = \mathbf{M}^{(1)}[T]$ for every $T \in \mathcal{L}(W,E)$.*

Proof. Let $T \in \mathcal{L}(W,E)$. Given $n \in \mathbb{N}$ and $w_1, \ldots, w_n \in W$ we have, from the triangle inequality of the lattice norm $\|\cdot\|_E$ on E, that

$$\begin{aligned}\Big\|\sum_{j=1}^{n}|T(w_j)|\,\Big\|_E &\leq \sum_{j=1}^{n}\big\|\,|T(w_j)|\,\big\|_E \\ &= \sum_{j=1}^{n}\big\|T(w_j)\big\|_E \leq \sum_{j=1}^{n}\Big(\|T\|\cdot\|w_j\|_W\Big) = \|T\|\Big(\sum_{j=1}^{n}\|w_j\|_W\Big).\end{aligned}$$

So, $T \in \mathcal{K}^{(1)}(W,Z)$ and $\|T\| \geq \mathbf{M}^{(1)}[T]$. Consequently $\mathcal{L}(W,E) \subseteq \mathcal{K}^{(1)}(W,E)$.

On the other hand, every operator $T \in \mathcal{K}^{(1)}(W,E)$ is necessarily continuous and satisfies $\|T\| \leq \mathbf{M}^{(1)}[T]$ via (2.105) with $q := 1$ and $Z := E$. This establishes the lemma. □

To present various basic facts for convex and concave operators, we require some preparation. Given $0 < q < \infty$, let

$$\kappa_q := \begin{cases} 2^{(1/q)-1} & \text{if} \quad 0 < q < 1, \\ 1 & \text{if} \quad 1 \leq q < \infty. \end{cases} \tag{2.108}$$

Then

$$\Big(\sum_{j=1}^{n}|a_j+b_j|^q\Big)^{1/q} \le \kappa_q\bigg(\Big(\sum_{j=1}^{n}|a_j|^q\Big)^{1/q}+\Big(\sum_{j=1}^{n}|b_j|^q\Big)^{1/q}\bigg) \tag{2.109}$$

for all $a_1,\dots,a_n,b_1,\dots,b_n\in\mathbb{R}$ with $n\in\mathbb{N}$; see [83, p. 20], for example.

Lemma 2.51. *Let Z be a Banach lattice or a q-B.f.s. In the latter case, let K denote a constant satisfying the "triangle inequality"* (Q3) *for Z. Fix $n\in\mathbb{N}$ and $z_1,\dots,z_n\in Z$. Assume that $0<q<\infty$.*

(i) *Given positive numbers $a_1,\dots,a_n$ with $\sum_{j=1}^{n}a_j=1$ and $0<r<q<\infty$, we have*

$$\Big(\sum_{j=1}^{n}a_j|z_j|^r\Big)^{1/r} \le \Big(\sum_{j=1}^{n}a_j|z_j|^q\Big)^{1/q}. \tag{2.110}$$

(ii) *If $b_1,\dots,b_n$ are positive numbers, then*

$$\Big(\sum_{j=1}^{n}|b_jz_j|^q\Big)^{1/q} = \Big(\sum_{j=1}^{n}b_j^q|z_j|^q\Big)^{1/q}. \tag{2.111}$$

(iii) *Given a positive number b, it follows that*

$$\Big(\sum_{j=1}^{n}|bz_j|^q\Big)^{1/q} = b\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}. \tag{2.112}$$

(iv) *Let $\zeta_1,\dots,\zeta_n$ be elements of Z such that $|z_j|\le|\zeta_j|$ for each $j=1,\dots,n$. Then*

$$\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q} \le \Big(\sum_{j=1}^{n}|\zeta_j|^q\Big)^{1/q}. \tag{2.113}$$

(v) *Let $\zeta_1,\dots,\zeta_n\in Z$. Then, with κ_q given by* (2.108), *we have*

$$\Big(\sum_{j=1}^{n}|z_j+\zeta_j|^q\Big)^{1/q} \le \kappa_q\bigg(\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}+\Big(\sum_{j=1}^{n}|\zeta_j|^q\Big)^{1/q}\bigg) \tag{2.114}$$

and also

$$\bigg\|\Big(\sum_{j=1}^{n}|z_j+\zeta_j|^q\Big)^{1/q}\bigg\|_Z \le \kappa_qK\bigg(\bigg\|\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}\bigg\|_Z+\bigg\|\Big(\sum_{j=1}^{n}|\zeta_j|^q\Big)^{1/q}\bigg\|_Z\bigg), \tag{2.115}$$

with the understanding that $K=1$ if Z is a Banach lattice.

(vi) *Let* $r > 0$ *satisfy* $K = 2^{1/r}$. *Then we have*

$$\Big\| \Big(\sum_{j=1}^{n} |z_j|^q \Big)^{1/q} \Big\|_Z \le 4^{1/r} n^{(1/q)+(1/r)} \sum_{j=1}^{n} \|z_j\|_Z, \tag{2.116}$$

with the understanding that $(1/r) = 0$ *if* $K = 1$ (*i.e.,* Z *is a Banach lattice*).

Proof. (i) First assume that Z is a Banach lattice with real part $Z_{\mathbb{R}}$. Define functions $f_r : \mathbb{R}^n \to \mathbb{R}$ and $f_q : \mathbb{R}^n \to \mathbb{R}$ by

$$f_r(t_1, \ldots, t_n) := \Big(\sum_{j=1}^{n} a_j |t_j|^r \Big)^{1/r} \quad \text{and} \quad f_q(t_1, \ldots, t_n) := \Big(\sum_{j=1}^{n} a_j |t_j|^q \Big)^{1/q}$$

for $(t_1, \ldots, t_n) \in \mathbb{R}^n$. Then $f_r, f_q \in \mathcal{H}_n$. The observation that $\sum_{i=1}^n a_i = 1$, together with $0 < (r/q) < 1$ and

$$1 = (r/q) + 1 - (r/q) = (r/q) + (q-r)/q,$$

enable us to apply Hölder's inequalities to obtain

$$\begin{aligned} f_r(t_1, \ldots, t_n) &= \Big(\sum_{j=1}^{n} a_j |t_j|^r \Big)^{1/r} = \Big(\sum_{j=1}^{n} \big(a_j^{r/q} |t_j|^r \big) a_j^{1-(r/q)} \Big)^{1/r} \\ &\le \Big(\sum_{j=1}^{n} \big(a_j^{r/q} |t_j|^r \big)^{q/r} \Big)^{r/(qr)} \Big(\sum_{j=1}^{n} \big(a_j^{1-(r/q)} \big)^{q/(q-r)} \Big)^{(q-r)/(qr)} \\ &= \Big(\sum_{j=1}^{n} a_j |t_j|^q \Big)^{1/q} = f_q(t_1, \ldots, t_n) \end{aligned}$$

for all $(t_1, \ldots, t_n) \in \mathbb{R}^n$. In other words, $f_r \le f_q$ pointwise on $\mathbb{R}^n$. So, Lemma 2.45 applied to $x_j := |z_j| \in Z_{\mathbb{R}}$, for $j = 1, \ldots, n$, gives $\tau_n(f_r) \le \tau_n(f_q)$. But,

$$\tau_n(f_r) = \Big(\sum_{j=1}^{n} a_j |z_j|^r \Big)^{1/r} \quad \text{and} \quad \tau_n(f_q) = \Big(\sum_{j=1}^{n} a_j |z_j|^q \Big)^{1/q}$$

and so (2.110) is clear.

Assume now that Z is a q-B.f.s. Since $z_1, \ldots, z_n$ are scalar functions, the fact from above that $\big(\sum_{j=1}^n a_j |t_j|^r \big)^{1/r} \le \big(\sum_{j=1}^n a_j |t_j|^q \big)^{1/q}$ for $(t_1, \ldots, t_n) \in \mathbb{R}^n$ can be directly applied pointwise to yield (2.110).

(ii) This is obvious for the case when Z is a q-B.f.s. So, assume that Z is a complex Banach lattice. Define $g_1 : \mathbb{R}^n \to \mathbb{R}$ and $g_2 : \mathbb{R}^n \to \mathbb{R}$ by

$$g_1(t_1, \ldots, t_n) := \Big(\sum_{j=1}^{n} |b_j t_j|^q \Big)^{1/q} \quad \text{and} \quad g_2(t_1, \ldots, t_n) := \Big(\sum_{j=1}^{n} b_j^q\, |t_j|^q \Big)^{1/q}$$

for $(t_1,\dots,t_n)\in\mathbb{R}^n$. Then $g_1,g_2\in\mathcal{H}_n$. Hence, via the linear operator τ_n given in Lemma 2.45 applied to the elements $x_j:=|z_j|$, for $j=1,\dots,n$, in the real Banach lattice $Z_{\mathbb{R}}$, the elements $\Big(\sum_{j=1}^n \big|b_j\,|z_j|\,\big|^q\Big)^{1/q}$ and $\Big(\sum_{j=1}^n b_j^q\,\big|\,|z_j|\,\big|^q\Big)^{1/q}$ of $Z_{\mathbb{R}}$ are defined by

$$\Big(\sum_{j=1}^n \big|b_j|z_j|\,\big|^q\Big)^{1/q} := \tau_n(g_1) \qquad \text{and} \qquad \Big(\sum_{j=1}^n b_j^q\big|\,|z_j|\,\big|^q\Big)^{1/q} := \tau_n(g_2).$$

But, $g_1=g_2$ on $\mathbb{R}^n$. Since $\big|b_j\,z_j\,\big|=\big|b_j|z_j|\,\big|$ and $\big|\,|z_j|\,\big|=|z_j|$ for each $j=1,\dots,n$, we obtain (2.111).

(iii) Again this is clear when Z is a q-B.f.s. So, assume that Z is a complex Banach lattice. By part (ii) with $b_j:=b$ for $j=1,\dots,n$, we have

$$\Big(\sum_{j=1}^n \big|b\,z_j\big|^q\Big)^{1/q} = \Big(\sum_{j=1}^n b^q\big|z_j\big|^q\Big)^{1/q} = \Big(b^q\sum_{j=1}^n \big|z_j\big|^q\Big)^{1/q}.$$

That the last term coincides with $b\Big(\sum_{j=1}^n \big|z_j\big|^q\Big)^{1/q}$ can again be proved by Lemma 2.45, once we observe that

$$\Big(b^q\sum_{j=1}^n \big|t_j\big|^q\Big)^{1/q} = b\Big(\sum_{j=1}^n \big|t_j\big|^q\Big)^{1/q}, \qquad (t_1,\dots,t_n)\in\mathbb{R}^n.$$

(iv) This is obvious when Z is a q-B.f.s. So, assume that Z is a complex Banach lattice. Define two functions $h_1,h_2:\mathbb{R}^{2n}\to\mathbb{R}$ by

$$h_1(t_1,\dots,t_n,t_{n+1},\dots,t_{2n}) := \Big(\sum_{j=1}^n |t_j|^q\Big)^{1/q}$$

and

$$h_2(t_1,\dots,t_n,t_{n+1},\dots,t_{2n}) := \Big(\sum_{j=1}^n \Big(|t_j|\vee|t_{n+j}|\Big)^q\Big)^{1/q}$$

for $(t_1,\dots,t_n,t_{n+1},\dots,t_{2n})\in\mathbb{R}^{2n}$. Then both h_1 and h_2 are continuous and homogeneous of degree 1, that is, $h_1,h_2\in\mathcal{H}_{2n}$. We now apply Lemma 2.45 (with $2n$ in place of n) to the vectors $|z_1|,\dots,|z_n|,|\zeta_1|,\dots,|\zeta_n|\in Z_{\mathbb{R}}$. Then, since $h_1\le h_2$ pointwise on $\mathbb{R}^{2n}$, it follows that $\tau_{2n}(h_1)\le\tau_{2n}(h_2)$ in $Z_{\mathbb{R}}$ and hence, in Z. Since $|z_j|\vee|\zeta_j|=|\zeta_j|$ for $j=1,\dots,n$, the inequality (2.113) follows.

(v) Since $|z_j+\zeta_j|\le|z_j|+|\zeta_j|$ for $j=1,\dots,n$, we apply (2.113) to obtain

$$\Big(\sum_{j=1}^n \big|z_j+\zeta_j\big|^q\Big)^{1/q} \le \Big(\sum_{j=1}^n \Big(|z_j|+|\zeta_j|\Big)^q\Big)^{1/q}.$$

Define two functions $h_3 : \mathbb{R}^{2n} \to \mathbb{R}$ and $h_4 : \mathbb{R}^{2n} \to \mathbb{R}$ by

$$h_3(t_1, \dots, t_n, t_{n+1}, \dots, t_{2n}) := \Big(\sum_{j=1}^{n} \big|t_j + t_{n+j}\big|^q \Big)^{1/q}$$

and

$$h_4(t_1, \dots, t_n, t_{n+1}, \dots, t_{2n}) := \kappa_q \Big(\Big(\sum_{j=1}^{n} \big|t_j\big|^q \Big)^{1/q} + \Big(\sum_{j=1}^{n} \big|t_{n+j}\big|^q \Big)^{1/q} \Big)$$

for all $(t_1, \dots, t_n, t_{n+1}, \dots, t_{2n}) \in \mathbb{R}^n$. Then $h_3, h_4 \in \mathcal{H}_{2n}$ and (2.109) implies that $h_3 \le h_4$ pointwise on $\mathbb{R}^{2n}$. So, if Z is a q-B.f.s., then we can apply this inequality pointwise to the functions $|z_1|, \dots, |z_n|, |\zeta_1|, \dots, |\zeta_n| \in Z$ to deduce (2.114). For the case when Z is a complex Banach lattice, Lemma 2.45 (with $2n$ in place of n) applied to the elements $|z_1|, \dots, |z_n|, |\zeta_1|, \dots, |\zeta_n|$ of the real Banach lattice $Z_{\mathbb{R}}$, yields that $\tau_{2n}(h_3) \le \tau_{2n}(h_4)$, that is, (2.114) again holds. Now (2.115) is clear from (Q3) and (2.114) if Z is a q-B.f.s. and from (2.114) if Z is a complex Banach lattice.

(vi) Define two functions $h_5 : \mathbb{R}^n \to \mathbb{R}$ and $h_6 : \mathbb{R}^n \to \mathbb{R}$ by

$$h_5(t_1, \dots, t_n) := \Big(\sum_{j=1}^{n} \big|t_j\big|^q \Big)^{1/q} \quad \text{and} \quad h_6(t_1, \dots, t_n) := n^{1/q} \sum_{j=1}^{n} |t_j|$$

for $(t_1, \dots, t_n) \in \mathbb{R}^n$. Then, h_5, $h_6 \in \mathcal{H}_n$. Observe that $h_5 \le h_6$ because

$$\begin{aligned} h_5(t_1, \dots, t_n) = \Big(\sum_{j=1}^{n} \big|t_j\big|^q \Big)^{1/q} &\le \Big(\sum_{j=1}^{n} \max_{1 \le k \le n} \big\{ \big|t_k\big|^q \big\} \Big)^{1/q} \\ &= n^{1/q} \max_{1 \le k \le n} |t_k| \ \le \ n^{1/q} \sum_{k=1}^{n} |t_k| \ = \ h_6(t_1, \dots, t_n) \end{aligned}$$

for $(t_1, \dots, t_n) \in \mathbb{R}^n$. Suppose that Z is a Banach lattice. Then Lemma 2.45 applied to the elements $|z_1|, \dots, |z_n| \in Z_{\mathbb{R}} \subseteq Z$ yields that $\tau_n(h_5) \le \tau_n(h_6)$ and, as a consequence, we have

$$\begin{aligned} \Big\| \Big(\sum_{j=1}^{n} |z_j|^q \Big)^{1/q} \Big\|_Z &= \big\| \tau_n(h_5) \big\|_Z \le \big\| \tau_n(h_6) \big\|_Z \\ &= \Big\| n^{1/q} \Big(\sum_{j=1}^{n} |z_j| \Big) \Big\|_Z = n^{1/q} \Big\| \sum_{j=1}^{n} |z_j| \Big\|_Z . \end{aligned}$$

This is precisely (2.116) when $(1/r)=0$. Now suppose that Z is a q-B.f.s. Let $|\!|\!|\cdot|\!|\!|$ denote an F-norm satisfying (2.5). Then (F4) implies that

$$\begin{aligned}\Big\|\sum_{j=1}^{n}|z_j|\Big\|_Z &\le 4^{1/r}\Big|\!\Big|\!\Big|\sum_{j=1}^{n}|z_j|\Big|\!\Big|\!\Big|^{1/r} \le 4^{1/r}\Big(\sum_{j=1}^{n}|\!|\!|\,|z_j|\,|\!|\!|\Big)^{1/r}\\ &\le 4^{1/r}\Big(n\max_{1\le j\le n}|\!|\!|\,|z_j|\,|\!|\!|\Big)^{1/r} \le 4^{1/r}n^{1/r}\sum_{j=1}^{n}|\!|\!|\,|z_j|\,|\!|\!|^{1/r}\\ &\le 4^{1/r}n^{1/r}\sum_{j=1}^{n}\big\|\,|z_j|\,\big\|_Z = 4^{1/r}n^{1/r}\sum_{j=1}^{n}\|z_j\|_Z.\end{aligned}$$

Now (2.116) is a consequence of this inequality and the pointwise inequality $\big(\sum_{j=1}^{n}|z_j|^q\big)^{1/q}\le n^{1/q}\sum_{j=1}^{n}|z_j|$ (which follows from $h_5\le h_6$). □

In the notation of Definition 2.46(i), assume that $T:W\to Z$ satisfies the inequalities (2.103) only for those elements coming from some dense subspace of W. A natural question is whether or not we can then deduce that T is q-convex. The answer is affirmative as we now show. The conclusion is the same for the corresponding question regarding a q-concave operator.

Lemma 2.52. *Let W be a quasi-Banach space and let Z be either a Banach lattice or a q-B.f.s. Suppose that $0<q<\infty$.*

(i) *Let $T:W\to Z$ be a continuous linear operator. If there exist a constant $C>0$ and dense subspace W_0 of W such that*

$$\Big\|\Big(\sum_{j=1}^{n}|T(u_j)|^q\Big)^{1/q}\Big\|_Z \le C\Big(\sum_{j=1}^{n}\|u_j\|_W^q\Big)^{1/q} \tag{2.117}$$

for all $n\in\mathbb{N}$ and $u_1,\dots,u_n\in W_0$, then T is q-convex.

(ii) *Let $S:Z\to W$ be a continuous linear operator. If there exist a constant $C>0$ and dense subspace Z_0 of Z such that*

$$\Big(\sum_{j=1}^{n}\|S(v_j)\|_W^q\Big)^{1/q} \le C\Big\|\sum_{j=1}^{n}|v_j|^q\Big\|_Z^{1/q} \tag{2.118}$$

for all $n\in\mathbb{N}$ and $v_1,\dots,v_n\in Z_0$, then S is q-concave.

Proof. Let K_W and K_Z be positive constants appearing in the "triangle inequalities" of W and Z, respectively (see (Q3)). Let $r>0$ satisfy $K_Z=2^{1/r}$. It is to be understood that $(1/r)=0$ if $K_Z=1$ (i.e., Z is a Banach lattice). Furthermore, let $\kappa_q>0$ be the constant as given in (2.108). Given $n\in\mathbb{N}$

and $x_1, \dots, x_n, y_1, \dots, y_n \in W$ we have, from (2.109) with $a_j := \|x_j\|_W$ and $b_j := \|y_j\|_W$ for $j = 1, \dots, n$, that

$$\Big(\sum_{j=1}^{n} \|x_j + y_j\|_W^q\Big)^{1/q} \le \Big(\sum_{j=1}^{n} K_W^q \big(\|x_j\|_W + \|y_j\|_W\big)^q\Big)^{1/q}$$
$$\le \kappa_q K_W \bigg(\Big(\sum_{j=1}^{n} \|x_j\|_W^q\Big)^{1/q} + \Big(\sum_{j=1}^{n} \|y_j\|_W^q\Big)^{1/q}\bigg). \qquad (2.119)$$

(i) Let $n \in \mathbb{N}$ and $w_1, \dots, w_n \in W$. Fix any $\varepsilon > 0$. Choose $u_1, \dots, u_n$ from the dense subspace W_0 of W such that

$$\|w_j - u_j\|_W \le \varepsilon / n^{1+(1/q)+(1/r)}, \qquad j = 1, \dots, n.$$

Apply Lemma 2.51(v) to obtain

$$\Big\|\Big(\sum_{j=1}^{n} |T(w_j)|^q\Big)^{1/q}\Big\|_Z = \Big\|\Big(\sum_{j=1}^{n} |T(w_j - u_j) + T(u_j)|^q\Big)^{1/q}\Big\|_Z$$
$$\le \kappa_q K_Z \bigg(\Big\|\Big(\sum_{j=1}^{n} |T(w_j - u_j)|^q\Big)^{1/q}\Big\|_Z + \Big\|\Big(\sum_{j=1}^{n} |T(u_j)|^q\Big)^{1/q}\Big\|_Z\bigg). \qquad (2.120)$$

Moreover, it follows from Lemma 2.51(vi) that

$$\Big\|\Big(\sum_{j=1}^{n} |T(w_j - u_j)|^q\Big)^{1/q}\Big\|_Z \le 4^{1/r} n^{(1/q)+(1/r)} \sum_{j=1}^{n} \big\|T(w_j - u_j)\big\|_Z$$
$$\le 4^{1/r} n^{(1/q)+(1/r)} \|T\| \sum_{j=1}^{n} \|w_j - u_j\|_W$$
$$\le \big(4^{1/r} n^{(1/q)+(1/r)} \|T\|\big) \big(n \cdot \varepsilon / n^{1+(1/q)+(1/r)}\big) = 4^{1/r} \varepsilon \|T\|. \qquad (2.121)$$

On the other hand, from (2.119) (with $x_j := (u_j - w_j)$ and $y_j := w_j$ for $j = 1, \dots, n$) and (2.117), we have that

$$\Big\|\Big(\sum_{j=1}^{n} |T(u_j)|^q\Big)^{1/q}\Big\|_Z \le C \Big(\sum_{j=1}^{n} \|u_j\|_W^q\Big)^{1/q} = C\Big(\sum_{j=1}^{n} \|(u_j - w_j) + w_j\|_W^q\Big)^{1/q}$$
$$\le C\kappa_q K_W \bigg(\Big(\sum_{j=1}^{n} \|u_j - w_j\|_W^q\Big)^{1/q} + \Big(\sum_{j=1}^{n} \|w_j\|_W^q\Big)^{1/q}\bigg)$$
$$\le C\kappa_q K_W \bigg(\Big(n \cdot \big(\varepsilon / n^{1+(1/q)+(1/r)}\big)^q\Big)^{1/q} + \Big(\sum_{j=1}^{n} \|w_j\|_W^q\Big)^{1/q}\bigg)$$
$$\le C\kappa_q K_W \varepsilon + C\kappa_q K_W \Big(\sum_{j=1}^{n} \|w_j\|_W^q\Big)^{1/q}. \qquad (2.122)$$

Now (2.120), (2.121) and (2.122) yield that

$$\begin{aligned}
&\Big\|\Big(\sum_{j=1}^{n}|T(w_j)|^q\Big)^{1/q}\Big\|_Z \\
&\quad\le \kappa_q K_Z\bigg(\Big\|\Big(\sum_{j=1}^{n}|T(w_j-u_j)|^q\Big)^{1/q}\Big\|_Z+\Big\|\Big(\sum_{j=1}^{n}|T(u_j)|^q\Big)^{1/q}\Big\|_Z\bigg) \\
&\quad\le \kappa_q K_Z\bigg(4^{1/r}\|T\|\varepsilon+C\kappa_q K_W\varepsilon+C\kappa_q K_W\Big(\sum_{j=1}^{n}\|w_j\|_W^q\Big)^{1/q}\bigg) \\
&\quad= \kappa_q K_Z\Big(4^{1/r}\|T\|+C\kappa_q K_W\Big)\varepsilon \;+\; C\kappa_q^2 K_W K_Z\Big(\sum_{j=1}^{n}\|w_j\|_W^q\Big)^{1/q}.
\end{aligned}$$

Since ε is arbitrary, this implies that

$$\Big\|\Big(\sum_{j=1}^{n}|T(w_j)|^q\Big)^{1/q}\Big\|_Z \;\le\; C\kappa_q^2 K_W K_Z\Big(\sum_{j=1}^{n}\|w_j\|_W^q\Big)^{1/q},$$

which implies that T is q-convex.

(ii) The proof is similar to that of part (i). Given are a number $n\in\mathbb{N}$ and vectors $z_1,\dots,z_n\in Z$. Let $\varepsilon>0$ be fixed. The dense subspace Z_0 contains elements $v_1,\dots,v_n$ such that

$$\|z_j-v_j\|_Z\le \varepsilon/n^{1+(1/q)+(1/r)},\qquad j=1,\dots,n. \tag{2.123}$$

Then (2.119), with $x_j:=S(z_j-v_j)$ and $y_j:=S(v_j)$ for $j=1,\dots,n$, implies that

$$\begin{aligned}
\Big(\sum_{j=1}^{n}\|S(z_j)\|_W^q\Big)^{1/q} &= \Big(\sum_{j=1}^{n}\|S(z_j-v_j)+S(v_j)\|_W^q\Big)^{1/q} \\
\le \kappa_q K_W\bigg(\Big(\sum_{j=1}^{n}\|S(z_j-v_j)\|_W^q\Big)^{1/q} &+ \Big(\sum_{j=1}^{n}\|S(v_j)\|_W^q\Big)^{1/q}\bigg).
\end{aligned} \tag{2.124}$$

Now we have

$$\begin{aligned}
\Big(\sum_{j=1}^{n}\|S(z_j-v_j)\|_W^q\Big)^{1/q} &\le \Big(\sum_{j=1}^{n}\|S\|^q\cdot\|z_j-v_j\|_Z^q\Big)^{1/q} \\
&\le\; \|S\|\left(n^{1/q}\cdot\varepsilon/n^{1+(1/q)+(1/r)}\right)\;\le\;\|S\|\varepsilon.
\end{aligned} \tag{2.125}$$

It follows from (2.118) and Lemma 2.51(v) that

$$\begin{aligned}
\Big(\sum_{j=1}^{n}\|S(v_j)\|_W^q\Big)^{1/q} &\le C\Big\|\Big(\sum_{j=1}^{n}|v_j|^q\Big)^{1/q}\Big\|_Z = C\Big\|\Big(\sum_{j=1}^{n}|(v_j-z_j)+z_j|^q\Big)^{1/q}\Big\|_Z \\
\le C\kappa_q K_Z\bigg(\Big\|\Big(\sum_{j=1}^{n}|v_j-z_j|^q\Big)^{1/q}\Big\|_Z &+ \Big\|\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}\Big\|_Z\bigg).
\end{aligned} \tag{2.126}$$

Moreover, Lemma 2.51(vi) implies that

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{n}|v_j-z_j|^q\Big)^{1/q}\Big\|_Z &\leq 4^{1/r}n^{(1/q)+(1/r)}\sum_{j=1}^{n}\|v_j-z_j\|_Z\\ &\leq \big(4^{1/r}n^{(1/q)+(1/r)}\big)\big(n\cdot\varepsilon/n^{1+(1/q)+(1/r)}\big)=4^{1/r}\varepsilon.\end{aligned}$$

It then follows from (2.126) that

$$\Big(\sum_{j=1}^{n}\|S(v_j)\|_W^q\Big)^{1/q}\leq C\kappa_q K_Z\bigg(4^{1/r}\varepsilon+\Big\|\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}\Big\|_Z\bigg).$$

This inequality, together with (2.124) and (2.125), imply that

$$\begin{aligned}&\Big(\sum_{j=1}^{n}\|S(z_j)\|_W^q\Big)^{1/q}\\ &\quad\leq \kappa_q K_W\bigg(\Big(\sum_{j=1}^{n}\|S(z_j-v_j)\|_W^q\Big)^{1/q}+\Big(\sum_{j=1}^{n}\|S(v_j)\|_W^q\Big)^{1/q}\bigg)\\ &\quad\leq \kappa_q K_W\bigg(\|S\|\varepsilon+4^{1/r}C\kappa_q K_Z\,\varepsilon+C\kappa_q K_Z\Big\|\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}\Big\|_Z\bigg)\\ &\quad= \kappa_q K_W\Big(\|S\|+4^{1/r}C\kappa_q K_Z\Big)\varepsilon+C\kappa_q^2K_WK_Z\Big\|\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}\Big\|_Z.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we conclude that

$$\Big(\sum_{j=1}^{n}\|S(z_j)\|_W^q\Big)^{1/q}\leq C\kappa_q^2K_WK_Z\Big\|\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}\Big\|_Z.$$

In other words, S is q-concave. □

The following lemma provides a basic fact which will be needed later in this section.

Lemma 2.53. *Let W be a quasi-Banach space and let Z be either a Banach lattice or a q-B.f.s. Suppose that $0<q<\infty$.*

(i) *A linear operator $T:W\to Z$ is q-convex if and only if*

$$\sup\Big\|\Big(\sum_{j=1}^{n}a_j|T(w_j)|^q\Big)^{1/q}\Big\|_Z<\infty, \tag{2.127}$$

where the supremum is taken over all $n\in\mathbb{N}$, positive numbers $a_1,\dots,a_n$ with $\sum_{j=1}^{n}a_j=1$, and vectors $w_1,\dots,w_n\in W$ with $\|w_j\|_W=1$ for $j=1,\dots,n$. In this case, $\mathbf{M}^{(q)}[T]$ equals the supremum in (2.127).

(ii) *A non-zero linear operator $S : Z \to W$ is q-concave if and only if*

$$\inf \Big\| \Big(\sum_{j=1}^{n} a_j |z_j|^q \Big)^{1/q} \Big\|_Z > 0, \tag{2.128}$$

where the infimum is taken over all $n \in \mathbb{N}$, positive numbers $a_1, \dots, a_n$ with $\sum_{j=1}^{n} a_j = 1$ and vectors $z_j \in Z$ with $\|S(z_j)\|_W = 1$ for $j = 1, \dots, n$. In this case, the infimum in (2.128) equals $\big(\mathbf{M}_{(q)}[S]\big)^{-1}$.

Proof. (i) The definition of a q-convex operator yields that T is q-convex if and only if

$$\sup \Big\| \Big(\sum_{j=1}^{n} |T(v_j)|^q \Big)^{1/q} \Big\|_Z < \infty, \tag{2.129}$$

where the supremum is taken over all $n \in \mathbb{N}$ and $v_1, \dots, v_n \in W \setminus \{0\}$ with $\Big(\sum_{j=1}^{n} \|v_j\|_W^q \Big)^{1/q} = 1$; see also, for example, the proof of Proposition 1.d.5 in [99]. In this case, the supremum equals $\mathbf{M}^{(q)}[T]$. For such $v_1, \dots, v_n \in W \setminus \{0\}$, we have, with $w_j := v_j / \|v_j\|_W$ for $j = 1, \dots, n$, that

$$\Big(\sum_{j=1}^{n} |T(v_j)|^q \Big)^{1/q} = \Big(\sum_{j=1}^{n} \Big| \|v_j\|_W T(w_j) \Big|^q \Big)^{1/q} = \Big(\sum_{j=1}^{n} \|v_j\|_W^q |T(w_j)|^q \Big)^{1/q} \tag{2.130}$$

via Lemma 2.51(ii) with $b_j := \|v_j\|_W$ and $z_j := T(w_j)$ for $j = 1, \dots, n$. Since $\sum_{j=1}^{n} \|v_j\|_W^q = 1$, it follows from (2.130) that the supremum in (2.127) equals that in (2.129). This verifies statement (i).

(ii) We break up the proof into three steps.

Step 1. *The operator S is q-concave if and only if*

$$\inf \left(\Big\| \Big(\sum_{j=1}^{n} |x_j|^q \Big)^{1/q} \Big\|_Z \cdot \Big(\sum_{j=1}^{n} \|S(x_j)\|_W^q \Big)^{-1/q} \right) > 0, \tag{2.131}$$

where the infimum is taken over all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in Z \setminus S^{-1}(\{0\})$.

To verify this step observe, via the definition of a q-concave operator, that S is q-concave if and only if

$$\min \left\{ c > 0 : \frac{1}{c} \le \frac{\Big\| \Big(\sum_{j=1}^{n} |x_j|^q \Big)^{1/q} \Big\|_Z}{\Big(\sum_{j=1}^{n} \|S(x_j)\|_W^q \Big)^{1/q}} \text{ for all } x_1, \dots, x_n \in Z \setminus S^{-1}(\{0\}) \right\} > 0, \tag{2.132}$$

in which case $\mathbf{M}_{(q)}[S]$ equals the minimum. Let d_q denote the infimum in (2.131). First assume that S is q-concave. Since $\mathbf{M}_{(q)}[S]$ equals the minimum in (2.132), it is clear that

$$0 < \Big(\mathbf{M}_{(q)}[S] \Big)^{-1} \le d_q.$$

Conversely, assume that $d_q > 0$. Then

$$\Big(\sum_{j=1}^{n} \|S(x_j)\|_W^q\Big)^{1/q} \le \frac{1}{d_q}\Big\|\Big(\sum_{j=1}^{n} |x_j|^q\Big)^{1/q}\Big\|_Z$$

whenever $x_1, \ldots, x_n \in Z \setminus S^{-1}(\{0\})$ and $n \in \mathbb{N}$. Hence, we have established Step 1 together with the equality $d_q = \Big(\mathbf{M}_{(q)}[S]\Big)^{-1}$.

Step 2. *We have*

$$d_q = \inf \Big\|\Big(\sum_{j=1}^{n} |y_j|^q\Big)^{1/q}\Big\|_Z, \tag{2.133}$$

where the infimum is taken over all $n \in \mathbb{N}$ *and* $y_1, \ldots, y_n \in Z$ *such that*

$$\sum_{j=1}^{n} \|S(y_j)\|_W^q = 1.$$

Let d_q^* denote the right-hand side of (2.133). Clearly we have $d_q \le d_q^*$. To prove the reverse inequality, let $\varepsilon > 0$. Then there exist $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in Z \setminus S^{-1}(\{0\})$ such that

$$\Big\|\Big(\sum_{j=1}^{n} |x_j|^q\Big)^{1/q}\Big\|_Z \cdot \Big(\sum_{j=1}^{n} \|S(x_j)\|_W^q\Big)^{-1/q} < d_q + \varepsilon. \tag{2.134}$$

Let $a := \Big(\sum_{j=1}^{n} \|S(x_j)\|_W^q\Big)^{1/q}$ and $y_j := a^{-1}x_j$ for $j = 1, \ldots, n$. Then

$$\Big(\sum_{j=1}^{n} \|S(y_j)\|_W^q\Big)^{1/q} = a^{-1}\Big(\sum_{j=1}^{n} \|S(x_j)\|_W^q\Big)^{1/q} = 1.$$

Moreover, it follows from Lemma 2.51(iii), with $b := a^{-1}$, and (2.134) that

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{n} |y_j|^q\Big)^{1/q}\Big\|_Z &= \Big\|\Big(\sum_{j=1}^{n} |a^{-1}x_j|^q\Big)^{1/q}\Big\|_Z \\ &= \Big\|a^{-1}\Big(\sum_{j=1}^{n} |x_j|^q\Big)^{1/q}\Big\|_Z = a^{-1}\Big\|\Big(\sum_{j=1}^{n} |x_j|^q\Big)^{1/q}\Big\|_Z < d_q + \varepsilon.\end{aligned}$$

Therefore, $d_q^* \le d_q + \varepsilon$. Since ε is arbitrary, we have $d_q^* \le d_q$, which establishes Step 2.

Step 3. *With* d_q^* *denoting the infimum in the right-hand side of* (2.133) *it follows that*

$$d_q^* = \inf \Big\|\Big(\sum_{j=1}^{n} a_j|z_j|^q\Big)^{1/q}\Big\|_Z \tag{2.135}$$

where the infimum is taken over all $n \in \mathbb{N}$*, positive numbers* $a_1, \dots, a_n$ *with* $\sum_{j=1}^n a_j = 1$*, and* $z_1, \dots, z_n \in Z$ *with* $\|S(z_j)\|_W = 1$ *for* $j = 1, \dots, n$*.*

To verify this, let d_q^{**} denote the right-hand side of (2.135). We first show that $d_q^* \geq d_q^{**}$. Fix $n \in \mathbb{N}$ and take $y_1, \dots, y_n \in Z \setminus S^{-1}(\{0\})$ with $\sum_{j=1}^n \|S(y_j)\|_W^q = 1$. Let $a_j := \|S(y_j)\|_W^q$ and $z_j := \|S(y_j)\|_W^{-1} \cdot y_j$ for $j = 1, \dots, n$. Then $\sum_{j=1}^n a_j = 1$ and $\|S(z_j)\|_W = 1$ for $j = 1, \dots, n$. Apply Lemma 2.51(ii), with $(a_j)^{1/q}$ in place of b_j for $j = 1, \dots, n$ to deduce that

$$\Big\| \Big(\sum_{j=1}^n |y_j|^q \Big)^{1/q} \Big\|_Z = \Big\| \Big(\sum_{j=1}^n |(a_j)^{1/q} z_j|^q \Big)^{1/q} \Big\|_Z$$
$$= \Big\| \Big(\sum_{j=1}^n (a_j^{1/q})^q \, |z_j|^q \Big)^{1/q} \Big\|_Z = \Big\| \Big(\sum_{j=1}^n a_j \, |z_j|^q \Big)^{1/q} \Big\|_Z \geq d_q^{**}.$$

This implies that $d_q^* \geq d_q^{**}$.

We shall prove the reverse inequality in a similar fashion. Fix $n \in \mathbb{N}$ and take positive numbers $a_1, \dots, a_n$ with $\sum_{j=1}^n a_j = 1$ and vectors $x_1, \dots, x_n \in Z$ with $\|S(x_j)\|_W = 1$ for $j = 1, \dots, n$. Let

$$z_j := (a_j)^{1/q} x_j, \qquad j = 1, \dots, n.$$

Then $\sum_{j=1}^n \|S(z_j)\|_W^q = \sum_{j=1}^n a_j \, \|S(x_j)\|_W^q = 1$. Now, since $|\alpha z| = \alpha |z|$ for each $\alpha \geq 0$ and $z \in Z$, it follows that

$$\Big\| \Big(\sum_{j=1}^n a_j \, |x_j|^q \Big)^{1/q} \Big\|_Z = \Big\| \Big(\sum_{j=1}^n a_j \, |(a_j)^{-1/q} z_j|^q \Big)^{1/q} \Big\|_Z$$
$$= \Big\| \Big(\sum_{j=1}^n a_j \, (a_j^{-1/q})^q \, |z_j|^q \Big)^{1/q} \Big\|_Z = \Big\| \Big(\sum_{j=1}^n |z_j|^q \Big)^{1/q} \Big\|_Z \geq d_q^*,$$

which implies that $d_q^{**} \geq d_q^*$. Therefore, Step 3 holds.

Combining Steps 1, 2 and 3 establishes part (ii). □

Proposition 2.54. *Suppose that* W *is a quasi-Banach space and that* Z *is either a Banach lattice or a q-B.f.s.*

(i) *Given* $0 < q < \infty$*, the collection* $\mathcal{K}^{(q)}(W, Z)$ *is a linear subspace of* $\mathcal{L}(W, Z)$*.*

(ii) *Whenever* $0 < r < q < \infty$*, we have the inclusion*

$$\mathcal{K}^{(q)}(W, Z) \subseteq \mathcal{K}^{(r)}(W, Z),$$

that is, every q-convex operator is r-convex. Moreover,

$$\mathbf{M}^{(r)}[T] \leq \mathbf{M}^{(q)}[T], \qquad T \in \mathcal{K}^{(q)}(W, Z). \tag{2.136}$$

(iii) *Given* $0 < q < \infty$, *the collection* $\mathcal{K}_{(q)}(Z, W)$ *is a linear subspace of* $\mathcal{L}(Z, W)$.

(iv) *Whenever* $0 < r < q < \infty$, *we have the inclusion*

$$\mathcal{K}_{(r)}(Z, W) \subseteq \mathcal{K}_{(q)}(Z, W),$$

that is, every r*-concave operator is* q*-concave. Moreover,*

$$\mathbf{M}_{(q)}[S] \leq \mathbf{M}_{(r)}[S], \qquad S \in \mathcal{K}_{(r)}(Z, W). \tag{2.137}$$

Proof. (i) Let $T_1, T_2 \in \mathcal{K}^{(q)}(W, Z)$. To prove that the sum $(T_1 + T_2) : W \to Z$ is also q-concave, fix $n \in \mathbb{N}$ and $w_1, \dots, w_n \in W$. We have from (2.115) that

$$\begin{aligned}
&\Big\| \Big(\sum_{j=1}^{n} |(T_1 + T_2)(w_j)|^q \Big)^{1/q} \Big\|_Z \\
&\leq \kappa_q K \bigg(\Big\| \Big(\sum_{j=1}^{n} |T_1(w_j)|^q \Big)^{1/q} \Big\|_Z + \Big\| \Big(\sum_{j=1}^{n} |T_2(w_j)|^q \Big)^{1/q} \Big\|_Z \bigg) \\
&= \kappa_q K \Big(\mathbf{M}^{(q)}[T_1] + \mathbf{M}^{(q)}[T_2] \Big) \Big(\sum_{j=1}^{n} \|w_j\|_W^q \Big)^{1/q},
\end{aligned}$$

which implies that the sum $(T_1 + T_2)$ is q-concave. So $\mathcal{K}^{(q)}(W, Z)$ is closed under addition.

The fact that $\mathcal{K}^{(q)}(W, Z)$ is closed under scalar multiplication is easily verified. So, (i) holds.

(ii) Let $T \in \mathcal{K}^{(q)}(W, Z)$. Given $n \in \mathbb{N}$, positive numbers $a_1, \dots, a_n$ with $\sum_{j=1}^{n} a_j = 1$ and vectors $w_j \in W$ with $\|w_j\|_W = 1$ for $j = 1, \dots, n$, Lemma 2.51(i) implies that

$$\Big\| \Big(\sum_{j=1}^{n} a_j |T(w_j)|^r \Big)^{1/r} \Big\|_Z \leq \Big\| \Big(\sum_{j=1}^{n} a_j |T(w_j)|^q \Big)^{1/q} \Big\|_Z.$$

This and Lemma 2.53(i) with r in place of q implies that T is r-convex and (2.136) holds.

(iii) Let $S_1, S_2 \in \mathcal{K}_{(q)}(Z, W)$. We shall show that their sum $(S_1 + S_2) : Z \to W$ is also q-concave. To this end, fix $n \in \mathbb{N}$ and $z_1, \dots, z_n \in Z$. Let K denote a constant satisfying $\|w_1 + w_2\|_W \leq K\big(\|w_1\|_W + \|w_2\|_W\big)$ for all elements w_1, w_2 of the quasi-Banach space W. Then

$$\begin{aligned}
\Big(\sum_{j=1}^{n} \|(S_1 + S_2)(z_j)\|_W^q \Big)^{1/q} &\leq \bigg(\sum_{j=1}^{n} \Big(K \Big(\big\|S_1(z_j)\big\|_W + \big\|S_2(z_j)\big\|_W \Big) \Big)^q \bigg)^{1/q} \\
&= K \bigg(\sum_{j=1}^{n} \Big(\big\|S_1(z_j)\big\|_W + \big\|S_2(z_j)\big\|_W \Big)^q \bigg)^{1/q}.
\end{aligned}$$

This and (2.109) give

$$\begin{aligned}
&\left(\sum_{j=1}^{n}\left\|(S_1+S_2)(z_j)\right\|_W^q\right)^{1/q}\\
&\quad\le \kappa_q K\left(\left(\sum_{j=1}^{n}\left\|S_1(z_j)\right\|_W^q\right)^{1/q}+\left(\sum_{j=1}^{n}\left\|S_2(z_j)\right\|_W^q\right)^{1/q}\right)\\
&\quad\le \kappa_q K\Big(\mathbf{M}_{(q)}[S_1]+\mathbf{M}_{(q)}[S_2]\Big)\left(\sum_{j=1}^{n}|z_j|^q\right)^{1/q}
\end{aligned}$$

because S_1 and S_2 are q-concave. Thus (S_1+S_2) is q-concave, which implies that $\mathcal{K}_{(q)}(Z,W)$ is closed under addition.

Since $\mathcal{K}_{(q)}(Z,W)$ can easily be shown to be closed under scalar multiplication, we conclude that $\mathcal{K}_{(q)}(Z,W)$ is a linear subspace of $\mathcal{L}(Z,W)$.

(iv) Let $S\in\mathcal{K}_{(r)}(Z,W)\setminus\{0\}$. We can apply Lemma 2.53(ii), with r in place of q, to deduce that

$$\Big(\mathbf{M}_{(r)}[S]\Big)^{-1}=\inf\left\|\left(\sum_{j=1}^{n}a_j|z_j|^r\right)^{1/r}\right\|_Z>0,$$

where the infimum is taken over all $n\in\mathbb{N}$, positive numbers $a_1,\dots,a_n$ with $\sum_{j=1}^n a_j=1$ and vectors $z_1,\dots,z_n\in Z$ with $\left\|S(z_j)\right\|_W=1$ for $j=1,\dots,n$. For such $n\in\mathbb{N}$, numbers $a_1,\dots,a_n$ and vectors $z_1,\dots,z_n\in Z$, it follows from Lemma 2.51(i) that

$$0<\Big(\mathbf{M}_{(r)}[S]\Big)^{-1}\le\left\|\left(\sum_{j=1}^{n}a_j|z_j|^r\right)^{1/r}\right\|_Z\le\left\|\left(\sum_{j=1}^{n}a_j\,|z_j|^q\right)^{1/q}\right\|_Z.$$

Hence, Lemma 2.53(ii) implies that S is q-concave and $(\mathbf{M}_{(r)}[S])^{-1}\le(\mathbf{M}_{(q)}[S])^{-1}$, that is, (2.137) holds. □

Corollary 2.55. *Let Z be a Banach lattice or a q-B.f.s. Assume that $0<r<q<\infty$.*

(i) *If Z is q-convex, then Z is r-convex and $\mathbf{M}^{(r)}[Z]\le\mathbf{M}^{(q)}[Z]$.*

(ii) *If Z is r-concave, then Z is q-concave and $\mathbf{M}_{(q)}[Z]\le\mathbf{M}_{(r)}[Z]$.*

Proof. (i) This follows from Proposition 2.54(ii) with $W:=Z$ because the identity id_Z on Z satisfies $\mathrm{id}_Z\in\mathcal{K}^{(q)}(Z,Z)\subseteq\mathcal{K}^{(r)}(Z,Z)$.

(ii) We can conclude that $\mathrm{id}_Z\in\mathcal{K}_{(r)}(Z,Z)\subseteq\mathcal{K}_{(q)}(Z,Z)$ from Proposition 2.54(iv) with $W:=Z$. So, (ii) holds. □

To discuss the composition of a convex or concave operator with another operator, let us introduce a new class of operators. Fix $0<q<\infty$. For each $k=1,2$, let Z_k be either a Banach lattice or a q-B-f.s. Note that we are allowed to include the case when Z_1 is a Banach lattice and Z_2 is a q-B.f.s. or vice versa. Let

$\Lambda_q(Z_1, Z_2)$ denote the class of all operators $U \in \mathcal{L}(Z_1, Z_1)$ for which there exists a constant $C > 0$ satisfying

$$\Big\| \Big(\sum_{j=1}^{n} |U(z_j)|^q \Big)^{1/q} \Big\|_{Z_2} \le C \Big\| \Big(\sum_{j=1}^{n} |z_j|^q \Big)^{1/q} \Big\|_{Z_1}, \qquad n \in \mathbb{N}, \quad z_1, \ldots, z_n \in Z_1. \tag{2.138}$$

For each $U \in \Lambda_q(Z_1, Z_2)$, let C_U denote the smallest constant $C > 0$ satisfying (2.138).

Lemma 2.56. *Let $0 < q < \infty$. For each $k = 1, 2$, assume that Z_k is a Banach lattice or a q-B.f.s. Then $\Lambda_q(Z_1, Z_2)$ is a linear subspace of $\mathcal{L}(Z_1, Z_2)$.*

Proof. As it is routine to prove that $\Lambda_q(Z_1, Z_2)$ is closed under scalar multiplication, we shall verify only that $\Lambda_q(Z_1, Z_2)$ is closed under addition. Let $U_1, U_2 \in \Lambda_q(Z_1, Z_2)$. To show that $(U_1 + U_2) \in \Lambda_q(Z_1, Z_2)$, fix $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in Z_1$. Then, from (2.115) and the definitions of C_{U_k} for $k = 1, 2$, it follows that

$$\begin{aligned}
\Big\| \Big(\sum_{j=1}^{n} \big|(U_1 + U_2)(z_j)\big|^q \Big)^{1/q} \Big\|_{Z_2} &= \Big\| \Big(\sum_{j=1}^{n} \big|U_1(z_j) + U_2(z_j)\big|^q \Big)^{1/q} \Big\|_{Z_2} \\
&\le \kappa_q K \bigg(\Big\| \Big(\sum_{j=1}^{n} \big|U_1(z_j)\big|^q \Big)^{1/q} \Big\|_{Z_2} + \Big\| \Big(\sum_{j=1}^{n} \big|U_2(z_j)\big|^q \Big)^{1/q} \Big\|_{Z_2} \bigg) \\
&\le \kappa_q K \big(C_{U_1} + C_{U_2} \big) \Big\| \Big(\sum_{j=1}^{n} |z_j|^q \Big)^{1/q} \Big\|_{Z_1},
\end{aligned}$$

where K satisfies (Q3) for the space Z_2. Hence, $\big(U_1 + U_2\big) \in \Lambda_q(Z_1, Z_2)$. $\square$

Lemma 2.57. *Let $0 < q < \infty$. For each $k = 1, 2$, assume that Z_k is either a Banach lattice or a q-B.f.s.*

(i) *If an operator $U \in \mathcal{L}(Z_1, Z_2)$ satisfies*

$$\Big(\sum_{j=1}^{n} |U(z_j)|^q \Big)^{1/q} \le \Big| U\Big(\Big(\sum_{j=1}^{n} |z_j|^q \Big)^{1/q} \Big) \Big|, \qquad z_1, \ldots, z_n \in Z_1, \quad n \in \mathbb{N}, \tag{2.139}$$

in the order of Z_2, then $U \in \Lambda_q(Z_1, Z_2)$.

(ii) *Assume further that $1 \le q < \infty$ and that Z_1 and Z_2 are Banach lattices.*

(a) *Every positive operator $U \in \mathcal{L}(Z_1, Z_2)$ satisfies*

$$\Big(\sum_{j=1}^{n} |U(z_j)|^q \Big)^{1/q} \le U\Big(\Big(\sum_{j=1}^{n} |z_j|^q \Big)^{1/q} \Big), \qquad z_1, \ldots, z_n \in Z_1, \quad n \in \mathbb{N}, \tag{2.140}$$

and hence, $U \in \Lambda_q(Z_1, Z_2)$.

(b) *Every linear combination of positive operators from Z_1 into Z_2 belongs to $\Lambda_q(Z_1, Z_2)$.*

Proof. (i) Since $\|\cdot\|_{Z_2}$ is a lattice quasi-norm, it follows from (2.139) that

$$\Big\|\Big(\sum_{j=1}^{n}\big|U(z_j)\big|^q\Big)^{1/q}\Big\|_{Z_2} \le \Big\|U\Big(\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}\Big)\Big\|_{Z_2} \le \|U\|\cdot\Big\|\Big(\sum_{j=1}^{n}|z_j|^q\Big)^{1/q}\Big\|_{Z_1}$$

for all $z_1,\dots,z_n \in Z_1$ with $n\in\mathbb{N}$. So, $U\in\Lambda_q(Z_1,Z_2)$.

(ii) (a) First, for a fixed $z\in Z_1$, we shall show that

$$\big|U(z)\big| \le U(|z|). \tag{2.141}$$

Write $z = x+iy$ with $x,y\in(Z_1)_{\mathbb{R}}$. Then

$$\big|U(z)\big| = \sup_{0\le\theta<2\pi}\Big|(\cos\theta)\,U(x)+(\sin\theta)\,U(y)\Big| = \sup_{0\le\theta<2\pi}\Big|U\big((\cos\theta)x+(\sin\theta)y\big)\Big|; \tag{2.142}$$

apply (2.14) after observing that $U(z) = U(x)+i\,U(y)$ and that $U(x),U(y)\in(Z_2)_{\mathbb{R}}$. Fix $0\le\theta<2\pi$. By (2.14) we have

$$\pm\big((\cos\theta)x+(\sin\theta)y\big) \le |z|$$

and, since U is positive, it follows that

$$\pm U\big((\cos\theta)x+(\sin\theta)y\big) = U\Big(\pm\big((\cos\theta)x+(\sin\theta)y\big)\Big) \le U(|z|).$$

This implies that

$$\Big|U\big((\cos\theta)x+(\sin\theta)y\big)\Big| \le U(|z|) \tag{2.143}$$

via the general fact that $|\xi| = \xi\vee(-\xi)$ for every $\xi\in(Z_2)_{\mathbb{R}}$. Then (2.14), (2.142) and (2.143) yield (2.141).

Fix $n\in\mathbb{N}$ and $z_1,\dots,z_n\in Z_1$. Then

$$\Big(\sum_{j=1}^{n}\big|U(z_j)\big|^q\Big)^{1/q} \le \Big(\sum_{j=1}^{n}\big|U(|z_j|)\big|^q\Big)^{1/q} \tag{2.144}$$

by (2.141) and Lemma 2.51(iv). Since $U\big((Z_1)_{\mathbb{R}}\big)\subseteq(Z_2)_{\mathbb{R}}$, let $U_{\mathbb{R}}:(Z_1)_{\mathbb{R}}\to(Z_2)_{\mathbb{R}}$ denote the restriction of U to $(Z_1)_{\mathbb{R}}$, with codomain $(Z_2)_{\mathbb{R}}$. Then $U_{\mathbb{R}}$ is a positive operator between two real Banach lattices and hence,

$$\Big(\sum_{j=1}^{n}\big|U_{\mathbb{R}}(x_j)\big|^q\Big)^{1/q} \le U_{\mathbb{R}}\Big(\Big(\sum_{j=1}^{n}|x_j|^q\Big)^{1/q}\Big), \qquad x_1,\dots,x_n\in Z_{\mathbb{R}}; \tag{2.145}$$

see [92, p. 8] or the proof of Proposition 1.d.9 in [99]. Now, (2.144) and (2.145) (with $x_j := |z_j|$ for $j=1,\dots,n$) yield (2.140).

(b) This is a consequence of Lemma 2.56 and part (a) above. □

Remark 2.58. In Lemma 2.57(ii) we have not assumed that Z_2 is Dedekind complete. If Z_2 were Dedekind complete, then (2.141) would be a consequence of the general facts that $|U(z)| \le |U|(|z|)$ and that $|U| = U$, where $|U|$ denotes the modulus of U; see [165, Lemma 92.5]. $\square$

Example 2.59. Let $X(\mu)$ and $Y(\mu)$ be q-B.f.s.' over a positive, finite measure space (Ω, Σ, μ). Then the multiplication operator

$$M_g \in \bigcap_{0<q<\infty} \Lambda_q\big(X(\mu), Y(\mu)\big), \qquad g \in \mathcal{M}\big(X(\mu), Y(\mu)\big).$$

In fact, fix $g \in \mathcal{M}\big(X(\mu), Y(\mu)\big)$ and $0 < q < \infty$. Given $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in X(\mu)$, it is clear that

$$\Big(\sum_{j=1}^{n} |g f_j|^q\Big)^{1/q} = \Big| g \Big(\sum_{j=1}^{n} |f_j|^q\Big)^{1/q} \Big|$$

pointwise on Ω. That is,

$$\Big(\sum_{j=1}^{n} \Big|M_g(f_j)\Big|^q\Big)^{1/q} = \Big| M_g\Big(\Big(\sum_{j=1}^{n} |f_j|^q\Big)^{1/q}\Big)\Big|. \tag{2.146}$$

So, $M_g \in \Lambda_q\big(X(\mu), Y(\mu)\big)$ via Lemma 2.57(i) with $Z_1 := X(\mu)$, $Z_2 := Y(\mu)$ and $U := M_g$.

Note that, if $X(\mu)$ and $Y(\mu)$ are B.f.s.' and $1 \le q < \infty$, then Lemma 2.57(ii) can also be applied to deduce that $M_g \in \Lambda_q\big(X(\mu), Y(\mu)\big)$ because

$$M_g = M_{\mathrm{Re}(g)^+} - M_{\mathrm{Re}(g)^-} + i\Big(M_{\mathrm{Im}(g)^+} - M_{\mathrm{Im}(g)^-}\Big). \qquad \square$$

Example 2.60. Let $\mu : 2^{\mathbb{N}} \to [0, \infty)$ be a finite measure with $\mu(\{n\}) > 0$ for every $n \in \mathbb{N}$. Let $0 < r < \infty$ and let $U : \ell^r(\mu) \to \ell^r(\mu)$ denote the unit right shift operator, that is,

$$U(f)(k) := \begin{cases} 0 & \text{if } \ k = 1, \\ f(k-1) & \text{if } \ k \ge 2, \end{cases}$$

for every $f \in \ell^r(\mu)$. In the notation of sequences, we can rewrite this as

$$U : (a_1, a_2, \ldots) \longmapsto (0, a_1, a_2, \ldots), \qquad (a_n)_{n=1}^{\infty} \in \ell^r(\mu).$$

(i) Let $0 < q < \infty$. Then U satisfies

$$\Big(\sum_{j=1}^{n} \big|U(f_j)\big|^q\Big)^{1/q} = U\Big(\big(\sum_{j=1}^{n} |f_j|^q\big)^{1/q}\Big), \qquad f_1, \ldots, f_n \in \ell^r(\mu), \quad n \in \mathbb{N}. \tag{2.147}$$

So, from Lemma 2.57(i) it follows that $U \in \Lambda_q\big(\ell^r(\mu), \ell^r(\mu)\big)$.

(ii) Let $g \in \ell^\infty = \mathcal{M}\big(\ell^r(\mu), \ell^r(\mu)\big)$ (see (2.75) with $X(\mu) := \ell^r(\mu)$). Then the compositions $M_g \circ U$ and $U \circ M_g$ are operators belonging to $\Lambda_q\big(\ell^r(\mu), \ell^r(\mu)\big)$. Again, this can be proved by applying Lemma 2.57(i) together with (2.146) and (2.147). $\square$

Given a complex Banach lattice Z, its dual Banach space Z^* is actually a (complex) Banach lattice with respect to the dual norm $\|\cdot\|_{Z^*}$, which is also a lattice norm, and

$$\big\langle z, |z^*|\big\rangle \;=\; \sup\Big\{\big|\langle y, z^*\rangle\big| : y \in Z \text{ and } |y| \le z\Big\}, \qquad z \in Z^+, \quad z^* \in Z^*, \quad (2.148)$$

[149, Ch. IV, Corollary 3 of Theorem 1.8] or [165, pp. 323–324]. Here, the Banach space dual Z^* of Z coincides with the order dual of Z (which is the complex vector lattice of all $\mathbb{C}$-valued, order bounded linear functionals on Z). The following inequality is a consequence of (2.148):

$$\big|\langle z, z^*\rangle\big| \;\le\; \big\langle |z|, |z^*|\big\rangle, \qquad z \in Z, \quad z^* \in Z^*; \qquad (2.149)$$

see [166, Theorem 36.4] with $F := \mathbb{R}$ there and note (by the discussion on p. 239 of [166]) that Z there only has to be Archimedean and uniformly complete, which is satisfied by Banach lattices.

Example 2.61. (i) Let $1 \le q < \infty$. A continuous linear operator T from a Banach lattice Z into a Banach space W is said to be *absolutely q-summing* if there exists a constant $C > 0$ such that

$$\Big(\sum_{j=1}^{n} \big\|T(z_j)\big\|_W^q\Big)^{1/q} \;\le\; C \sup_{z^* \in \mathbf{B}[Z^*]} \Big(\sum_{j=1}^{n} \big|\langle z_j, z^*\rangle\big|^q\Big)^{1/q} \qquad (2.150)$$

for all $n \in \mathbb{N}$ and $z_1, \dots, z_n \in Z$. Detailed information concerning absolutely summing operators between Banach spaces is available, for example, in [41]. Let $\Pi_q(Z, W)$ denote the class of all absolutely q-summing operators from Z into W. Then

$$\Pi_q(Z, W) \;\subseteq\; \mathcal{K}_{(q)}(Z, W). \qquad (2.151)$$

For the case when Z and W are both over $\mathbb{R}$, this has been verified in [99, p. 56]. Their proof can be adapted to the complex case. In fact, let $z^* \in Z^*$. Its modulus $|z^*| : Z \to \mathbb{C}$ is a positive linear functional and hence, Lemma 2.57(ii)(a) (with $U := |z^*|$) together with (2.149) imply that

$$\begin{aligned}
\Big(\sum_{j=1}^{n} \big|\langle z_j, z^*\rangle\big|^q\Big)^{1/q} &\le \Big(\sum_{j=1}^{n} \big\langle |z_j|, |z^*|\big\rangle^q\Big)^{1/q} \le \Big\langle \Big(\sum_{j=1}^{n} |z_j|^q\Big)^{1/q}, \; |z^*|\Big\rangle \\
&\le \big\|\, |z^*| \,\big\|_{Z^*} \Big\| \Big(\sum_{j=1}^{n} |z_j|^q\Big)^{1/q} \Big\|_Z \;=\; \|z^*\|_{Z^*} \Big\| \Big(\sum_{j=1}^{n} |z_j|^q\Big)^{1/q} \Big\|_Z
\end{aligned}$$

whenever $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in Z$. Now let $T \in \Pi_q(Z, W)$ and $C > 0$ be a constant satisfying (2.150). Then the previous inequalities and (2.150) yield

$$\Big(\sum_{j=1}^{n} \|T(z_j)\|_W^q \Big)^{1/q} \leq C \Big\| \Big(\sum_{j=1}^{n} |z_j|^q \Big)^{1/q} \Big\|_Z$$

for all $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in Z$. Thus, T is q-concave, that is, $T \in \mathcal{K}_{(q)}(Z, W)$. Hence, (2.151) holds.

(ii) Assume that $2 < r < q < \infty$. Then

$$\Pi_q\big(\ell^1, \ell^r\big) \;=\; \mathcal{K}_{(q)}\big(\ell^1, \ell^r\big) \;=\; \mathcal{L}\big(\ell^1, \ell^r\big).$$

This is a consequence of part (i) and the fact that

$$\Pi_q(\ell^1, \ell^r) \;=\; \mathcal{L}(\ell^1, \ell^r); \tag{2.152}$$

see [31, Corollary 24.6]. Moreover, (2.152) implies that

$$\Pi_q(Z, W) \;=\; \mathcal{L}(Z, W) \tag{2.153}$$

provided Z is an $\mathcal{L}^1$-space and W is an $\mathcal{L}^r$-space; see [41, p. 60] for the definition of $\mathcal{L}^p$-spaces for $1 \leq p < \infty$. In fact, this can be proved by adapting the proof of [41, Theorem 3.1] (which is Grothendieck's Theorem stating that every continuous linear operator from an $\mathcal{L}^1$-space into an $\mathcal{L}^2$-space is absolutely 1-summing). In particular, since $L^1([0,1])$ and $L^r([0,1])$ are an $\mathcal{L}^1$-space and an $\mathcal{L}^r$-space respectively (see [41, Theorem 3.2]), we have by (2.153) that

$$\Pi_q\big(L^1([0,1]), L^r([0,1])\big) = \mathcal{K}_{(q)}\big(L^1([0,1]), L^r([0,1])\big) = \mathcal{L}\big(L^1([0,1]), L^r([0,1])\big).$$

□

It would be interesting to find further criteria for a continuous linear operator to belong to the class Λ_q, in addition to those given in Lemma 2.57. For the special case $q = 2$, it turns out that every continuous linear operator between Banach lattices belongs to the class Λ_2. For real Banach lattices, this is due to J.L. Krivine [92, Theorem 3]. Let K_{G} denote Grothendieck's constant.

Lemma 2.62. *Let Z_1 and Z_2 be Banach lattices. If $U \in \mathcal{L}(Z_1, Z_2)$, then*

$$\Big\| \Big(\sum_{j=1}^{n} \big|U(z_j)\big|^2 \Big)^{1/2} \Big\|_{Z_2} \leq 4 K_{\mathrm{G}} \|U\| \cdot \Big\| \Big(\sum_{j=1}^{n} |z_j|^2 \Big)^{1/2} \Big\|_{Z_1} \tag{2.154}$$

whenever $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in Z_1$. Consequently, $\mathcal{L}(Z_1, Z_2) = \Lambda_2(Z_1, Z_2)$.

Proof. Fix $n \in \mathbb{N}$. Observe that

$$\begin{aligned}
&\Big\|\Big(\sum_{j=1}^{n}|\zeta_j^{(1)}+\zeta_j^{(2)}+\zeta_j^{(3)}+\zeta_j^{(4)}|^2\Big)^{1/2}\Big\|_{Z_2}\\
&\le \Big\|\Big(\sum_{j=1}^{n}|\zeta_j^{(1)}+\zeta_j^{(2)}|^2\Big)^{1/2}\Big\|_{Z_2}+\Big\|\Big(\sum_{j=1}^{n}|\zeta_j^{(3)}+\zeta_j^{(4)}|^2\Big)^{1/2}\Big\|_{Z_2}\\
&\le \Big\|\Big(\sum_{j=1}^{n}|\zeta_j^{(1)}|^2\Big)^{1/2}\Big\|_{Z_2}+\Big\|\Big(\sum_{j=1}^{n}|\zeta_j^{(2)}|^2\Big)^{1/2}\Big\|_{Z_2}\\
&\qquad+\ \Big\|\Big(\sum_{j=1}^{n}|\zeta_j^{(3)}|^2\Big)^{1/2}\Big\|_{Z_2}+\Big\|\Big(\sum_{j=1}^{n}|\zeta_j^{(4)}|^2\Big)^{1/2}\Big\|_{Z_2}
\end{aligned}\tag{2.155}$$

whenever $\zeta_j^{(l)} \in Z_2$ for $j = 1,\dots,n$ and $l = 1,2,3,4$, where we have used (2.115) twice with $\kappa_2 := 1$ and $K := 1$.

There exist operators $U_1, U_2 \in \mathcal{L}(Z_1, Z_2)$ such that $U_k\big((Z_1)_{\mathbb{R}}\big) \subseteq (Z_2)_{\mathbb{R}}$ for $k = 1,2$ and $U = U_1 + i\,U_2$; see [165, §92], for example. Let

$$(U_1)_{\mathbb{R}} : (Z_1)_{\mathbb{R}} \to (Z_2)_{\mathbb{R}} \quad \text{and} \quad (U_2)_{\mathbb{R}} : (Z_1)_{\mathbb{R}} \to (Z_2)_{\mathbb{R}}$$

denote the restriction of U_1 and U_2 to $(Z_1)_{\mathbb{R}}$ with codomain $(Z_2)_{\mathbb{R}}$. It follows from [99, Theorem 1.f.14] that

$$\begin{aligned}
&\Big\|\Big(\sum_{j=1}^{n}\big|(U_k)_{\mathbb{R}}(\xi_j)\big|^2\Big)^{1/2}\Big\|_{(Z_2)_{\mathbb{R}}}\\
&\qquad\le\ K_{\mathrm{G}}\big\|(U_k)_{\mathbb{R}}\big\|\cdot\Big\|\Big(\sum_{j=1}^{n}|\xi_j|^2\Big)^{1/2}\Big\|_{(Z_1)_{\mathbb{R}}},\quad k=1,2,
\end{aligned}\tag{2.156}$$

whenever $\xi_1,\dots,\xi_n \in (Z_1)_{\mathbb{R}}$. Fix $z_1,\dots,z_n \in Z_1$. For each $j = 1,\dots,n$, write $z_j = x_j + iy_j$ with $x_j, y_j \in (Z_1)_{\mathbb{R}}$, so that

$$U(z_j) \ =\ (U_1)_{\mathbb{R}}(x_j)\ +\ i\,(U_2)_{\mathbb{R}}(x_j)\ +\ i\,(U_1)_{\mathbb{R}}(y_j)\ -\ (U_2)_{\mathbb{R}}(y_j).$$

Then we can apply (2.155) to obtain that

$$\begin{aligned}
&\Big\|\Big(\sum_{j=1}^{n}\big|U(z_j)\big|^2\Big)^{1/2}\Big\|_{Z_2}\\
&\le\sum_{k=1}^{2}\left(\Big\|\Big(\sum_{j=1}^{n}\Big|(U_k)_{\mathbb{R}}(x_j)\Big|^2\Big)^{1/2}\Big\|_{Z_2}+\Big\|\Big(\sum_{j=1}^{n}\Big|(U_k)_{\mathbb{R}}(y_j)\Big|^2\Big)^{1/2}\Big\|_{Z_2}\right).
\end{aligned}\tag{2.157}$$

Given $k = 1, 2$, since $\|\xi\|_{(Z_2)_{\mathbb{R}}} = \|\xi\|_{Z_2}$ for every $\xi \in (Z_2)_{\mathbb{R}} \subseteq Z$ and $\|(U_k)_{\mathbb{R}}\| \leq \|U_k\|$, it follows from (2.156) that

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{n}\Big|(U_k)_{\mathbb{R}}(x_j)\Big|^2\Big)^{1/2}\Big\|_{Z_2} &\leq K_{\mathrm{G}}\|(U_k)_{\mathbb{R}}\| \cdot \Big\|\Big(\sum_{j=1}^{n}|x_j|^2\Big)^{1/2}\Big\|_{Z_1} \\ &\leq K_{\mathrm{G}}\|U_k\| \cdot \Big\|\Big(\sum_{j=1}^{n}|z_j|^2\Big)^{1/2}\Big\|_{Z_1}. \end{aligned} \tag{2.158}$$

The last inequality follows from Lemma 2.51(iv) because $|x_j| \leq |z_j|$ for every $j = 1, \dots, n$. Similarly,

$$\Big\|\Big(\sum_{j=1}^{n}\Big|(U_k)_{\mathbb{R}}(y_j)\Big|^2\Big)^{1/2}\Big\|_{Z_2} \leq K_{\mathrm{G}}\|U_k\| \cdot \Big\|\Big(\sum_{j=1}^{n}|z_j|^2\Big)^{1/2}\Big\|_{Z_1}. \tag{2.159}$$

So, since $\|U_k\| \leq \|U\|$ for $k = 1, 2$, the inequality (2.154) follows from (2.157), (2.158) and (2.159). □

Now we present results concerning the composition of convex operators with other continuous linear operators.

Proposition 2.63. *Suppose that W is a quasi-Banach space and that $0 < q < \infty$.*

(i) *Let W_1 be another quasi-Banach space and let $U \in \mathcal{L}(W, W_1)$. Suppose that Z is either a Banach lattice or a q-B.f.s. If $T \in \mathcal{K}^{(q)}(W_1, Z)$, then the composition $T \circ U : W \to Z$ is also q-convex and $\mathbf{M}^{(q)}[T \circ U] \leq \big(\mathbf{M}^{(q)}[T]\big)\, \|U\|$.*

(ii) *For each $k = 1, 2$, assume that Z_k is either a Banach lattice or a q-B.f.s. If $T \in \mathcal{K}^{(q)}(W, Z_1)$ and $U \in \Lambda_q(Z_1, Z_2)$, then $U \circ T \in \mathcal{K}^{(q)}(W, Z_2)$.*

(iii) *Let Z_1 and Z_2 be Banach lattices. If $T \in \mathcal{K}^{(2)}(W, Z_1)$ and $U \in \mathcal{L}(Z_1, Z_2)$, then $U \circ T \in \mathcal{K}^{(2)}(W, Z_2)$.*

Proof. (i) Given $n \in \mathbb{N}$ and $w_1, \dots, w_n \in W$, the q-convexity of T gives that

$$\begin{aligned}&\Big\|\Big(\sum_{j=1}^{n}|(T \circ U)(w_j)|^q\Big)^{1/q}\Big\|_Z \leq \Big(\mathbf{M}^{(q)}[T]\Big)\Big(\sum_{j=1}^{n}\|U(w_j)\|_{W_1}^q\Big)^{1/q} \\ &\leq \Big(\mathbf{M}^{(q)}[T]\Big)\Big(\sum_{j=1}^{n}\|U\|^q \cdot \|w_j\|_W^q\Big)^{1/q} = \Big(\mathbf{M}^{(q)}[T]\Big)\|U\|\Big(\sum_{j=1}^{n}\|w_j\|_W^q\Big)^{1/q}.\end{aligned}$$

Thus, part (i) holds.

(ii) Fix $n \in \mathbb{N}$ and $w_1, \dots, w_n \in W$. Since T is q-convex and U belongs to the class $\Lambda_q(Z_1, Z_2)$, it follows that

$$\Big\| \Big(\sum_{j=1}^{n} \big|(U \circ T)(w_j)\big|^q \Big)^{1/q} \Big\|_{Z_2} \leq C_U \Big\| \Big(\sum_{j=1}^{n} \big|T(w_j)\big|^q \Big)^{1/q} \Big\|_{Z_1}$$
$$\leq C_U \Big(\mathbf{M}^{(q)}[T]\Big) \Big(\sum_{j=1}^{n} \|w_j\|_W^q \Big)^{1/q}.$$

This establishes part (ii).

(iii) Apply Lemma 2.62 and the 2-convexity of T to obtain that

$$\Big\| \Big(\sum_{j=1}^{n} \big|(U \circ T)(w_j)\big|^2 \Big)^{1/2} \Big\|_{Z_2} \leq\ 4 K_{\mathrm{G}} \|U\| \Big(\mathbf{M}^{(2)}[T]\Big) \Big(\sum_{j=1}^{n} \|w_j\|_W^2 \Big)^{1/2}$$

whenever $w_1, \dots, w_n \in W$ and $n \in \mathbb{N}$. So, $U \circ T$ is 2-convex. □

Some consequences of Proposition 2.63 are now provided, which will be needed later.

Corollary 2.64. *Let W be a quasi-Banach space and Z be either a Banach lattice or a q-B.f.s. If Z is q-convex for some $0 < q < \infty$, then*

$$\mathcal{L}(W, Z) \ = \ \mathcal{K}^{(q)}(W, Z).$$

Proof. Let $U \in \mathcal{L}(W, Z)$. By definition, the identity operator id_Z on Z is q-convex. So, Proposition 2.63(i), with $W_1 := Z$ and $T := \mathrm{id}_Z$, implies that $U = \mathrm{id}_Z \circ U$ is q-convex. So, the corollary holds because we already know that $\mathcal{K}^{(q)}(W, Z) \subseteq \mathcal{L}(W, Z)$. □

Corollary 2.65. *Assume that $1 \leq q < \infty$ and Z_1 is a q-convex Banach lattice. If Z_2 is any Banach lattice, then every positive operator $U : Z_1 \to Z_2$ is q-convex.*

Proof. We have $U \in \Lambda_q(Z_1, Z_2)$ via Lemma 2.57(ii)(a). So, the q-convexity of the identity operator id_{Z_1} on Z_1 implies that

$$U = U \circ \mathrm{id}_{Z_1} \ \in\ \mathcal{K}^{(q)}(Z_1, Z_2)$$

by Proposition 2.63(ii) with $W := Z_1$ and $T := \mathrm{id}_{Z_1}$. □

Corollary 2.66. *Let $X(\mu)$ and $Y(\mu)$ be q-B.f.s.' over a positive, finite measure space (Ω, Σ, μ). If $X(\mu)$ is q-convex for some $0 < q < \infty$ and if $g \in \mathcal{M}\big(X(\mu), Y(\mu)\big)$, then $M_g : X(\mu) \to Y(\mu)$ is a q-convex operator.*

Proof. We know from Example 2.59 that $M_g \in \Lambda_q\big(X(\mu), Y(\mu)\big)$. By assumption, the identity $\mathrm{id}_{X(\mu)}$ is q-convex. So, apply Proposition 2.63(ii) (with $W := X(\mu)$, $Z_1 := X(\mu)$, $Z_2 := Y(\mu)$, $T := \mathrm{id}_{X(\mu)}$ and $U := M_g$) to deduce that

$$M_g = M_g \circ \mathrm{id}_{X(\mu)} \ \in\ \mathcal{K}^{(q)}\big(X(\mu), Y(\mu)\big).$$ □

Corollary 2.67. *Let Z_1 be a 2-convex Banach lattice and Z_2 be a Banach lattice. Then*

$$\mathcal{L}(Z_1, Z_2) \; = \; \mathcal{K}^{(2)}(Z_1, Z_2).$$

Proof. Let $U \in \mathcal{L}(Z_1, Z_2)$. The identity id_{Z_1} on Z_1 is 2-convex by assumption. So, from Proposition 2.63(iii) with $W := Z_1$ and $T := \mathrm{id}_{Z_1}$, it follows that

$$U = U \circ \mathrm{id}_{Z_1} \in \mathcal{K}^{(2)}(Z_1, Z_2).$$

The reverse containment $\mathcal{K}^{(2)}(Z_1, Z_2) \subseteq \mathcal{L}(Z_1, Z_2)$ always holds. $\square$

Now we present results for the composition of a concave operator with a continuous linear operator.

Proposition 2.68. *Suppose that W is a quasi-Banach space and that $0 < q < \infty$.*

- (i) *Let Z be either a Banach lattice or a q-B.f.s., and let $S \in \mathcal{K}_{(q)}(Z, W_1)$ for some $0 < q < \infty$ and quasi-Banach space W_1. If $U \in \mathcal{L}(W_1, W)$, then $U \circ S \in \mathcal{K}_{(q)}(Z, W)$ and $\mathbf{M}_{(q)}[U \circ S] \le \|U\| \left(\mathbf{M}_{(q)}[S]\right)$.*
- (ii) *For each $k = 1, 2$, assume that Z_k is either a Banach lattice or a q-B.f.s. If $U \in \Lambda_q(Z_1, Z_2)$ and $S \in \mathcal{K}_{(q)}(Z_2, W)$, then $S \circ U \in \mathcal{K}_{(q)}(Z_1, W)$.*
- (iii) *Let Z_1 and Z_2 be Banach lattices. If $U \in \mathcal{L}(Z_1, Z_2)$ and $S \in \mathcal{K}_{(2)}(Z_2, W)$, then $S \circ U \in \mathcal{K}_{(2)}(Z_1, W)$.*

Proof. (i) Given $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in Z$, the q-concavity of S yields that

$$\begin{aligned}
\Big(\sum_{j=1}^{n} \big\|(U \circ S)(z_j)\big\|_W^q\Big)^{1/q} &\le \Big(\sum_{j=1}^{n} \|U\|^q \cdot \big\|S(z_j)\big\|_{W_1}^q\Big)^{1/q} \\
= \|U\| \Big(\sum_{j=1}^{n} \big\|S(z_j)\big\|_{W_1}^q\Big)^{1/q} &\le \|U\| \left(\mathbf{M}_{(q)}[S]\right) \Big\|\Big(\sum_{j=1}^{n} |z_j|^q\Big)^{1/q}\Big\|_Z .
\end{aligned}$$

So, part (i) holds.

(ii) Let $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in Z_1$. Since $U \in \Lambda_q(Z_1, Z_2)$ and S is q-concave, it follows from Lemma 2.57 that

$$\begin{aligned}
\Big(\sum_{j=1}^{n} \big\|(S \circ U)(z_j)\big\|_W^q\Big)^{1/q} &\le \left(\mathbf{M}_{(q)}[S]\right) \Big\|\Big(\sum_{j=1}^{n} |U(z_j)|^q\Big)^{1/q}\Big\|_{Z_2} \\
&\le C_U \left(\mathbf{M}_{(q)}[S]\right) \Big\|\Big(\sum_{j=1}^{n} |z_j|^q\Big)^{1/q}\Big\|_{Z_1},
\end{aligned}$$

which establishes part (ii).

(iii) Apply Lemma 2.62 and the 2-concavity of S to obtain

$$\Big(\sum_{j=1}^{n}\big\|(S\circ U)(z_j)\big\|_W^2\Big)^{1/2} \le 4K_{\mathrm{G}}\Big(\mathbf{M}_{(2)}[S]\Big)\|U\|\cdot\Big\|\Big(\sum_{j=1}^{n}|z_j|^2\Big)^{1/2}\Big\|_{Z_1},$$

whenever $z_1,\dots,z_n\in Z$ and $n\in\mathbb{N}$. So, $S\circ U$ is 2-convex. □

We now present some consequences of Proposition 2.68 which will be useful in later chapters. The proofs of these corollaries will be omitted because we can adapt the proofs of Corollaries 2.64 to 2.67; all we need to do is to apply Proposition 2.68 in place of Proposition 2.63.

Corollary 2.69. *Let Z be either a Banach lattice or a q-B.f.s. If Z is q-concave for some $0<q<\infty$, then*

$$\mathcal{L}(Z,W) = \mathcal{K}_{(q)}(Z,W)$$

for every quasi-Banach space W.

Corollary 2.70. *Let Z_1 be a Banach lattice. If Z_2 is a q-concave Banach lattice for some $1\le q<\infty$ and $U: Z_1\to Z_2$ is a positive operator, then U is a q-concave operator.*

It is not possible to remove the assumption that U is positive in Corollary 2.70. In fact, there exists a non-1-concave, continuous linear operator from a Banach lattice into a 1-concave Banach lattice; see Example 3.75.

Corollary 2.71. *Let $X(\mu)$ and $Y(\mu)$ be q-B.f.s.' over a positive, finite measure space (Ω,Σ,μ). If $Y(\mu)$ is q-concave for some $0<q<\infty$ and $g\in\mathcal{M}\big(X(\mu),Y(\mu)\big)$, then the multiplication operator $M_g: X(\mu)\to Y(\mu)$ is q-concave.*

Corollary 2.72. *Let Z_1 and Z_2 be Banach lattices. If Z_1 is 2-concave, then*

$$\mathcal{L}(Z_1,Z_2) = \mathcal{K}_{(2)}(Z_1,Z_2).$$

Let us consider the convexity and concavity properties of some particular spaces.

Example 2.73. Given are a number $0<q<\infty$ and a (possibly infinite) measure space (Ω,Σ,η). Assume that there exists a pairwise disjoint, infinite sequence $\{A(j)\}_{j=1}^{\infty}\subseteq\Sigma$ such that $0<\eta\big(A(j)\big)<\infty$ for every $j\in\mathbb{N}$.

(i) The q-B.f.s. $L^q(\eta)$ is both q-convex and q-concave with

$$\mathbf{M}^{(q)}\big[L^q(\eta)\big] = 1 \qquad\text{and}\qquad \mathbf{M}_{(q)}\big[L^q(\eta)\big] = 1. \tag{2.160}$$

This can be obtained directly from the fact that

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{n}|f_j|^q\Big)^{1/q}\Big\|_{L^q(\eta)} &= \Big(\int_\Omega\Big(\sum_{j=1}^{n}|f_j|^q\Big)\,d\eta\Big)^{1/q}\\ &= \Big(\sum_{j=1}^{n}\int_\Omega|f_j|^q\,d\eta\Big)^{1/q} = \Big(\sum_{j=1}^{n}\|f_j\|_{L^q(\eta)}^q\Big)^{1/q}\end{aligned} \tag{2.161}$$

for all $n\in\mathbb{N}$ and $f_1,\dots,f_n\in L^q(\eta)$.

(i-a) Assume that $0 < r < q$. Then $L^q(\eta)$ is r-convex and $\mathbf{M}^{(r)}[L^q(\eta)] = 1$ because (2.107), Corollary 2.55(i) and (2.160) yield that

$$1 \;\le\; \mathbf{M}^{(r)}\big[L^q(\eta)\big] \;\le\; \mathbf{M}^{(q)}\big[L^q(\eta)\big] \;=\; 1.$$

(i-b) Assume that $0 < q < r$. Observe from (2.107), Corollary 2.55(ii) and (2.160) that

$$1 \;\le\; \mathbf{M}_{(r)}\big[L^q(\eta)\big] \;\le\; \mathbf{M}_{(q)}\big[L^q(\eta)\big] \;=\; 1.$$

Consequently, $L^q(\eta)$ is r-concave and $\mathbf{M}_{(r)}[L^q(\eta)] = 1$.

(ii) Let $f_j := \|\chi_{A(j)}\|_{L^q(\eta)}^{-1} \cdot \chi_{A(j)}$ for $j \in \mathbb{N}$. Let $0 < r < \infty$. Since

$$\Big(\sum_{j=1}^{n} |f_j|^r\Big)^{1/r} \;=\; \Big(\sum_{j=1}^{n} |f_j|^q\Big)^{1/q} \;=\; \sum_{j=1}^{n} \eta(A_j)^{-1/q}\chi_{A(j)}, \qquad n \in \mathbb{N},$$

we have that

$$\Big\|\Big(\sum_{j=1}^{n} |f_j|^r\Big)^{1/r}\Big\|_{L^q(\eta)} \;=\; \Big\|\Big(\sum_{j=1}^{n} |f_j|^q\Big)^{1/q}\Big\|_{L^q(\eta)} \;=\; n^{1/q}, \qquad n \in \mathbb{N}. \tag{2.162}$$

Moreover, $\|f_j\|_{L^q(\eta)} = 1$ for all $j = 1, \dots, n$ and so

$$\Big(\sum_{j=1}^{n} \|f_j\|_{L^q(\eta)}^r\Big)^{1/r} \;=\; n^{1/r}, \qquad n \in \mathbb{N}. \tag{2.163}$$

(ii-a) If $0 < q < r < \infty$, then $L^q(\eta)$ is not r-convex. To show this, assume on the contrary that $L^q(\eta)$ is r-convex. Then it follows from (2.162) and (2.163) that

$$\begin{aligned} n^{1/q} &= \Big\|\Big(\sum_{j=1}^{n} |f_j|^r\Big)^{1/r}\Big\|_{L^q(\eta)} \le \Big(\mathbf{M}^{(r)}[L^q(\eta)]\Big)\Big(\sum_{j=1}^{n} \|f_j\|_{L^q(\eta)}^r\Big)^{1/r} \\ &= \Big(\mathbf{M}^{(r)}[L^q(\eta)]\Big) n^{1/r}, \end{aligned}$$

whenever $n \in \mathbb{N}$. This is impossible because $(1/q) > (1/r)$.

(ii-b) If $0 < r < q < \infty$, then $L^q(\eta)$ is not r-concave. In fact, the r-concavity of $L^q(\eta)$ would imply, via (2.162) and (2.163), that

$$n^{1/r} \;\le\; \Big(\mathbf{M}_{(r)}[L^q(\eta)]\Big) n^{1/q}, \qquad n \in \mathbb{N}.$$

This is impossible because $(1/r) > (1/q)$. □

According to Example 2.73(i), for $L^2([0,1])$ equipped with its usual (lattice) norm, we have $\mathbf{M}^{(2)}\big[L^2([0,1])\big] = 1$ and $\mathbf{M}_{(2)}\big[L^2([0,1])\big] = 1$. This may not be the case if we equip $L^2([0,1])$ with an equivalent (lattice) norm.

Example 2.74. Let μ denote Lebesgue measure on $[0,1]$. We shall discuss the 2-convexity and 2-concavity constants of $L^2(\mu) = L^2([0,1])$ when it is endowed with an equivalent (lattice) norm. To be precise, $\|\cdot\|_{L^2(\mu)}$ denotes the usual L^2-norm on $L^2(\mu)$. Let $A \in \mathcal{B}([0,1])$ satisfy $\mu(A) > 0$ and $\mu\big([0,1] \setminus A\big) > 0$, and set $B := [0,1] \setminus A$.

(i) Let Z_1 denote $L^2(\mu)$ equipped with the non-negative functional $\|\cdot\|_{Z_1}$ given by

$$\|f\|_{Z_1} := \big\|f\chi_A\big\|_{L^2(\mu)} + \big\|f\chi_B\big\|_{L^2(\mu)}, \qquad f \in L^2(\mu);$$

it is straightforward to prove that $\|\cdot\|_{Z_1}$ defines a lattice norm. Moreover, $\|\cdot\|_{Z_1}$ is equivalent to the original norm on $L^2(\mu)$ because

$$\|f\|_{L^2(\mu)} \le \|f\|_{Z_1} \le \sqrt{2}\|f\|_{L^2(\mu)}, \qquad f \in L^2(\mu). \tag{2.164}$$

So, Z_1 is also 2-convex and 2-concave. We shall show that

$$\mathbf{M}^{(2)}[Z_1] = \sqrt{2} \qquad \text{and} \qquad \mathbf{M}_{(2)}[Z_1] = 1.$$

To this end, fix $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in Z_1 = L^2(\mu)$. Then, via (2.161) (with $\eta := \mu$ and $q := 2$) and both inequalities in (2.164) we have that

$$\begin{aligned}\bigg\|\Big(\sum_{j=1}^n |f_j|^2\Big)^{1/2}\bigg\|_{Z_1} &\le \sqrt{2}\bigg\|\Big(\sum_{j=1}^n |f_j|^2\Big)^{1/2}\bigg\|_{L^2(\mu)} \\ &= \sqrt{2}\Big(\sum_{j=1}^n \|f_j\|_{L^2(\mu)}^2\Big)^{1/2} \le \sqrt{2}\Big(\sum_{j=1}^n \|f_j\|_{Z_1}^2\Big)^{1/2},\end{aligned}$$

which implies that $\mathbf{M}^{(2)}[Z_1] \le \sqrt{2}$. To establish the reverse inequality consider

$$g_1 := \big(\mu(A)\big)^{-1/2}\chi_A \quad \text{and} \quad g_2 := \big(\mu(B)\big)^{-1/2}\chi_B.$$

Then

$$\Big\|\Big(|g_1|^2 + |g_2|^2\Big)^{1/2}\Big\|_{Z_1} = \big\|g_1 + g_2\big\|_{Z_1} = \|g_1\|_{L^2(\mu)} + \|g_2\|_{L^2(\mu)} = 2$$

and

$$\Big(\|g_1\|_{Z_1}^2 + \|g_2\|_{Z_1}^2\Big)^{1/2} = \Big(\|g_1\|_{L^2(\mu)}^2 + \|g_2\|_{L^2(\mu)}^2\Big)^{1/2} = \sqrt{2}.$$

In other words

$$\Big\|\Big(|g_1|^2 + |g_2|^2\Big)^{1/2}\Big\|_{Z_1} = \sqrt{2}\Big(\|g_1\|_{Z_1}^2 + \|g_2\|_{Z_1}^2\Big)^{1/2}.$$

So, $\mathbf{M}^{(2)}[Z_1] \ge \sqrt{2}$ and therefore, $\mathbf{M}^{(2)}[Z_1] = \sqrt{2}$.

Let us turn our attention to the concavity constant $\mathbf{M}_{(2)}[Z_1]$. Still with $f_1, \ldots, f_n \in Z_1$ and $n \in \mathbb{N}$, we have that

$$\begin{aligned}\Big(\sum_{j=1}^n \|f_j\|_{Z_1}^2\Big)^{1/2} &= \Big(\sum_{j=1}^n \Big(\|f_j\chi_A\|_{L^2(\mu)} + \|f_j\chi_B\|_{L^2(\mu)}\Big)^2\Big)^{1/2} \\ &\le \Big(\sum_{j=1}^n \|f_j\chi_A\|_{L^2(\mu)}^2\Big)^{1/2} + \Big(\sum_{j=1}^n \|f_j\chi_B\|_{L^2(\mu)}^2\Big)^{1/2} \\ &= \Big\|\Big(\sum_{j=1}^n |f_j\chi_A|^2\Big)^{1/2}\Big\|_{L^2(\mu)} + \Big\|\Big(\sum_{j=1}^n |f_j\chi_B|^2\Big)^{1/2}\Big\|_{L^2(\mu)}\end{aligned}$$

by the Cauchy-Schwarz inequality in $\mathbb{R}^n$ and (2.161) applied with $\eta := \mu$ and $q := 2$. On the other hand,

$$\begin{aligned}&\Big\|\Big(\sum_{j=1}^n |f_j\chi_A|^2\Big)^{1/2}\Big\|_{L^2(\mu)} + \Big\|\Big(\sum_{j=1}^n |f_j\chi_B|^2\Big)^{1/2}\Big\|_{L^2(\mu)} \\ &= \Big\|\Big(\sum_{j=1}^n |f_j|^2\Big)^{1/2}\chi_A\Big\|_{L^2(\mu)} + \Big\|\Big(\sum_{j=1}^n |f_j|^2\Big)^{1/2}\chi_B\Big\|_{L^2(\mu)} = \Big\|\Big(\sum_{j=1}^n |f_j|^2\Big)^{1/2}\Big\|_{Z_1}\end{aligned}$$

via the definition of $\|\cdot\|_{Z_1}$. Therefore,

$$\Big(\sum_{j=1}^n \|f_j\|_{Z_1}^2\Big)^{1/2} \le \Big\|\Big(\sum_{j=1}^n |f_j|^2\Big)^{1/2}\Big\|_{Z_1},$$

which implies that $\mathbf{M}_{(2)}[Z_1] \le 1$. Recalling the general fact from (2.107) that $\mathbf{M}_{(2)}[Z_1] \ge 1$, we have $\mathbf{M}_{(2)}[Z_1] = 1$.

(ii) Let Z_∞ denote $L^2(\mu)$ equipped with the non-negative functional $\|\cdot\|_{Z_\infty}$ defined by

$$\|f\|_{Z_\infty} := \max\Big\{\|f\chi_A\|_{L^2(\mu)},\ \|f\chi_B\|_{L^2(\mu)}\Big\}, \qquad f \in Z_\infty.$$

It is clear that $\|\cdot\|_{Z_\infty}$ is a lattice norm. Moreover, $\|\cdot\|_{Z_\infty}$ is equivalent to $\|\cdot\|_{L^2(\mu)}$ because

$$\|f\|_{Z_\infty} \le \|f\|_{L^2(\mu)} \le \sqrt{2}\|f\|_{Z_\infty}, \qquad f \in L^2(\mu) = Z_\infty.$$

Hence, $(Z_\infty, \|\cdot\|_{Z_\infty})$ is 2-convex and 2-concave. One can prove that

$$\mathbf{M}^{(2)}[Z_\infty] = 1 \quad \text{and} \quad \mathbf{M}_{(2)}[Z_\infty] = \sqrt{2}$$

by adapting the arguments in part (i). □

Let us show how convexity and concavity properties alter when forming the p-th power of a q-B.f.s. For a real q-B.f.s, the following result has been given in [30, Lemma 2].

Proposition 2.75. *Let $X(\mu)$ be a q-B.f.s. over a positive, finite measure space (Ω, Σ, μ). Suppose that $0 < p < \infty$ and $0 < q < \infty$.*

(i) *If $X(\mu)$ is q-convex, then its p-th power $X(\mu)_{[p]}$ is (q/p)-convex and*

$$\mathbf{M}^{(q/p)}\big[X(\mu)_{[p]}\big] = \Big(\mathbf{M}^{(q)}[X(\mu)]\Big)^p.$$

(ii) *If $X(\mu)$ is q-concave, then its p-th power $X(\mu)_{[p]}$ is (q/p)-concave and*

$$\mathbf{M}_{(q/p)}\big[X(\mu)_{[p]}\big] = \Big(\mathbf{M}_{(q)}[X(\mu)]\Big)^p.$$

Proof. (i) Let $f_1, \ldots, f_n \in X(\mu)_{[p]}$ with $n \in \mathbb{N}$. Since $\big|f_j\big|^{1/p} \in X(\mu)$ for $j = 1, \ldots, n$, it follows from (2.54) that $\big(\sum_{j=1}^n \big(|f_j|^{1/p}\big)^q\big)^{1/q} \in X(\mu)$ and hence, that

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^n |f_j|^{q/p}\Big)^{p/q}\Big\|_{X(\mu)_{[p]}} &= \Big\|\Big(\sum_{j=1}^n \big(\big|f_j\big|^{1/p}\big)^q\Big)^{1/q}\Big\|_{X(\mu)}^p \\ &\le \left(\mathbf{M}^{(q)}[X(\mu)]\ \Big(\sum_{j=1}^n \big\|\,\big|f_j\big|^{1/p}\big\|_{X(\mu)}^q\Big)^{1/q}\right)^p \\ &= \Big(\mathbf{M}^{(q)}[X(\mu)]\Big)^p \Big(\sum_{j=1}^n \big\|f_j\big\|_{X(\mu)_{[p]}}^{q/p}\Big)^{p/q}.\end{aligned}$$

Therefore, $X(\mu)_{[p]}$ is (q/p)-convex and

$$\mathbf{M}^{(q/p)}\big[X(\mu)_{[p]}\big] \ \le\ \Big(\mathbf{M}^{(q)}[X(\mu)]\Big)^p. \tag{2.165}$$

To prove the reverse inequality to (2.165), let $Y(\mu) := X(\mu)_{[p]}$. Then $X(\mu)$ is the $(1/p)$-th power of the (q/p)-convex space $Y(\mu)$. Apply (2.165), with $Y(\mu)$ in place of $X(\mu)$ and with q in place of (q/p), to deduce that

$$\mathbf{M}^{(q)}[X(\mu)] = \mathbf{M}^{(p\cdot q/p)}\big[Y(\mu)_{[1/p]}\big] \le \Big(\mathbf{M}^{(q/p)}[Y(\mu)]\Big)^{1/p} = \Big(\mathbf{M}^{(q/p)}\big[X(\mu)_{[p]}\big]\Big)^{1/p},$$

that is,

$$\Big(\mathbf{M}^{(q)}[X(\mu)]\Big)^p \ \le\ \mathbf{M}^{(q/p)}\big[X(\mu)_{[p]}\big].$$

This establishes the desired identity $\mathbf{M}^{(q/p)}\big[X(\mu)_{[p]}\big] = \Big(\mathbf{M}^{(q)}[X(\mu)]\Big)^p$.

(ii) To prove that $X(\mu)_{[p]}$ is (q/p)-concave, fix $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X(\mu)_{[p]}$. Since $\left|f_j\right|^{1/p} \in X(\mu)$ for $j = 1, \dots, n$, we have that

$$\begin{aligned}
\Big(\sum_{j=1}^{n} \|f_j\|_{X(\mu)_{[p]}}^{q/p}\Big)^{p/q} &= \Big(\sum_{j=1}^{n} \|\,|f_j|^{1/p}\|_{X(\mu)}^{q}\Big)^{p/q} \\
&\le \bigg(\Big(\mathbf{M}_{(q)}[X(\mu)]\Big)\,\Big\|\Big(\sum_{j=1}^{n} \big(\left|f_j\right|^{1/p}\big)^{q}\Big)^{1/q}\Big\|_{X(\mu)}\bigg)^{p} \\
&= \Big(\mathbf{M}_{(q)}[X(\mu)]\Big)^{p}\,\Big\|\Big(\sum_{j=1}^{n} \left|f_j\right|^{q/p}\Big)^{p/q}\Big\|_{X(\mu)_{[p]}}.
\end{aligned}$$

Consequently, $X(\mu)_{[p]}$ is (q/p)-concave and

$$\mathbf{M}_{(q/p)}\big[X(\mu)_{[p]}\big] \;\le\; \Big(\mathbf{M}_{(q)}[X(\mu)]\Big)^{p}. \tag{2.166}$$

To prove the reverse inequality, take $Y(\mu) = X(\mu)_{[p]}$ as above. Recalling that $X(\mu)$ is the $(1/p)$-th power of the (q/p)-concave space $Y(\mu)$, we can apply (2.166), with $Y(\mu)$ in place of $X(\mu)$ and with q in place of (q/p), to deduce that

$$\mathbf{M}_{(q)}[X(\mu)] = \mathbf{M}_{(p\cdot q/p)}\big[Y(\mu)_{[1/p]}\big] \le \Big(\mathbf{M}_{(q/p)}[Y(\mu)]\Big)^{1/p} = \Big(\mathbf{M}_{(q/p)}\big[X(\mu)_{[p]}\big]\Big)^{1/p},$$

that is,

$$\Big(\mathbf{M}_{(q)}[X(\mu)]\Big)^{p} \le\; \mathbf{M}_{(q/p)}\big[X(\mu)_{[p]}\big].$$

Thereby the desired identity $\mathbf{M}_{(q/p)}\big[X(\mu)_{[p]}\big] \;=\; \Big(\mathbf{M}_{(q)}[X(\mu)]\Big)^{p}$ is established. □

In the case when $1 < p < \infty$, Proposition 2.75 implies that the p-th power $X(\mu)_{[p]}$ has a weaker concavity property than $X(\mu)$ and has a stronger convexity property than $X(\mu)$; see Corollary 2.55.

Let us introduce the Lorentz function spaces.

Example 2.76. Let (Ω, Σ, μ) be a non-atomic, positive, finite measure space. Given $f \in L^0_{\mathbb{R}}(\mu)$, let us adopt the abbreviation

$$\mu\big(|f| > c\big) :=\; \mu\big(\{\omega \in \Omega : |f(\omega)| > c\}\big), \qquad c > 0.$$

Define a function $f^* : \big[0,\, \mu(\Omega)\big) \to [0, \infty)$ by

$$f^*(t) :=\; \inf\big\{c > 0 : \mu\big(|f| > c\big) \le t\big\}, \qquad t \in \big(0,\, \mu(\Omega)\big].$$

The function f^*, which is called the *decreasing rearrangement* of f, is right-continuous and decreasing, [13, Ch. 2, Proposition 1.7], [69, Proposition 1.4.5].

(i) Fix $n \in \mathbb{N}$ and pairwise disjoint non-μ-null sets $A_1, \dots, A_n \in \Sigma$. Given a strictly decreasing finite sequence of positive numbers $\{a_j\}_{j=1}^n$, let

$$b_j := \sum_{k=1}^{j} \mu(A_k) \in \big(0, \mu(\Omega)\big] \quad \text{for} \quad j = 1, \dots, n.$$

If $n = 1$, then

$$\big(a_1 \chi_{A_1}\big)^* = a_1 \chi_{(0,\, b_1)} \qquad \text{on} \quad \big(0,\, \mu(\Omega)\big]$$

and if $n \geq 2$, then

$$\Big(\sum_{j=1}^{n} a_j \chi_{A_j}\Big)^* = a_1 \chi_{(0,\, b_1)} + \sum_{j=2}^{n} a_j \chi_{[b_{j-1},\, b_j)}, \qquad \text{on} \quad \big(0,\, \mu(\Omega)\big];$$

see [13, p. 38], [69, p. 46].

(ii) Given $0 < q < \infty$ and $0 < r < \infty$, the real *Lorentz space* $L^{q,r}_{\mathbb{R}}(\mu)$ is defined as the space of all $f \in L^0_{\mathbb{R}}(\mu)$ satisfying

$$\|f\|_{q,r} := \Big(\frac{r}{q} \int_0^{\mu(\Omega)} \big|f^*(t)\big|^r\, t^{(r/q)-1} dt\Big)^{1/r} < \infty;$$

see [13, Ch. 4, §4], [69, Ch. 1, §1.4], for example. It turns out that $L^{q,r}_{\mathbb{R}}(\mu)$ is an order ideal of the real vector lattice $L^0_{\mathbb{R}}(\mu)$ and the functional $\|\cdot\|_{q,r} : L^{q,r}_{\mathbb{R}}(\mu) \to [0,\infty)$ is a real lattice quasi-norm for which $L^{q,r}_{\mathbb{R}}(\mu)$ is complete; see for example [69, Theorem 1.4.11]. By part (i), all positive Σ-simple functions belong to $L^{q,r}_{\mathbb{R}}(\mu)$ and hence, so do all real Σ-simple functions. Thus, $(L^{q,r}_{\mathbb{R}}(\mu), \|\cdot\|_{q,r})$ is a real q-B.f.s. based on (Ω, Σ, μ).

The (complex) Lorentz space $L^{q,r}(\mu)$ is defined as the complexification

$$L^{q,r}(\mu) := L^{q,r}_{\mathbb{R}}(\mu) + i L^{q,r}_{\mathbb{R}}(\mu)$$

and becomes a q-B.f.s. over (Ω, Σ, μ) with respect to the lattice quasi-norm

$$f \mapsto \|f\|_{L^{q,r}(\mu)} := \big\| |f| \big\|_{q,r}, \qquad f \in L^{q,r}(\mu);$$

see the arguments after Lemma 2.4 or [69, Ch. 1, §1.4].

If $1 \leq r \leq q < \infty$, then $\|\cdot\|_{L^{q,r}(\mu)}$ is a lattice norm with respect to which $L^{q,r}(\mu)$ is a B.f.s., [13, Ch. 4, Theorem 4.3]. The lattice quasi-norm $\|\cdot\|_{L^{q,r}(\mu)}$ fails the triangle inequality if we omit the requirement that $r \leq q$, [69, p. 50]. However, for $q > 1$ and all $r \geq 1$ there is a lattice norm $\|\cdot\|_{(q,r)}$ such that $L^{q,r}(\mu)$ is a B.f.s. relative to $\|\cdot\|_{(q,r)}$, [13, Ch. 4, Theorem 4.6], [69, pp. 64–65]. For the remaining possibilities of $0 < q, r < \infty$, the q-B.f.s. $(L^{q,r}(\mu), \|\cdot\|_{L^{q,r}(\mu)})$ is non-normable. For instance, the case $0 < r < q < 1$ can be proved by (ii-c) and (vi) below; see [69, Ch. 1].

We now consider some special cases which will be needed later.

(ii-a) When $q > 0$ and $r = q$, we have $L^{q,q}(\mu) = L^q(\mu)$ with $\|\cdot\|_{L^{q,q}(\mu)} = \|\cdot\|_{L^q(\mu)}$ for $q < 1$, [13, p. 216].

(ii-b) With $0 < q < \infty$ and $0 < r_1 < r_2 < \infty$, we have

$$L^{q,r_1}(\mu) \subseteq L^{q,r_2}(\mu);$$

see [69, Proposition 1.4.10].

(ii-c) Now, if $0 < r < q < \infty$, then (ii-a) and (ii-b) above yield that

$$L^{q,r}(\mu) \subseteq L^{q,q}(\mu) = L^q(\mu).$$

(iii) Let us return to the general case when $0 < q < \infty$ and $0 < r < \infty$. Then $L^{q,r}(\mu)$ is σ-o.c. In fact, suppose that $f \in L^{q,r}(\mu)^+$ and take functions $f_n \in L^{q,r}(\mu)^+$, for $n \in \mathbb{N}$, satisfying $f_n \uparrow f$. Then $f_n^* \uparrow f^*$ pointwise on $\big(0, \mu(\Omega)\big]$; for example, see [13, Ch. 2, Proposition 1.7], [69, Proposition 1.4.5], [154, Lemma 3.5]. So, $\|f - f_n\|_{L^{q,r}(\mu)} = \|f - f_n\|_{q,r} \to 0$ as $n \to \infty$ by the Monotone Convergence Theorem and the definition of $\|\cdot\|_{q,r}$. In other words, $L^{q,r}(\mu)$ is σ-o.c.

(iv) Given $0 < q < \infty$ and $0 < r < \infty$, we shall show that the identity

$$L^{q,r}(\mu)_{[p]} = L^{(q/p),(r/p)}(\mu) \tag{2.167}$$

holds when $1 \le p < \infty$; for the case of real spaces, this is stated in [30, p. 159]. By recalling the definition of the p-th power of a q-B.f.s. as given in (2.46), the identity (2.167) will follow once we establish the fact that a function $f \in L^0_{\mathbb{R}}(\mu)^+$ belongs to $L^{q,r}(\mu)_{[p]}$ if and only if $f \in L^{(q/p),(r/p)}(\mu)$. So, fix $f \in L^0_{\mathbb{R}}(\mu)^+$. According to [13, Ch. 2, Proposition 1.7], we have $(f^*)^{1/p} = (f^{1/p})^*$ on $\big(0, \mu(\Omega)\big]$. It follows that

$$\Big(\frac{r}{q}\int_0^{\mu(\Omega)} \big[f^*(t)\big]^{r/p} t^{(r/q)-1} dt\Big)^{p/r} = \Big[\Big(\frac{r}{q}\int_0^{\mu(\Omega)} \big[(f^{1/p})^*(t)\big]^r t^{(r/q)-1} dt\Big)^{1/r}\Big]^p,$$

which implies that $f \in L^{(q/p),(r/p)}(\mu)$ if and only if $f^{1/p} \in L^{q,r}(\mu)$. Since $f^{1/p} \in L^{q,r}(\mu)$ is equivalent to $f \in L^{q,r}(\mu)_{[p]}$, we have established the fact that $f \in L^{(q/p),(r/p)}(\mu)$ if and only if $f \in L^{q,r}(\mu)_{[p]}$. So, (2.167) holds.

(v) Let us discuss convexity and concavity properties of $L^{q,r}(\mu)$ for $0 < q < \infty$ and $0 < r < \infty$. It follows from [30, p. 159] that the real Lorentz space $L^{q,r}_{\mathbb{R}}(\mu)$ is u-convex provided $0 < u < q$ and $0 < u \le r$. So, $L^{q,r}(\mu)$ is also u-convex via Lemma 2.49(ii).

(vi) Given $0 < r < 1$, we shall show that the Lorentz space $L^{1,r}(\mu)$ is *not* 1-convex. To make our arguments more transparent, assume further that $\mu(\Omega) = 1$ because the general case can be derived from this special case. Fixing $n \in \mathbb{N}$ we shall construct functions $f_1, \ldots, f_n \in L^{1,r}(\mu)^+$ satisfying

$$\Big\|\sum_{j=1}^n f_j\Big\|_{L^{1,r}(\mu)} = \Big(1 + (n-2)\cdot(2^r - 1)\Big)^{1/r}, \tag{2.168}$$

which will then be used to verify that $L^{1,r}(\mu)$ is not 1-convex. Since μ is non-atomic, we can select pairwise disjoint sets $A_1, \ldots, A_n \in \Sigma$ such that

$$\mu(A_1) = 1/2^n \quad \text{and} \quad \mu(A_k) = 1/2^{(n-k+2)} \quad \text{for} \quad k = 2, \ldots, n. \tag{2.169}$$

Then

$$\sum_{k=1}^{j} \mu(A_k) = 1/2^{(n-j+1)}, \qquad j = 1, \ldots, n. \tag{2.170}$$

Given $j = 1, \ldots, n$, we have

$$\|\chi_{A_j}\|_{L^{1,r}(\mu)} = \Big(r\int_0^{\mu(A_j)} t^{r-1}dt\Big)^{1/r} = \mu(A_j)$$

because $(\chi_{A_j})^* = \chi_{(0,\,\mu(A_j))}$ on $[0,1]$ via part (i). Let $f_j := \mu(A_j)^{-1}\chi_{A_j}$, so that $\|f_j\|_{L^{1,r}(\mu)} = 1$. Now, part (i), (2.169) and (2.170) give

$$\begin{aligned}
\Big(\sum_{j=1}^{n} f_j\Big)^* &= \mu(A_1)^{-1}\chi_{(0,1/2^n)} + \mu(A_2)^{-1}\chi_{[1/2^n,\,1/2^{(n-1)})} + \cdots \\
&\qquad\qquad \cdots + \mu(A_n)^{-1}\chi_{[1/2^2,\,1/2)} \\
&= 2^n\chi_{(0,\,1/2^n)} + 2^n\chi_{[1/2^n,\,1/2^{(n-1)})} + \cdots + 2^2\chi_{[1/2^2,\,1/2)} \\
&= 2^n\chi_{(0,\,1/2^n)} + \sum_{k=2}^{n} 2^k\chi_{[1/2^k,\,1/2^{(k-1)})}.
\end{aligned}$$

So, we have

$$\begin{aligned}
\Big\|\sum_{j=1}^{n} f_j\Big\|_{L^{1,r}(\mu)}^r &= r\int_0^1 \Big[\Big(\sum_{j=1}^{n} f_j\Big)^*(t)\Big]^r t^{r-1}dt \\
&= \int_0^{1/2^n} r\,2^{nr}t^{r-1}dt + \sum_{k=2}^{n}\int_{1/2^k}^{1/2^{(k-1)}} r\,2^{kr}t^{r-1}\,dt \\
&= 1 + \sum_{k=2}^{n} 2^{kr}\cdot 2^{-kr}(2^r-1) \\
&= 1 + (n-2)\cdot(2^r-1),
\end{aligned}$$

from which (2.168) follows.

To prove that $L^{1,r}(\mu)$ is not 1-convex, assume on the contrary that $L^{1,r}(\mu)$ is 1-convex. It then follows from Proposition 2.23(iv) that $L^{1,r}(\mu)$ admits an equivalent lattice norm $\|\!|\cdot|\!\|_{L^{1,r}(\mu)}$. That is, there exist constants $a, b > 0$ for which

$$a|\!|\!|f|\!|\!|_{L^{1,r}(\mu)} \le \|f\|_{L^{1,r}(\mu)} \le b|\!|\!|f|\!|\!|_{L^{1,r}(\mu)}, \qquad f \in L^{1,r}(\mu).$$

Since $\|f_j\|_{L^{1,r}(\mu)} = 1$ for $j = 1, \ldots, n$, it follows that

$$\Big\| \sum_{j=1}^{n} f_j \Big\|_{L^{1,r}(\mu)} \leq b \Big\| \sum_{j=1}^{n} f_j \Big\|_{L^{1,r}(\mu)}$$
$$\leq b \sum_{j=1}^{n} \|f_j\|_{L^{1,r}(\mu)} \leq \big(b/a\big) \sum_{j=1}^{n} \|f_j\|_{L^{1,r}(\mu)} = \big(b/a\big) n.$$

This together with (2.168) give

$$\Big(1 + (n-2)\cdot(2^r - 1)\Big)^{1/r} \leq \big(b/a\big) n,$$

which is impossible because $(1/r) > 1$ and $n \in \mathbb{N}$ is arbitrarily fixed. Therefore, we conclude that $(L^{1,r}(\mu), \|\cdot\|_{L^{1,r}(\mu)})$ is not 1-convex.

(vii) Suppose that $0 < r < q < \infty$. Then $L^{q,r}(\mu)$ is not q-convex. In fact, if $L^{q,r}(\mu)$ were q-convex, then $L^{q,r}(\mu)_{[q]}$ would be 1-convex via Proposition 2.75(i). However, $L^{q,r}(\mu)_{[q]} = L^{1,r/q}(\mu)$ via part (iv) and $L^{1,r/q}(\mu)$ is not 1-convex via part (vi). Thus, $L^{q,r}(\mu)$ is not q-convex. □

Proposition 2.77. *Let E be a Banach lattice.*

(i) *For each $0 < q \leq 1$, we have that E is necessarily q-convex and*

$$\mathbf{M}^{(q)}[E] = 1. \tag{2.171}$$

(ii) *Suppose that E is infinite-dimensional. Then E fails to be q-concave for every $0 < q < 1$.*

Proof. (i) By Lemma 2.50 applied to the identity operator id_E on E and with $W := E$, we have that id_E is necessarily 1-convex, that is, E is 1-convex and that $\mathbf{M}^{(1)}[E] = \mathbf{M}^{(1)}[\mathrm{id}_E] = 1$. Now let $0 < q < 1$. From the fact that id_E is 1-convex and Proposition 2.54(ii), it follows that id_E is q-convex and $M^{(q)}[E] \leq M^{(1)}[E] = 1$. On the other hand, $M^{(q)}[E] \geq 1$ via (2.107), which establishes (2.171).

(ii) Assume, on the contrary, that E is q-concave for some $0 < q < 1$. Then E is also 1-concave via Corollary 2.55(ii). This, together with part (i), imply that E is both 1-convex and 1-concave. Then, so is its real part $E_{\mathbb{R}}$; see Lemma 2.49. This enables us to assume that $E_{\mathbb{R}}$ equals the space $L^1_{\mathbb{R}}(\eta)$ of all $\mathbb{R}$-valued functions integrable with respect to some $[0,\infty]$-valued measure η on some measurable space (Ω, Σ) and that the norm $\|\cdot\|_{E_{\mathbb{R}}}$ is equivalent to the usual L^1-norm. Indeed, according to [99, p. 59], there exist such a measure η and a surjective (real) isomorphism $R : E_{\mathbb{R}} \to L^1_{\mathbb{R}}(\eta)$ which preserves the lattice operations.

Now, $E_{\mathbb{R}} = L^1_{\mathbb{R}}(\eta)$ being infinite-dimensional, there exist non-η-null, pairwise disjoint sets $A(j) \in \Sigma$, for $j \in \mathbb{N}$, such that $\chi_{A(j)} \in L^1_{\mathbb{R}}(\eta) = E_{\mathbb{R}}$ for every $j \in \mathbb{N}$. So, we can apply Example 2.73(ii-b) to see that $L^1(\eta)$ is not q-concave and hence, neither is its real part $L^1_{\mathbb{R}}(\eta) = E_{\mathbb{R}}$. Therefore, E is not q-concave. □

The following result, eventually to be applied to the L^p-space of a vector measure (see Proposition 3.28(i) of Chapter 3), is a consequence of Propositions 2.75 and 2.77.

Corollary 2.78. *Let $X(\mu)$ be a B.f.s. over a positive, finite measure space (Ω, Σ, μ). Given $1 < p < \infty$, the $(1/p)$-power $X(\mu)_{[1/p]}$ of $X(\mu)$ is a p-convex B.f.s. with p-convexity constant 1.*

Proof. That $X(\mu)_{[1/p]}$ is a B.f.s. follows from Proposition 2.23(i) and that $X(\mu)$ is 1-convex follows from Proposition 2.77(i) with $E = X(\mu)$. By Proposition 2.75 (with $q := 1$ and with $(1/p)$ in place of p), the $(1/p)$-th power $X(\mu)_{[1/p]}$ is p-convex and $\mathbf{M}^{(p)}\big[X(\mu)_{[1/p]}\big] = \Big(\mathbf{M}^{(1)}[X(\mu)]\Big)^{1/p}$. On the other hand, recall that $\mathbf{M}^{(1)}[X(\mu)]=1$ via Proposition 2.77(i) with $E:=X(\mu)$. Thus, $\mathbf{M}^{(p)}\big[X(\mu)_{[1/p]}\big]=1$. □

By Proposition 2.77(ii), the identity operator on any infinite-dimensional Banach lattice fails to be q-concave whenever $0 < q < 1$. A natural question arises of whether or not there exists any q-concave operator (on some Banach lattice) when $0 < q < 1$. The following example provides an affirmative answer (for $q = 1/2$); it will be generalized in Lemma 2.80.

Example 2.79. Let $\mu : 2^{\mathbb{N}} \to [0, \infty)$ be any finite measure such that $\mu(\{n\}) > 0$ for each $n \in \mathbb{N}$ and define

$$\varphi(n) := \mu(\{n\}) < \infty, \qquad n \in \mathbb{N}. \tag{2.172}$$

In particular, the function φ on $\mathbb{N}$ belongs to ℓ^1. Fix $g \in \ell^1$.

Then $g \in \mathcal{M}\big(\ell^2(\mu), \ell^1(\mu)\big)$ because $g \in \ell^\infty$ and so $g \cdot \ell^2(\mu) \subseteq \ell^2(\mu) \subseteq \ell^1(\mu)$.

(i) We shall show that $M_g : \ell^2(\mu) \to \ell^1(\mu)$ is $(1/2)$-concave. It follows that

$$\big\|gf\big\|_{\ell^1(\mu)} = \big\|gf\varphi\big\|_{\ell^1} \le \Big(\sum_{k=1}^{\infty} |g(k)f(k)\varphi(k)|^{1/2}\Big)^2, \qquad f \in \ell^2(\mu), \tag{2.173}$$

because of the definition of $\|\cdot\|_{\ell^1(\mu)}$ and the inequality $\sum_{k=1}^\infty |a_k| \le \big(\sum_{k=1}^\infty |a_k|^{1/2}\big)^2$ for every complex sequence $\{a_k\}_{k=1}^\infty$. Fix $n \in \mathbb{N}$ and $f_1, \dots, f_n \in \ell^2(\mu)$. Then, (2.173) implies that

$$\begin{aligned}
&\Big(\sum_{j=1}^{n}\big\|M_g(f_j)\big\|_{\ell^1(\mu)}^{1/2}\Big)^2 = \Big(\sum_{j=1}^{n}\big\|gf_j\big\|_{\ell^1(\mu)}^{1/2}\Big)^2 \\
&\le \Big(\sum_{j=1}^{n}\sum_{k=1}^{\infty}\big|g(k)f_j(k)\varphi(k)\big|^{1/2}\Big)^2 \\
&= \left(\sum_{k=1}^{\infty}\Big[\Big(\sum_{j=1}^{n}|f_j(k)|^{1/2}\Big)\big|\varphi(k)\big|^{1/2}\Big]\big|g(k)\big|^{1/2}\right)^2
\end{aligned}$$

$$\le \Big(\sum_{k=1}^{\infty} \Big[\Big(\sum_{j=1}^{n} |f_j(k)|^{1/2}\Big)^2 \varphi(k)\Big]\Big)\Big(\sum_{k=1}^{\infty} |g(k)|\Big)$$
$$= \Big\|\Big(\sum_{j=1}^{n} |f_j|^{1/2}\Big)^2\Big\|_{\ell^1(\mu)} \cdot \|g\|_{\ell^1} \;\le\; \|g\|_{\ell^1} \cdot \|\chi_{\mathbb{N}}\|_{\ell^2(\mu)} \cdot \Big\|\Big(\sum_{j=1}^{n} |f_j|^{1/2}\Big)^2\Big\|_{\ell^2(\mu)},$$

where the Cauchy-Schwarz inequality is applied twice (once in ℓ^2 and once in $\ell^2(\mu)$). So, M_g is (1/2)-concave.

(ii) Let us also show that M_g is (1/2)-concave by factorizing it through $\ell^{1/2}(\mu)$. Such a factorization cannot be seen from the computation in part (i). First observe that

$$g/\varphi \in \ell^1(\mu) \subseteq \ell^{2/3}(\mu) = \mathcal{M}\big(\ell^2(\mu), \ell^{1/2}(\mu)\big) \tag{2.174}$$

because (2.80) gives that $\mathcal{M}\big(\ell^2(\mu), \ell^{1/2}(\mu)\big) = \ell^{2/3}(\mu)$. Next we show that

$$\varphi \in \mathcal{M}\big(\ell^{1/2}(\mu),\, \ell^1(\mu)\big). \tag{2.175}$$

For this, let $f \in \ell^{1/2}(\mu)$. Since $\varphi^2 \in \ell^\infty$ we have $\varphi(\varphi f) = \varphi^2 f \in \ell^{1/2} \subseteq \ell^1$, and hence, $\varphi f \in \ell^1(\mu)$. Thus, (2.175) holds. So, we can write

$$M_g = M_\varphi \circ M_{g/\varphi}$$

with $M_{g/\varphi} \in \mathcal{M}\big(\ell^2(\mu), \ell^{1/2}(\mu)\big)$ and $M_\varphi \in \mathcal{M}\big(\ell^{1/2}(\mu), \ell^1(\mu)\big)$; see (2.174) and (2.175). Thus, M_g factorizes through $\ell^{1/2}(\mu)$ according to the following diagram:

$$\begin{tikzcd}
\ell^2(\mu) \arrow[rr, "M_g"] \arrow[dr, "M_{g/\varphi}"'] & & \ell^1(\mu) \,. \\
& \ell^{1/2}(\mu) \arrow[ur, "M_\varphi"'] &
\end{tikzcd}$$

Now, recall from Example 2.73(i) that $\ell^{1/2}(\mu)$ is (1/2)-concave. So, it follows from Corollary 2.71 that the $\ell^{1/2}(\mu)$-valued operator $M_{g/\varphi}$ is (1/2)-concave. Now apply Proposition 2.68(i) to conclude that $M_g = M_\varphi \circ M_{g/\varphi}$ is (1/2)-concave. □

In the above example, the requirement that μ is purely atomic is crucial. In fact, if μ is non-atomic, then the dual space $L^{1/2}(\mu)^* = \{0\}$ (see Example 2.10) and so $\mathcal{L}\big(L^{1/2}(\mu),\, L^1(\mu)\big) = \{0\}$ (see Lemma 2.9). Thus, the only continuous linear operator from $L^2(\mu)$ into $L^1(\mu)$, which factorizes through $L^{1/2}(\mu)$, is the zero operator.

To generalize Example 2.79, let us clarify some terminology. Given are positive numbers q, r, u and a positive, finite measure μ on $2^{\mathbb{N}}$. For a function $g \in \mathcal{M}\big(\ell^r(\mu), \ell^u(\mu)\big)$, we say that the multiplication operator $M_g : \ell^r(\mu) \to \ell^u(\mu)$ *factorizes through* $\ell^q(\mu)$ *via multiplications* if there exist functions

$$g_1 \in \mathcal{M}\big(\ell^r(\mu), \ell^q(\mu)\big) \quad \text{and} \quad g_2 \in \mathcal{M}\big(\ell^q(\mu), \ell^u(\mu)\big)$$

such that $g = g_2 g_1$ or, equivalently, that $M_g = M_{g_2} \circ M_{g_1}$. So, the class of all functions $g \in \mathcal{M}\big(\ell^r(\mu), \ell^u(\mu)\big)$ for which M_g factorizes through $\ell^q(\mu)$ via multiplications is identical to the product

$$\begin{aligned} &\mathcal{M}\big(\ell^q(\mu), \ell^u(\mu)\big) \cdot \mathcal{M}\big(\ell^r(\mu), \ell^q(\mu)\big) \\ &\qquad := \Big\{ g_2 g_1 : g_2 \in \mathcal{M}\big(\ell^q(\mu), \ell^u(\mu)\big),\ g_1 \in \mathcal{M}\big(\ell^r(\mu), \ell^q(\mu)\big) \Big\}, \end{aligned} \tag{2.176}$$

represented schematically by the following diagram:

$$\begin{tikzcd} \ell^r(\mu) \arrow[rr, "M_g"] \arrow[dr, "M_{g_1}"'] & & \ell^u(\mu) . \\ & \ell^q(\mu) \arrow[ur, "M_{g_2}"'] & \end{tikzcd}$$

Lemma 2.80. *Given a finite measure $\mu : 2^{\mathbb{N}} \to [0,\infty)$ with $\mu(\{n\}) > 0$ for all $n \in \mathbb{N}$, let $\varphi : \mathbb{N} \to (0,\infty)$ be the function given by* (2.172). *Suppose that q, r, u are positive numbers.*

(i) *Let $q < r$. If $w > 0$ is the number determined by $(1/r) + (1/w) = (1/q)$, then $\mathcal{M}\big(\ell^r(\mu),\, \ell^q(\mu)\big) = \ell^w(\mu) = \varphi^{-(1/w)} \cdot \ell^w$.*

(ii) *If $q \le u$, then $\mathcal{M}\big(\ell^q(\mu), \ell^u(\mu)\big) = \varphi^{(1/q)-(1/u)} \cdot \ell^\infty$.*

(iii) *Let $q < r$ and $q \le u$. Given a function $g \in \mathcal{M}\big(\ell^r(\mu), \ell^u(\mu)\big)$, the multiplication operator $M_g : \ell^r(\mu) \to \ell^u(\mu)$ factorizes through $\ell^q(\mu)$ via multiplications if and only if $g \in \varphi^{(1/q)-(1/u)} \cdot \ell^w(\mu)$, in which case M_g is q-concave.*

Proof. (i) The first equality in (i) is a special case of (2.80) in Example 2.30(i). The fact that the multiplication operator $M_{\varphi^{1/w}} : \ell^w(\mu) \to \ell^w$ is a surjective linear isometry ensures the second equality.

(ii) Observe that $\varphi \in \ell^1$ and the multiplication operators $M_{\varphi^{1/q}} : \ell^q(\mu) \to \ell^q$ and $M_{\varphi^{-1/u}} : \ell^u \to \ell^u(\mu)$ are surjective linear isometries. Let

$$\mathcal{M}\big(\ell^q, \ell^s\big) := \big\{ f \in \mathbb{C}^{\mathbb{N}} : f \cdot \ell^q \subseteq \ell^s \big\}, \qquad 0 < s \le \infty. \tag{2.177}$$

Then

$$\mathcal{M}\big(\ell^q(\mu), \ell^u(\mu)\big) = \varphi^{(1/q)-(1/u)} \cdot \mathcal{M}(\ell^q, \ell^u). \tag{2.178}$$

On the other hand, $q \le u$ implies that

$$\ell^\infty \subseteq \mathcal{M}(\ell^q, \ell^u) \subseteq \mathcal{M}(\ell^q, \ell^\infty).$$

Since it is easy to verify that $\mathcal{M}(\ell^q, \ell^\infty) = \ell^\infty$, it follows that $\ell^\infty = \mathcal{M}(\ell^q, \ell^u)$. This and (2.178) establish part (ii).

(iii) The fact that $M_g : \ell^r(\mu) \to \ell^u(\mu)$ factorizes through $\ell^q(\mu)$ via multiplications if and only if $g \in \varphi^{(1/q)-(1/u)} \cdot \ell^w(\mu) = \varphi^{-(1/w)} \cdot \ell^w$ follows from the identities

$$\mathcal{M}\big(\ell^q(\mu), \ell^u(\mu)\big) \cdot \mathcal{M}\big(\ell^r(\mu), \ell^q(\mu)\big) = \big(\varphi^{(1/q)-(1/u)} \cdot \ell^\infty\big) \cdot \ell^w(\mu) = \varphi^{(1/q)-(1/u)} \cdot \ell^w(\mu)$$

because of parts (i), (ii) and the identity $\ell^\infty \cdot \big(\ell^w(\mu)\big) = \ell^w(\mu)$.

Assume now that $g \in \varphi^{(1/q)-(1/u)} \cdot \ell^w(\mu)$ and write $g = \varphi^{(1/q)-(1/u)} \cdot g_1$ for some $g_1 \in \ell^w(\mu)$. Since $\ell^q(\mu)$ is q-concave (via Example 2.73(i)), we can apply Corollary 2.71 to deduce that $M_{g_1} : \ell^r(\mu) \to \ell^q(\mu)$ is q-concave. Now, the function $g_2 := \varphi^{(1/q)-(1/u)}$ belongs to $\varphi^{(1/q)-(1/u)} \cdot \ell^\infty = \mathcal{M}\big(\ell^q(\mu), \ell^u(\mu)\big)$. In particular, M_{g_2} is continuous. So, it follows from Proposition 2.68(i) that $M_g = M_{g_2} \circ M_{g_1}$ is q-concave. $\square$

A natural question arises from (iii) in Lemma 2.80. Suppose that $0 < q < r < \infty$ and $q \le u < \infty$. If a function $g \in \mathcal{M}\big(\ell^r(\mu), \ell^u(\mu)\big)$ has the property that $M_g : \ell^r(\mu) \to \ell^u(\mu)$ is q-concave, does it follow that M_g factorizes through $\ell^q(\mu)$ via multiplications? By applying a Maurey–Rosenthal type theorem, we shall show in Chapter 6 that the answer is affirmative whenever $u \ge 1$; see Example 6.31(iii).

Now let the assumption be as in Example 2.79. We claim that the factorization in part (ii) of Example 2.79 is a special case of Lemma 2.80 with $q := 1/2$, $r := 2$ and $u := 1$. Observe, for this choice of r, q, that $w = 2/3$. For $g \in \ell^1$, in which case $g = \varphi \cdot (g/\varphi)$, we have $(g/\varphi) \in \ell^1(\mu) \subseteq \ell^{2/3}(\mu)$ and $\varphi \in \varphi^{2-1} \cdot \ell^\infty$ (as $(1/q) - (1/u) = 2 - 1$). So, Lemma 2.80(iii) provides the factorization $M_g = M_\varphi \circ M_{(g/\varphi)}$ through $\ell^{1/2}(\mu)$, which is exactly what is presented in Example 2.79(ii).

Chapter 3

Vector Measures and Integration Operators

It is evident from Chapter 1 that the theory of vector measures and their integration theory play a vital role in this text, both by providing various techniques and as a structural approach in general. Ever since the fundamental monographs [42], [86] appeared some 30 years ago, there has been an ever growing literature on this topic. The first aim of this chapter is to collect together those aspects of the existing theory that are relevant to our needs. Since many of these results are only formulated over real spaces, we will need to extend them (as also done in Chapter 2) to the setting of spaces over $\mathbb{C}$. In some cases this can be achieved by the usual complexification arguments but, for others, entirely new proofs need to be provided. In addition, many new results are also developed and thereby appear for the first time. This is particularly the case in relation to the spaces $L^p(\nu)$ and $L^p_{\mathrm{w}}(\nu)$, for $1 \leq p < \infty$ and ν a Banach-space-valued vector measure, and the associated integration operators defined on them. These results are essential for Chapter 5, where we develop the theory of p-th power factorable operators defined on suitable function spaces.

Particular emphasis is placed on identifying and characterizing certain "ideal properties" of the integration operator, such as compactness, weak compactness and complete continuity. The q-concavity of the Banach lattices $L^p(\nu)$ is analyzed in the final section, together with that of the associated integration operator. The collective properties of the integration operator will pervade all subsequent chapters, especially Chapters 5, 6 and 7 dealing with applications. In order to make more transparent some of the subtleties that occur between these properties (in relation to integration operators), we have included many relevant and non-trivial examples.

In conclusion, we wish to highlight the fact, for the reader with a particular interest in vector measures and integration, that the new material in this chapter alone provides a rich supply of open problems and topics which are worthy of further research.

3.1 Vector measures

Throughout this section, let E be a complex Banach space with norm $\|\cdot\|_E$ unless stated otherwise. We sometimes consider the case when E is a Banach lattice; this will be indicated explicitly. Our Banach lattices are over $\mathbb{C}$ unless otherwise stated (see [165, Exercise 100.15]). Let (Ω, Σ) be a measurable space, and let $\nu : \Sigma \to E$ be a *vector measure*, that is, it is a σ-additive set function. The *variation* measure of ν, denoted by $|\nu| : \Sigma \to [0, \infty]$, is defined as for scalar measures, [42, Ch. I, Definition 1.4]; it is the smallest $[0,\infty]$-valued measure dominating ν in the sense that $\|\nu(A)\|_E \le |\nu|(A)$ for every $A \in \Sigma$. Given $x^* \in E^*$, let $\langle \nu, x^* \rangle : \Sigma \to \mathbb{C}$ denote the scalar measure

$$\langle \nu, x^* \rangle : A \mapsto \langle \nu(A),\, x^* \rangle, \quad A \in \Sigma;$$

its variation measure $|\langle \nu, x^* \rangle| : \Sigma \to [0, \infty)$ is necessarily finite. The *semivariation* $\|\nu\|$ of ν is the set function defined by

$$\|\nu\|(A) := \sup_{x^* \in \mathbf{B}[E^*]} |\langle \nu, x^* \rangle|(A), \qquad A \in \Sigma. \tag{3.1}$$

Then, $\|\nu\|(A) < \infty$ for every $A \in \Sigma$, which follows, for example, from [42, Ch. I, Corollary 1.19]. A useful fact is that

$$\|\nu\|(A) \;=\; \sup \Big\| \sum_{j=1}^{n} a_j \cdot \nu(A_j) \Big\|_E, \qquad A \in \Sigma, \tag{3.2}$$

where the supremum is taken over all choices of scalars $a_j \in \mathbb{C}$ with $|a_j| \le 1$ $(j = 1, \ldots, n)$, Σ-partitions $\{A_j\}_{j=1}^n$ of A, and $n \in \mathbb{N}$. This is given in [42, Ch. I, Proposition 1.11] when the scalar field is real. To adapt the proof to the complex case we only need to extend the function $\operatorname{sgn} : \mathbb{R} \to [0, \infty)$ to $\mathbb{C}$ via the formulae $\operatorname{sgn} a := a/|a|$ for $a \in \mathbb{C} \setminus \{0\}$ and $\operatorname{sgn} 0 := 0$ so that $a = |a| \cdot \operatorname{sgn} a$ for every $a \in \mathbb{C}$; if $a \in \mathbb{R} \setminus \{0\}$, then $|a| = (1/\operatorname{sgn} a) \cdot a = (\operatorname{sgn} a) \cdot a$, which has been used in the proof of [42, Ch. I, Proposition 1.11]. It follows from (3.1) that

$$0 \;\le\; \|\nu\|(A) \;\le\; |\nu|(A) \;\le\; \infty, \qquad A \in \Sigma. \tag{3.3}$$

Another useful fact is that

$$\sup_{B \in \Sigma \cap A} \|\nu(B)\|_E \;\le\; \|\nu\|(A) \;\le\; 4 \sup_{B \in \Sigma \cap A} \|\nu(B)\|_E, \qquad A \in \Sigma; \tag{3.4}$$

see again [42, Ch. I, Proposition 1.11]. The semivariation $\|\nu\|$ may not be σ-additive. Indeed, $\|\nu\|$ is σ-additive if and only if $\|\nu\| = |\nu|$ on Σ because $|\nu|$ is the smallest scalar measure dominating ν. In this case we necessarily have that $|\nu|(A) < \infty$ for every $A \in \Sigma$, that is, ν has *finite variation.* To provide an example, we recall that a Banach-lattice-valued vector measure is called *positive* if its range lies in the positive cone. A complex Banach lattice E is said to be an *abstract L^1-space* if $\|x+y\|_E = \|x\|_E + \|y\|_E$ whenever $x, y \in E^+$ satisfy $x \wedge y = 0$, [94, §15, Definition 1]. It follows that

$$\|x+y\|_E \;=\; \|x\|_E \;+\; \|y\|_E, \qquad x, y \in E^+, \tag{3.5}$$

[94, §25, Theorem 1]. A typical example of an abstract L^1-space is the L^1-space of some $[0,\infty]$-valued measure. Conversely, every abstract L^1-space can be represented as such a space; for the details, see [94, Ch. 5, §15, Theorem 3].

Example 3.1. Let E be an abstract L^1-space and $\nu : \Sigma \to E$ be a *positive* vector measure. Whenever $\{A_j\}_{j=1}^n$ with $n \in \mathbb{N}$ is an arbitrary finite Σ-partition of $A \in \Sigma$, we have

$$\sum_{j=1}^{n} \big\|\nu(A_j)\big\|_E \;=\; \Big\|\sum_{j=1}^{n} \nu(A_j)\Big\|_E \;=\; \big\|\nu(A)\big\|_E \;\le\; \|\nu\|(A).$$

So, $|\nu|(A) \le \big\|\nu(A)\big\|_E \le \|\nu\|(A)$. This, together with (3.3), imply that

$$|\nu|(A) \;=\; \big\|\nu(A)\big\|_E \;=\; \|\nu\|(A), \qquad A \in \Sigma. \tag{3.6}$$

In other words, $|\nu| = \|\nu\|$ on Σ and $\|\nu\|$ is σ-additive. Note that ν necessarily has finite variation. $\square$

The Orlicz–Pettis Theorem, [42, Ch. I, Corollary 2.4], says that a Banach-space-valued, finitely additive set function $\eta : \Sigma \to E$ defined on a σ-algebra Σ is σ-additive if and only if it is scalarly σ-additive, that is, $\langle \eta, x^* \rangle : \Sigma \to \mathbb{C}$ is σ-additive for *every* $x^* \in E^*$. It would be useful if the σ-additivity of η were guaranteed by checking the scalar σ-additivity for a smaller subset than the whole space E^*. The following result, which we call the generalized Orlicz–Pettis Theorem, tells us when we can do so (at least in particular situations).

Lemma 3.2. *The following conditions for a Banach space E are equivalent.*

(i) *The Banach space E does not contain an isomorphic copy of ℓ^∞.*

(ii) *Given a measurable space (Ω, Σ), any finitely additive set function $\eta : \Sigma \to E$ and any total subset H of E^*, the set function η is σ-additive if and only if $\langle \eta, x^* \rangle : \Sigma \to \mathbb{C}$ is σ-additive for every $x^* \in H$.*

This result can be found in [40, Theorem 1.1]; see also [42, Ch. I, Corollary 4.7] and [161, § 0].

Let $\mathcal{R}(\nu)$ denote the range of the vector measure $\nu : \Sigma \to E$, that is,

$$\mathcal{R}(\nu) := \;\{\nu(A) : A \in \Sigma\}.$$

An important fact, given in [10, Theorem 2.9], [42, Ch. I, Corollary 2.7], is the following

Lemma 3.3. *The range of every Banach-space-valued vector measure is a relatively weakly compact set.*

Let $\mathcal{L}^0(\Sigma)$ denote the space of all $\mathbb{C}$-valued, Σ-measurable functions on Ω. According to [97, Definition 2.1], a function $f \in \mathcal{L}^0(\Sigma)$ is called *ν-integrable* if

(I-1) it is $\langle \nu, x^* \rangle$-integrable for every $x^* \in E^*$, and

(I-2) for every $A \in \Sigma$ there exists a unique element $\nu_f(A) \in E$ satisfying

$$\big\langle \nu_f(A),\, x^* \big\rangle \;=\; \int_A f\, d\langle \nu, x^* \rangle, \qquad\qquad x^* \in E^*.$$

The Orlicz–Pettis Theorem, [42, Ch. I, Corollary 2.4], ensures that the E-valued set function $\nu_f : A \mapsto \nu_f(A)$ on Σ is again a vector measure; it is called the *indefinite integral* of f relative to ν. We will also use the classical notation

$$\int_A f\, d\nu := \nu_f(A), \qquad A \in \Sigma.$$

Let $L^1(\nu)$ denote the space of all ν-integrable functions on Ω, equipped with the seminorm $\|\cdot\|_{L^1(\nu)}$ defined by

$$\|f\|_{L^1(\nu)} := \sup_{x^* \in \mathbf{B}[E^*]} \int_\Omega |f|\, d|\langle \nu, x^* \rangle|, \qquad f \in L^1(\nu). \tag{3.7}$$

According to [141, Theorem 6.13], applied to each scalar measure $\langle \nu_f, x^* \rangle$, for $x^* \in E^*$, it follows that

$$\|f\|_{L^1(\nu)} \;=\; \|\nu_f\|(\Omega), \qquad f \in L^1(\nu). \tag{3.8}$$

Each Σ-simple function $s : \Omega \to \mathbb{C}$ is ν-integrable in a natural way. In fact, if

$$s \;=\; \sum_{j=1}^{n} a_j \chi_{A_j} \in \operatorname{sim} \Sigma \tag{3.9}$$

for some $a_j \in \mathbb{C}$ and $A_j \in \Sigma$, $j = 1, \ldots, n$, and $n \in \mathbb{N}$, then s is ν-integrable with $\int_A s\, d\nu = \sum_{j=1}^n a_j \nu(A_j \cap A)$ for every $A \in \Sigma$. The fact that ν is σ-additive on the σ-algebra Σ guarantees that the integral $\int_A s\, d\nu$ does not depend on the choice of a_j's and A_j's ($j = 1, \ldots, n$ with $n \in \mathbb{N}$) satisfying (3.9). It follows from [97, Theorem 2.2] that every bounded function in $\mathcal{L}^0(\Sigma)$ is necessarily ν-integrable.

Every function $f \in L^1(\nu)$ satisfying $\|f\|_{L^1(\nu)} = 0$ is called *ν-null*. By $\mathcal{N}(\nu)$ we denote the subspace of $L^1(\nu)$ consisting of all ν-null functions. A set $A \in \Sigma$ is called ν-null if $\chi_A \in \mathcal{N}(\nu)$. The family of all ν-null sets is denoted by $\mathcal{N}_0(\nu)$. With $\mathcal{N}_0(|\nu|)$ denoting the family of all $|\nu|$-null sets, we have that $\mathcal{N}_0(\nu) = \mathcal{N}_0(|\nu|)$; in other words, the ν-null and $|\nu|$-null sets coincide.

Remark 3.4. The following observations will be useful:

(i) $A \in \Sigma$ is ν-null if and only if $\nu(\Sigma \cap A) := \{\nu(B \cap A) : B \in \Sigma\} = \{0\}$.

(ii) A function $f \in L^1(\nu)$ is ν-null if and only if $f^{-1}\big(\mathbb{C} \setminus \{0\}\big) \in \mathcal{N}_0(\nu)$. □

Unless stated otherwise, we identify $L^1(\nu)$ with its quotient space $L^1(\nu)/\mathcal{N}(\nu)$ so that the seminorm $\|\cdot\|_{L^1(\nu)}$ will also be identified with its associated quotient norm, as for the case of a scalar measure. Moreover, $L^1(\nu)$ is a vector lattice with respect to the ν-a.e. pointwise order and $\|\cdot\|_{L^1(\nu)}$ is then a lattice norm on $L^1(\nu)$.

In the following two theorems, we collect together some basic facts on ν-integrable functions and on the space $L^1(\nu)$.

Theorem 3.5. *Let $\nu : \Sigma \to E$ be a vector measure. The following assertions for a function $f \in \mathcal{L}^0(\Sigma)$ are equivalent.*

(i) *f is ν-integrable.*

(ii) *There exists a sequence $\{s_n\}_{n=1}^\infty \subseteq \operatorname{sim}\Sigma$ such that $s_n \to f$ pointwise as $n \to \infty$ and such that the sequence $\{\int_A s_n\, d\nu\}_{n=1}^\infty$ converges in E for every $A \in \Sigma$.*

(iii) *There exists a sequence $\{f_n\}_{n=1}^\infty \subseteq L^1(\nu)$ such that $f_n \to f$ pointwise as $n \to \infty$ and such that the sequence $\big\{\int_A f_n\, d\nu\big\}_{n=1}^\infty$ converges in E for every $A \in \Sigma$.*

If (ii) $\big($*resp.* (iii)$\big)$ *holds, then*

$$\int_A f\, d\nu = \lim_{n\to\infty} \int_A s_n\, d\nu$$

$\big($*resp.* $\int_A f\, d\nu = \lim_{n\to\infty} \int_A f_n\, d\nu\big)$ *for every $A \in \Sigma$.*

Proof. This has essentially been given in [97, Theorem 2.4] which does not include part (iii) above but, whose arguments can easily be adapted to show the equivalence of (iii) with the others. □

Remark 3.6. (i) The above theorem asserts that we can also define ν-integrability of a Σ-measurable function f via part (ii). In fact, this is exactly the original definition employed by R. Bartle, N. Dunford and J. Schwartz, [10, Definition 2.5]. Accordingly, some authors call “our” integral (as given in (I-1) and (I-2)) the Bartle–Dunford-Schwartz integral.

(ii) It is clear that the pointwise convergence $s_n \to f$ in (ii) (resp. $f_n \to f$ in (iii)) can be relaxed to μ-a.e. pointwise convergence. □

A finite measure $\mu : \Sigma \to [0, \infty)$ is called a *control measure* for a vector measure $\nu : \Sigma \to E$ if μ and ν are mutually absolutely continuous, that is, $\mu(A) \to 0$ if and only if $\nu(A) \to 0$. By a theorem of B.J. Pettis, this is equivalent to the identity $\mathcal{N}_0(\mu) = \mathcal{N}_0(\nu)$, [42, Ch. I, Theorem 2.1]. For the existence of control measures, see [10, Corollary 2.4]. Our definition of control measure differs from that in [42, p. 11]. It turns out (see Theorem 3.7 below) that $L^1(\nu)$ is a B.f.s. based on (Ω, Σ, μ), so that the constant function χ_Ω is a weak order unit (by Proposition 2.2(v)). This important fact, first explicitly pointed out by G.P. Curbera [21], plays an important role in the study of vector measures and related operators.

There is a special class of control measures for ν. In fact, via Rybakov's Theorem, [42, Ch. IX, Theorem 2.2], there exists a linear functional $x^* \in E^*$ such that the scalar measure $|\langle \nu, x^*\rangle| : \Sigma \to [0, \infty)$ is a control measure for ν. We call such an x^* a *Rybakov functional.* The collection of all Rybakov functionals is denoted by $\mathbf{R}_\nu[E^*]$.

Theorem 3.7. *Let $\nu : \Sigma \to E$ be a vector measure.*

(i) (*Lebesgue Dominated Convergence Theorem*) *Let $g \in L^1(\nu)^+$. If $\{f_n\}_{n=1}^\infty$ is a sequence in $\mathcal{L}^0(\Sigma)$ converging ν-a.e. to a function $f \in \mathcal{L}^0(\Sigma)$ and if $|f_n| \le g$ (ν-a.e.) for all $n \in \mathbb{N}$, then f is ν-integrable and $\lim_{n\to\infty} f_n = f$ in the norm $\|\cdot\|_{L^1(\nu)}$.*

(ii) *The normed space $L^1(\nu)$ is complete. Furthermore, $L^1(\nu)$ is a weakly compactly generated Banach space in which $\operatorname{sim}\Sigma$ is dense.*

(iii) *Given any control measure μ for ν, the space $L^1(\nu) \subseteq L^0(\mu)$ is a B.f.s. based on the measure space (Ω, Σ, μ) and the norm $\|\cdot\|_{L^1(\nu)}$ is an o.c. lattice norm. In particular, the constant function χ_Ω is a weak order unit of $L^1(\nu)$.*

(iv) *Let $x^* \in \mathbf{R}_\nu[E^*]$. Then $L^1(\nu) \subseteq L^1\big(|\langle \nu, x^*\rangle|\big)$ as vector sublattices of $L^0\big(|\langle \nu, x^*\rangle|\big)$ and the natural inclusion map is continuous.*

Proof. (i) See [98, Theorem 2.2].

(ii) Completeness of $L^1(\nu)$ over $\mathbb{R}$ can be found in [86, Ch. IV] and in [58], [130] over $\mathbb{C}$, while denseness of $\operatorname{sim}\Sigma$ follows from part (i). That $L^1(\nu)$ is weakly compactly generated has been proved in [21, Theorem 2]. The proof there is based on the facts that the set function

$$[\nu] : A \mapsto \chi_A \in L^1(\nu), \qquad A \in \Sigma, \tag{3.10}$$

is σ-additive via part (i) and hence, has relatively weakly compact range by Lemma 3.3, and that $\operatorname{sim}\Sigma$ is dense in $L^1(\nu)$.

(iii) It is clear from (3.7) that the norm $\|\cdot\|_{L^1(\nu)}$ is a lattice norm. Parts (i) and (ii) yield that $L^1(\nu)$ is a B.f.s. (over (Ω, Σ, μ)) with σ-o.c. norm. Hence, $\|\cdot\|_{L^1(\nu)}$ is an o.c. lattice norm; see Remark 2.5.

(iv) Since every $f \in L^1(\nu)$ is necessarily $|\langle \nu, x^*\rangle|$-integrable and

$$\int_\Omega |f|\, d|\langle \nu, x^*\rangle| \le \|f\|_{L^1(\nu)}$$

(see (3.7)), part (iv) follows. $\square$

Part (iii) of the above theorem has been presented explicitly in [21, Theorem 1] with an emphasis on the Banach lattice structure of $L^1(\nu)$. Theorem 8 of the same paper [21] provides a sort of "converse" of the above part (iii) in the sense that every o.c. Banach lattice with a weak order unit can be realized as $L^1(\nu)$ for a suitable vector measure ν. So, we are dealing with a large class of Banach lattices by investigating $L^1(\nu)$. Before presenting this result, let us clarify some terminology. The order continuity of the norm of a general Banach lattice is defined as in the case of a q-B.f.s. (see Remark 2.5). So, a Banach lattice is o.c. if and only if its real part is. Let Z and W be Banach lattices which are the complexification of real Banach lattices $Z_{\mathbb{R}}$ and $W_{\mathbb{R}}$, respectively. We say that an injective bicontinuous linear operator $T : Z \to W$ is a *lattice isomorphism* onto its range if $T(Z_{\mathbb{R}}) \subseteq W_{\mathbb{R}}$ and if T preserves the lattice operations, that is,

$$T(z_1 \vee z_2) = T(z_1) \vee T(z_2) \quad \text{and} \quad T(z_1 \wedge z_2) = T(z_1) \wedge T(z_2), \qquad z_1, z_2 \in Z_{\mathbb{R}}, \tag{3.11}$$

[149, pp. 135–136]. If, in addition, such a T is a linear isometry, then T is said to be a *lattice isometry*. The Banach lattices Z and W are said to be *lattice isomorphic* if there exists a surjective lattice isomorphism $T : Z \to W$. If such a T happens to be a linear isometry, then we say that Z and W are *lattice isometric*. Note that any linear operator $T : Z \to W$ satisfying $T(Z_{\mathbb{R}}) \subseteq W_{\mathbb{R}}$ and (3.11) is necessarily *positive* in the sense that $T(Z^+) \subseteq W^+$. We can similarly define a *real* lattice isomorphism and a *real* lattice isometry from $Z_{\mathbb{R}}$ into $W_{\mathbb{R}}$. In the notation above, let $S_{\mathbb{R}} : Z_{\mathbb{R}} \to W_{\mathbb{R}}$ be an $\mathbb{R}$-linear operator. Then it admits the *canonical extension* $S : Z \to W$ defined by

$$S(x + iy) := S_{\mathbb{R}}(x) + iS_{\mathbb{R}}(y), \qquad \text{for} \quad x + iy \in Z \quad \text{with} \quad x, y \in Z_{\mathbb{R}}; \tag{3.12}$$

see, for example, [149, p. 135], [165, §92].

Lemma 3.8. *Let Z and W be Banach lattices which are the complexification of real Banach lattices $Z_{\mathbb{R}}$ and $W_{\mathbb{R}}$, respectively. Then the following assertions hold.*

(i) *Let $T : Z \to W$ be a lattice isomorphism onto its range and let $T_{\mathbb{R}} : Z_{\mathbb{R}} \to W_{\mathbb{R}}$ denote the restriction of T to $Z_{\mathbb{R}}$, with codomain space $W_{\mathbb{R}}$. Then $T_{\mathbb{R}}$ is a real lattice isomorphism onto its range. If, in addition, T is surjective, then so is $T_{\mathbb{R}}$.*

(ii) *Let $T : Z \to W$ be a lattice isometry onto its range. Then the real lattice isomorphism $T_{\mathbb{R}} : Z_{\mathbb{R}} \to W_{\mathbb{R}}$ given in* (i) *is a real lattice isometry.*

(iii) *Let $S_{\mathbb{R}} : Z_{\mathbb{R}} \to W_{\mathbb{R}}$ be a continuous $\mathbb{R}$-linear operator. Then its canonical extension $S : Z \to W$ is a continuous linear operator, i.e., $S \in \mathcal{L}(Z, W)$.*

(iv) *Let $S_{\mathbb{R}} : Z_{\mathbb{R}} \to W_{\mathbb{R}}$ be a real lattice isomorphism onto its range. Then its canonical extension $S : Z \to W$ is a lattice isomorphism onto its range. If, in addition, $S_{\mathbb{R}}$ is surjective, then so is S.*

(v) *Assume further that both Z and W are o.c. If $S_{\mathbb{R}} : Z_{\mathbb{R}} \to W_{\mathbb{R}}$ is a real lattice isometry, then its canonical extension $S : Z \to W$ of $S_{\mathbb{R}}$ is a lattice isometry.*

Proof. (i) By definition $T(Z_{\mathbb{R}}) \subseteq W_{\mathbb{R}}$ and so the operator $T_{\mathbb{R}} : Z_{\mathbb{R}} \to W_{\mathbb{R}}$ is well defined. It is clear from (3.11) that $T_{\mathbb{R}}$ preserves order. Moreover, from the definition of the complexification of real Banach lattices (see Chapter 2), we have

$$\|x\|_{Z_{\mathbb{R}}} = \big\| \, |x| \, \big\|_{Z_{\mathbb{R}}} = \|x\|_Z, \qquad x \in Z_{\mathbb{R}} \subseteq Z, \tag{3.13}$$

and

$$\|u\|_{W_{\mathbb{R}}} = \big\| \, |u| \, \big\|_{W_{\mathbb{R}}} = \|u\|_W \qquad u \in W_{\mathbb{R}} \subseteq W. \tag{3.14}$$

So, the bicontinuity of T implies that of $T_{\mathbb{R}}$. Hence, $T_{\mathbb{R}}$ is a real lattice isomorphism.

Now assume that T is surjective. To prove that $T_{\mathbb{R}}$ is surjective, let $u \in W_{\mathbb{R}} \subseteq W$. Then $u = T(x+iy) = T(x) + iT(y)$ for some $x + iy \in Z$ with $x, y \in Z_{\mathbb{R}}$. Since $u, T(x), T(y) \in W_{\mathbb{R}}$, we have $iT(y) \in W_{\mathbb{R}} \cap (iW_{\mathbb{R}}) = \{0\}$ and hence, $y = 0$, that is, $u = T(x) = T_{\mathbb{R}}(x)$. Thus, $T_{\mathbb{R}}$ is surjective.

(ii) If T is a lattice isometry, then it follows from (3.13) and (3.14) that $T_{\mathbb{R}}$ is also a lattice isometry.

(iii) Given $x + iy \in Z$ with $x, y \in Z_{\mathbb{R}}$, it follows from (2.15) that

$$\begin{aligned} \big\|S(x+iy)\big\|_W &= \big\|S_{\mathbb{R}}(x) \ + \ iS_{\mathbb{R}}(y)\big\|_W \\ &\le \big\|S_{\mathbb{R}}(x)\big\|_W + \ \big\|iS_{\mathbb{R}}(y)\big\|_W \ = \ \big\|S_{\mathbb{R}}(x)\big\|_{W_{\mathbb{R}}} + \ \big\|S_{\mathbb{R}}(y)\big\|_{W_{\mathbb{R}}} \\ &\le \|S_{\mathbb{R}}\| \cdot \|x\|_{Z_{\mathbb{R}}} + \|S_{\mathbb{R}}\| \cdot \|y\|_{Z_{\mathbb{R}}} \ \le \ 2\,\|S_{\mathbb{R}}\| \cdot \|x+iy\|_Z, \end{aligned} \tag{3.15}$$

which implies that S is continuous.

(iv) Clearly S preserves the order because $S_{\mathbb{R}}$ does. The injectivity of S easily follows from that of $S_{\mathbb{R}}$. The range of S is expressed as

$$S(Z) \ = \ S_{\mathbb{R}}(Z_{\mathbb{R}}) + iS_{\mathbb{R}}(Z_{\mathbb{R}}). \tag{3.16}$$

So, every element of $S(Z)$ is of the form $u + iv$ with $u, v \in S_{\mathbb{R}}(Z_{\mathbb{R}})$. Then (3.12) gives

$$S\Big(S_{\mathbb{R}}^{-1}(u) + iS_{\mathbb{R}}^{-1}(v)\Big) \ = \ u + iv.$$

So, $S^{-1}(u+iv) = S_{\mathbb{R}}^{-1}(u) + iS_{\mathbb{R}}^{-1}(v)$. Therefore, a similar argument to (3.15) yields that

$$\big\|S^{-1}(u+iv)\,\big\|_Z \ \le \ 2\,\|S_{\mathbb{R}}^{-1}\| \cdot \|u+iv\|_W,$$

that is, S^{-1} is continuous on the range $S(Z) \subseteq W$ of S. This establishes the fact that S is a lattice isomorphism onto its range.

If $S_{\mathbb{R}}$ is surjective, then so is S via (3.16).

(v) Fix $x+iy \in Z$ with $x, y \in Z_{\mathbb{R}}$. Then $|x+iy| = \bigvee_{\theta \in [0,2\pi)} \big|(\cos\theta)x + (\sin\theta)y\big|$ in $Z_{\mathbb{R}} \subseteq Z$ via (2.14). Let $\mathbf{F}$ be the collection of all finite subsets of the interval $[0, 2\pi)$ directed by inclusion. Since Z is o.c., so is its real part $Z_{\mathbb{R}}$. It then follows that

$$
\begin{aligned}
|x+iy| &= \bigvee_{\Lambda\in\mathbf{F}} \Big(\bigvee_{\theta\in\Lambda} \big|(\cos\theta)x+(\sin\theta)y\big|\Big) \\
&= \lim_{\Lambda\in\mathbf{F}} \Big(\bigvee_{\theta\in\Lambda} \big|(\cos\theta)x+(\sin\theta)y)\big|\Big) \qquad (3.17)
\end{aligned}
$$

in $Z_{\mathbb{R}}$. On the other hand, since W is o.c. (hence, also $W_{\mathbb{R}}$ is o.c.) and since

$$
\big|S(x+iy)\,\big| = \big|S_{\mathbb{R}}(x)+iS_{\mathbb{R}}(y)\,\big| = \bigvee_{\theta\in[0,2\pi)} \Big|(\cos\theta)S_{\mathbb{R}}(x)+(\sin\theta)S_{\mathbb{R}}(y)\,\Big|,
$$

it follows that

$$
\begin{aligned}
\big|S(x+iy)\,\big| &= \bigvee_{\Lambda\in\mathbf{F}} \Big(\bigvee_{\theta\in\Lambda} \Big|\,(\cos\theta)S_{\mathbb{R}}(x)+(\sin\theta)S_{\mathbb{R}}(y)\,\Big|\,\Big) \\
&= \lim_{\Lambda\in\mathbf{F}} \Big(\bigvee_{\theta\in\Lambda} \Big|\,(\cos\theta)S_{\mathbb{R}}(x)+(\sin\theta)S_{\mathbb{R}}(y)\,\Big|\,\Big)
\end{aligned}
$$

in $W_{\mathbb{R}} \subseteq W$. Hence, we have

$$
\begin{aligned}
\big\|S(x+iy)\,\big\|_W &= \big\|\,|S(x+iy)|\,\big\|_{W_{\mathbb{R}}} \\
&= \lim_{\Lambda\in\mathbf{F}} \Big\| \bigvee_{\theta\in\Lambda} \big|(\cos\theta)S_{\mathbb{R}}(x)+(\sin\theta)S_{\mathbb{R}}(y)\big|\,\Big\|_{W_{\mathbb{R}}} \\
&= \lim_{\Lambda\in\mathbf{F}} \Big\| \bigvee_{\theta\in\Lambda} \big|S_{\mathbb{R}}\big((\cos\theta)x+(\sin\theta)y\big)\big|\,\Big\|_{W_{\mathbb{R}}}. \qquad (3.18)
\end{aligned}
$$

Since $S_{\mathbb{R}}$ preserves order, we have

$$
\Big|S_{\mathbb{R}}\big((\cos\theta)x+(\sin\theta)y\big)\,\Big| \;=\; S_{\mathbb{R}}\Big(\big|(\cos\theta)x+(\sin\theta)y\big|\Big), \qquad \theta\in[0,2\pi), \quad (3.19)
$$

in the real vector lattice $W_{\mathbb{R}}$, [2, Theorem 7.2]. Fix $\Lambda\in\mathbf{F}$. Since Λ is a finite set and $S_{\mathbb{R}}$ is a linear isometry, we have from (3.19) that

$$
\begin{aligned}
\Big\| \bigvee_{\theta\in\Lambda} \big|S_{\mathbb{R}}\big((\cos\theta)x+(\sin\theta)y\big)\big|\Big\|_{W_{\mathbb{R}}} &= \Big\|S_{\mathbb{R}}\Big(\bigvee_{\theta\in\Lambda} \big|(\cos\theta)x+(\sin\theta)y\big|\Big)\Big\|_{W_{\mathbb{R}}} \\
&= \Big\| \bigvee_{\theta\in\Lambda} \big|(\cos\theta)x+(\sin\theta)y\big|\Big\|_{Z_{\mathbb{R}}}. \qquad (3.20)
\end{aligned}
$$

Finally, it follows from (3.17), (3.18) and (3.20) that

$$
\begin{aligned}
\|S(x+iy)\|_W &= \lim_{\Lambda\in\mathbf{F}} \Big\| \bigvee_{\theta\in\Lambda} \big|S_{\mathbb{R}}\big((\cos\theta)x+(\sin\theta)y\big)\big|\Big\|_{W_{\mathbb{R}}} \\
&= \lim_{\Lambda\in\mathbf{F}} \Big\| \bigvee_{\theta\in\Lambda} \big|(\cos\theta)x+(\sin\theta)y\big|\Big\|_{Z_{\mathbb{R}}} = \|x+iy\|_Z.
\end{aligned}
$$

That is, S is a linear isometry. Since we already know that S is a lattice isomorphism, part (v) holds. □

When the scalar field is $\mathbb{R}$, the following result is due to G.P. Curbera, [21, Theorem 8]. We say that a positive element e of a Banach lattice E is a *weak order unit* of E if it is a weak order unit of the real part $E_{\mathbb{R}}$, which is a real Banach lattice (see Chapter 2), that is, $x \wedge (ne) \uparrow x$ for every $x \in E^+$, or, equivalently, $x \wedge e = 0$ implies that $x = 0$, [2, p. 36]. This agrees with the definition given in [149, p. 138].

Proposition 3.9. *Let E be an o.c. Banach lattice with a weak order unit. Then there exists an E-valued, positive vector measure ν such that E and $L^1(\nu)$ are lattice isometric.*

Proof. Write E as the complexification $E = E_{\mathbb{R}} + iE_{\mathbb{R}}$ of its real part $E_{\mathbb{R}}$ (see Chapter 2). There exist an order continuous real B.f.s. $X_{\mathbb{R}}(\mu)$ over a probability space (Ω, Σ, μ) and a surjective, real lattice isometry $S_{\mathbb{R}} : E_{\mathbb{R}} \to X_{\mathbb{R}}(\mu)$, [99, Theorem 1.b.14]. Let $X(\mu) := X_{\mathbb{R}}(\mu) + iX_{\mathbb{R}}(\mu)$ and endow $X(\mu)$ with the lattice norm defined by $\|f\|_{X(\mu)} := \big\| |f| \big\|_{X_{\mathbb{R}}(\mu)}$ for $f \in X(\mu)$, that is, $X(\mu)$ is the complexification of the real Banach lattice $X_{\mathbb{R}}(\mu)$. Since both E and $X(\mu)$ are o.c., it follows from Lemma 3.8(iv)–(v) that the canonical extension $S : E \to X(\mu)$ of $S_{\mathbb{R}}$ is a surjective lattice isometry. This enables us to assume that $E = X(\mu)$.

Since $X(\mu)$ is o.c., the set function $\nu : \Sigma \to X(\mu)$ defined by

$$\nu(A) := \chi_A, \qquad A \in \Sigma,$$

is a vector measure. By adapting the proof of Theorem 8 in [21] we can conclude that $L^1(\nu) = X(\mu)$ with equal norms. Alternatively, apply Corollary 3.66(ii) below. □

Let E be a Banach space and $\nu : \Sigma \to E$ be a vector measure. It follows from (3.4), with ν_f in place of ν, that

$$\sup_{A\in\Sigma} \Big\| \int_A f \, d\nu \Big\|_E \le \|f\|_{L^1(\nu)} \le 4 \sup_{A\in\Sigma} \Big\| \int_A f \, d\nu \Big\|_E, \qquad f \in L^1(\nu). \tag{3.21}$$

Consequently, the function

$$|\!|\!| \cdot |\!|\!|_{L^1(\nu)} : f \longmapsto \sup_{A\in\Sigma} \Big\| \int_A f \, d\nu \Big\|_E, \qquad f \in L^1(\nu), \tag{3.22}$$

is an equivalent norm on $L^1(\nu)$. However, (3.22) is not a lattice norm, in general, even in the case of a scalar measure. Let us give a counterexample for a "genuine" vector measure.

Example 3.10. Let $\mu : \mathcal{B}([0,1]) \to [0,1]$ be Lebesgue measure. Given $1 \le r < \infty$, write $L^r([0,1]) := L^r(\mu)$, as usual. With $E := L^r([0,1])$, define an E-valued, finitely additive set function ν_r on $\Sigma := \mathcal{B}([0,1])$ by

$$\nu_r(A)(t) := \int_0^t \chi_A(u) \, d\mu(u), \qquad t \in [0,1]. \tag{3.23}$$

Given $\{A_n\}_{n=1}^\infty \subseteq \Sigma$ with $A_n \downarrow \emptyset$, we have $\|\nu_r(A_n)\|_E \to 0$. In fact, $\nu_r(A_n)(\cdot) \in C([0,1])$ for $n \in \mathbb{N}$ and $\nu_r(A_n) \to 0$ in the uniform norm on $C([0,1])$, [129, p. 135]. So, ν_r is a vector measure. We shall present a systematic treatment of such measures later. For now, let $f := \chi_{[0,2^{-1}]} - \chi_{(2^{-1},1]}$, in which case $\int_A f\,d\nu_r = \nu_r\big(A\cap[0,2^{-1}]\big) - \nu_r\big(A\cap(2^{-1},1]\big)$, with $0 \le \nu_r\big(A\cap[0,2^{-1}]\big) \le \nu_r\big([0,2^{-1}]\big)$ and $0 \le \nu_r\big(A\cap(2^{-1},1]\big) \le \nu_r\big((2^{-1},1]\big)$, so that

$$\Big|\nu_r\big(A\cap[0,2^{-1}]\big) \;-\; \nu_r\big(A\cap(2^{-1},1]\big)\Big| \;\le\; \nu_r\big([0,2^{-1}]\big), \qquad A\in\Sigma.$$

That is,

$$\Big|\int_A f\,d\nu_r\Big| \le \int_{[0,2^{-1}]} f\,d\nu_r, \qquad A\in\Sigma. \tag{3.24}$$

We claim that

$$|\!|\!| f |\!|\!|_{L^1(\nu_r)} \;=\; \Big(\frac{1}{2}\Big)^{(r+1)/r}\cdot\Big(\frac{r+2}{r+1}\Big)^{1/r} \;<\; \Big(\frac{1}{r+1}\Big)^{1/r} \;=\; |\!|\!|\, |f|\, |\!|\!|_{L^1(\nu_r)}. \tag{3.25}$$

Indeed, (3.24) yields that

$$|\!|\!| f |\!|\!|_{L^1(\nu_r)} = \sup_{A\in\Sigma}\Big\|\int_A f\,d\nu_r\Big\|_E \;=\; \Big\|\nu_r\big([0,2^{-1}]\big)\Big\|_E \;=\; \Big(\frac{1}{2}\Big)^{(r+1)/r}\cdot\Big(\frac{r+2}{r+1}\Big)^{1/r}.$$

On the other hand, $|\!|\!|\, |f|\, |\!|\!|_{L^1(\nu_r)} = \big\|\nu_r([0,1])\big\|_E = (r+1)^{-1/r}$, which verifies (3.25). Accordingly, $|\!|\!| \cdot |\!|\!|_{L^1(\nu_r)}$ is not a lattice norm. $\square$

We can also define $\nu_\infty : \mathcal{B}([0,1]) \to L^\infty([0,1])$ by (3.23). The vector measure

$$\nu_r : \mathcal{B}([0,1]) \to L^r([0,1]), \qquad 1\le r\le\infty, \tag{3.26}$$

will be called the *Volterra measure of order* r. This terminology is, of course, derived from the Volterra integral operator $V_r : L^r([0,1]) \to L^r([0,1])$ defined by

$$V_r(f)(t) \;=\; \int_0^t f(u)\,du, \qquad t\in[0,1], \tag{3.27}$$

for $f \in L^r([0,1])$. It is clear that $\nu_r(A) = V_r(\chi_A)$, for $A \in \mathcal{B}([0,1]$. For each $1\le r\le\infty$, the operator V_r is known to be compact. Indeed, for $1 \le r < \infty$, this follows from [80, Satz 11.6]. For $r=\infty$, a direct calculation shows that

$$V_\infty(f) \;=\; \Big(\int_0^1 f(u)\,du\Big)\chi_{[0,1]} \;-\; V_1^*(f), \qquad f\in L^\infty([0,1]),$$

where $V_1^* \in \mathcal{L}\big(L^\infty([0,1])\big)$ is the dual operator of $V_1 \in \mathcal{L}\big(L^1([0,1])\big)$. Since V_1 is compact, so is V_1^* because of Schauder's Theorem, [46, VI Theorem 5.2]. Moreover, the rank-1 operator $T : f \mapsto \left(\int_0^1 f(u)\,du\right)\chi_{[0,1]}$ is surely compact in $L^\infty([0,1])$ and hence, $V_\infty = T - V_1^*$ is also compact.

There also exist vector measures ν for which $\|\!|\cdot|\!\|_{L^1(\nu)} = \|\cdot\|_{L^1(\nu)}$, in which case $\|\!|\cdot|\!\|_{L^1(\nu)}$ is a lattice norm. For example, the measure given by (3.10) has this property (this is a special case of Corollary 3.66(ii)). Another instance occurs in Example 3.24 below. Actually, we will see in Proposition 3.12 below that $\|\!|\cdot|\!\|_{L^1(\nu)}$ is a lattice norm if and only if it coincides with $\|\cdot\|_{L^1(\nu)}$. Of course, in general, a Banach lattice can have distinct but equivalent lattice norms. Just consider ℓ^1 with the usual norm $\|\varphi\|_{\ell^1} = \sum_{n=1}^\infty |\varphi(n)|$ and also the lattice norm given by $\|\!|\varphi|\!\|_{\ell^1} := \|\varphi\|_{\ell^1} + |\varphi(1)|$ for $\varphi \in \ell^1$. Let us first give a preliminary result.

Lemma 3.11. *Let $\nu : \Sigma \to E$ be a Banach-space-valued measure. Then*

$$\|f\|_{L^1(\nu)} = \sup\left\{\Big\|\int_\Omega sf\,d\nu\Big\|_E : s \in \operatorname{sim}\Sigma \ \textit{and} \ \sup_{\omega\in\Omega}|s(\omega)| \le 1\right\}, \qquad f \in L^1(\nu). \tag{3.28}$$

Proof. Let $f \in L^1(\nu)$. Given $s = \sum_{j=1}^n a_j\chi_{A_j} \in \operatorname{sim}\Sigma$ with $n \in \mathbb{N}$, scalars $a_j \in \mathbb{C}$ satisfying $|a_j| \le 1$ for $j = 1, \dots, n$, and pairwise disjoint sets $\{A_j\}_{j=1}^n \subseteq \Sigma$, we have

$$\int_\Omega sf\,d\nu = \sum_{j=1}^n a_j \cdot \nu_f(A_j).$$

Therefore (3.2), with the indefinite integral ν_f in place of ν, yields (3.28) because $\|f\|_{L^1(\nu)} = \|\nu_f\|(\Omega)$. $\square$

Proposition 3.12. *Let $\nu : \Sigma \to E$ be a Banach-space-valued measure.*

(i) *The following assertions are equivalent.*

  (a) *The norm $\|\!|\cdot|\!\|_{L^1(\nu)}$ on $L^1(\nu)$ as given by* (3.22) *is a lattice norm.*

  (b) $\|\!|f|\!\|_{L^1(\nu)} = \|f\|_{L^1(\nu)}$ *for every $f \in L^1(\nu)$.*

(ii) *If $\left\|\int_\Omega f\,d\nu\right\|_E = \|f\|_{L^1(\nu)}$ for every $f \in L^1(\nu)$, then* (a) *and* (b) *of part* (i) *hold.*

Proof. (i) (a) $\Rightarrow$ (b). Fix $f \in L^1(\nu)$. Let $s \in \operatorname{sim}\Sigma$ with $\sup_{\omega\in\Omega}|s(\omega)| \le 1$. Since $|sf| \le |f|$, it follows from (3.22) and (a) that

$$\Big\|\int_\Omega sf\,d\nu\Big\|_E \le \|\!|sf|\!\|_{L^1(\nu)} \le \|\!|f|\!\|_{L^1(\nu)}.$$

This and Lemma 3.11 imply that $\|f\|_{L^1(\nu)} \le \|\!|f|\!\|_{L^1(\nu)}$. Since we already know that the reverse inequality holds (see (3.21) and the definition of $\|\!|f|\!\|_{L^1(\nu)}$ in (3.22)), we have established (b).

(b) $\Rightarrow$ (a). This is clear because $\|\cdot\|_{L^1(\nu)}$ is a lattice norm.

(ii) The stated assumption, together with (3.21) and (3.22), imply that

$$\|f\|_{L^1(\nu)} = \Big\| \int_\Omega f\, d\nu \Big\|_E \leq \, |\!|\!| f |\!|\!|_{L^1(\nu)} \leq \, \|f\|_{L^1(\nu)}, \qquad f \in L^1(\nu),$$

which establishes (b) and hence, also (a) of (i). $\square$

It seems to be open whether or not the converse statement of part (ii) of the previous proposition holds. For another remark on part (ii) take note that, even if $\|f\|_{L^1(\nu)} = \big\|\int_\Omega f\, d\nu\big\|_E$ for every $f \in L^1(\nu)^+$, the norm $|\!|\!| \cdot |\!|\!|_{L^1(\nu)}$ may not be a lattice norm; for a counterexample see Example 3.10.

Lemma 3.13 below provides a simple but useful form of the norm $\|\cdot\|_{L^1(\nu)}$ for a *positive* vector measure ν.

Lemma 3.13. *Let E be a Banach lattice and let $\nu : \Sigma \to E$ be a positive vector measure. Then*

$$\|f\|_{L^1(\nu)} = \Big\| \int_\Omega |f|\, d\nu \Big\|_E, \qquad f \in L^1(\nu). \tag{3.29}$$

Proof. Given $x^* \in E^*$, the variation measure $\big|\langle \nu, x^* \rangle\big|$ satisfies the inequality

$$\big|\langle \nu, x^* \rangle\big|(A) \leq \big\langle \nu, |x^*| \big\rangle(A), \qquad A \in \Sigma, \tag{3.30}$$

because the assumption $\mathcal{R}(\nu) \subseteq E^+$ and (2.149) applied to $Z := E$ give

$$\begin{aligned} \big|\langle \nu, x^* \rangle(A)\big| &= \big|\langle \nu(A), x^* \rangle\big| \\ &\leq \big\langle |\nu(A)|, |x^*| \big\rangle = \big\langle \nu(A), |x^*| \big\rangle = \langle \nu, |x^*| \rangle\,(A), \qquad A \in \Sigma, \end{aligned}$$

and because $\big|\langle \nu, x^* \rangle\big|$ is the smallest $[0,\infty]$-valued measure dominating $\langle \nu, x^* \rangle$ (in the sense that $\big|\langle \nu, x^* \rangle(A)\big| \leq \big|\langle \nu, x^* \rangle\big|(A)$ for $A \in \Sigma$). Let $f \in L^1(\nu)$. We conclude from (3.7) and (3.30) that

$$\begin{aligned} \|f\|_{L^1(\nu)} &= \sup_{x^* \in \mathbf{B}[E^*]} \int_\Omega |f|\, d|\langle \nu, x^* \rangle| \leq \sup_{x^* \in \mathbf{B}[E^*]} \int_\Omega |f|\, d\big\langle \nu, |x^*| \big\rangle \\ &= \sup_{x^* \in \mathbf{B}[E^*]} \Big\langle \int_\Omega |f|\, d\nu,\ |x^*| \Big\rangle \leq \sup_{x^* \in \mathbf{B}[E^*]} \Big\| \int_\Omega |f|\, d\nu \Big\|_E \cdot \big\|\, |x^*|\, \big\|_{E^*} \\ &= \Big\| \int_\Omega |f|\, d\nu \Big\|_E \end{aligned}$$

because $\big\|\,|x^*|\,\big\|_{E^*} = \|x^*\|_{E^*} \leq 1$ for $x^* \in \mathbf{B}[E^*]$. Since we already know that $\big\|\int_\Omega |f|\, d\nu\big\|_E \leq \|f\|_{L^1(\nu)}$ via (3.21), the equality (3.29) holds. $\square$

Let μ be a control measure for the vector measure ν. Since $\mathcal{N}_0(\mu) = \mathcal{N}_0(\nu) = \mathcal{N}_0(|\nu|)$, the Banach lattice $L^1(|\nu|)$ is a complete normed function space based on the measure space (Ω, Σ, μ), in the terminology of Chapter 2. So, according to the definition, $L^1(|\nu|)$ is a B.f.s. over (Ω, Σ, μ) if and only if $\operatorname{sim}\Sigma \subseteq L^1(|\nu|)$ if and only if $|\nu|(\Omega) < \infty$, in which case $|\nu|$ is also a control measure for ν. Let us present some basic facts concerning $L^1(|\nu|)$; this space will play an important role in the sequel.

Lemma 3.14. *Let E be a Banach space and $\nu : \Sigma \to E$ be a vector measure.*

(i) *Every $|\nu|$-integrable function is ν-integrable, that is,*

$$L^1(|\nu|) \subseteq L^1(\nu). \tag{3.31}$$

Moreover, a function $f \in L^1(\nu)$ is $|\nu|$-integrable if and only if its indefinite integral $\nu_f : \Sigma \to E$ has finite variation, in which case

$$\|f\|_{L^1(\nu)} = \|\nu_f\|(\Omega) \leq |\nu_f|(\Omega) = \|f\|_{L^1(|\nu|)} < \infty. \tag{3.32}$$

(ii) *Assume, in addition, that E is an abstract L^1-space and that ν is positive.*

(a) *We have that $L^1(|\nu|) = L^1(\nu)$ with their given norms being equal.*

(b) *There exists $x_0^* \in (E^*)^+$ such that*

$$\|\nu(A)\|_E = \|\nu\|(A) = |\nu|(A) = \langle \nu, x_0^* \rangle(A), \qquad A \in \Sigma. \tag{3.33}$$

In particular, the finite measures $|\nu|$ and $\langle \nu, x_0^* \rangle$ on Σ are the same and, as a consequence, $L^1(|\nu|) = L^1(\nu) = L^1\big(\langle \nu, x_0^* \rangle\big)$ with their given norms being equal.

(iii) *We have $L^1(|\nu|) = L^1(\nu)$ with their given norms being equivalent if and only if $L^1(\nu)$ is lattice isomorphic to an abstract L^1-space.*

(iv) *We have $L^1(|\nu|) = L^1(\nu)$ with their given norms being equal if and only if $L^1(\nu)$ is an abstract L^1-space.*

Proof. (i) See [98, Theorem 4.2].

(ii) Let us prove (a). We already know that ν has finite variation via Example 3.1. Let $f \in L^1(\nu)^+$. Again by Example 3.1, now applied to the positive vector measure ν_f instead of ν, we have that $|\nu_f|(\Omega) < \infty$ and hence, $f \in L^1(|\nu|)$ via part (i). Therefore, $L^1(\nu)^+ \subseteq L^1(|\nu|)$ and consequently,

$$L^1(\nu) = \big(L^1(\nu)^+ - L^1(\nu)^+\big) + i\big(L^1(\nu)^+ - L^1(\nu)^+\big) \subseteq L^1(|\nu|). \tag{3.34}$$

This, together with (3.31), imply that $L^1(|\nu|) = L^1(\nu)$ as vector spaces. To prove that these two spaces have equal norms, let $s \in \operatorname{sim}\Sigma$. Write $s = \sum_{j=1}^n a_j \chi_{A_j}$ for some scalars $a_1, \dots, a_n \in \mathbb{C}$ and pairwise disjoint sets $A_1, \dots, A_n \in \Sigma$ with $n \in \mathbb{N}$. Since $|\nu|(A_j) = \|\nu(A_j)\|_E$ by (3.6) (with $A := A_j$ for $j = 1, \dots, n$), it follows from

Lemma 3.13 with $f := s$ and (3.5) that

$$\begin{aligned}\|s\|_{L^1(|\nu|)} &= \sum_{j=1}^{n} |a_j| \cdot |\nu|(A_j) \;=\; \sum_{j=1}^{n} |a_j| \cdot \big\|\nu(A_j)\big\|_E \\ &= \Big\| \sum_{j=1}^{n} |a_j| \cdot \nu(A_j) \Big\|_E \;=\; \Big\| \int_\Omega |s| \, d\nu \Big\|_E \;=\; \|s\|_{L^1(\nu)}.\end{aligned}$$

Thus, the norms $\|\cdot\|_{L^1(|\nu|)}$ and $\|\cdot\|_{L^1(\nu)}$ are equal on $\operatorname{sim}\Sigma$, which is dense in both $L^1(|\nu|)$ and $L^1(\nu)$. Hence, these norms coincide on the whole space $L^1(|\nu|) = L^1(\nu)$.

To prove (b), recall that the first two equalities in (3.33) have already been established in (3.6). To show that there exists $x_0^* \in (E^*)^+$ satisfying $\|\nu(A)\|_E = \langle \nu, x_0^* \rangle(A)$ for all $A \in \Sigma$, we use the fact that E is *lattice* isometric to the Lebesgue space $L^1(\eta)$ for some scalar measure $\eta : \mathcal{S} \to [0, \infty]$ defined on a measurable space $(\Lambda, \mathcal{S})$; see [94, Ch. 5, §15, Theorem 3], for instance. So, we may as well assume that $E = L^1(\eta)$ equipped with its usual L^1-norm $\|\cdot\|_{L^1(\eta)}$. Then, we have

$$\begin{aligned}\|\nu(A)\|_{L^1(\eta)} &= \int_\Lambda \nu(A)(\cdot)\, d\eta = \int_\Lambda \nu(A)(\cdot)\chi_\Lambda(\cdot)\, d\eta \\ &= \big\langle \nu(A),\, \chi_\Lambda \big\rangle = \big\langle \nu,\, \chi_\Lambda \big\rangle(A), \qquad A \in \Sigma,\end{aligned}$$

from which (3.33) follows. The rest of (b) is now easily verified.

(iii) See [20, Proposition 3.1] and [22, Proposition 2] for the real case. We need to prove this for the complex case. Since $L^1(|\nu|)$ is an abstract L^1-space, the "only if" portion is obvious.

So, assume that $L^1(\nu)$ is lattice isomorphic to an abstract L^1-space Z and that $T : L^1(\nu) \to Z$ is the corresponding surjective lattice isomorphism, which is necessarily positive. Let $f \in L^1(\nu)^+$. From the Lebesgue Dominated Convergence Theorem (see Theorem 3.7(i)) and continuity of T, it follows that the set function

$$\eta_f : A \mapsto T(f\chi_A), \qquad A \in \Sigma,$$

is a Z-valued, positive vector measure. Hence, η_f has finite variation via Example 3.1 (with $\nu := \eta_f$ there). Fix $A \in \Sigma$. Since $T^{-1} : Z \to L^1(\nu)$ is continuous, we have for the indefinite integral

$$\begin{aligned}\|\nu_f(A)\|_E &= \Big\| \int_A f\chi_A \, d\nu \Big\|_E \;\le\; \|f\chi_A\|_{L^1(\nu)} \\ &\le \|T^{-1}\| \cdot \big\|T(f\chi_A)\big\|_Z \;=\; \|T^{-1}\| \cdot \|\eta_f(A)\|_Z \\ &\le \|T^{-1}\| \cdot |\eta_f|(A).\end{aligned}$$

So, ν_f is dominated by the finite measure $|\eta_f| : \Sigma \to [0, \infty)$ and hence, ν_f has finite variation. Then part (i) implies that $f \in L^1(|\nu|)$. Therefore, $L^1(\nu)^+ \subseteq L^1(|\nu|)$ and hence, $L^1(\nu) \subseteq L^1(|\nu|)$ via (3.34). So, $L^1(\nu) = L^1(|\nu|)$ by part (i).

Since the identity map from $L^1(|\nu|)$ onto $L^1(\nu)$ is continuous by (3.32), the Open Mapping Theorem ensures that $L^1(|\nu|)$ and $L^1(\nu)$ are isomorphic Banach spaces, which establishes part (iii).

(iv) The "only if" portion is clear. So, assume that $L^1(\nu)$ is an abstract L^1-space. We claim that

$$|\nu|(A) \;=\; \|\chi_A\|_{L^1(\nu)}, \qquad A \in \Sigma. \tag{3.35}$$

In fact, given $A \in \Sigma$, let $\{A_j\}_{j=1}^n$ be any finite Σ-partition of A (with $n \in \mathbb{N}$). Since $L^1(\nu)$ is an abstract L^1-space, it follows that

$$\sum_{j=1}^n \big\|\nu(A_j)\big\|_E \;\le\; \sum_{j=1}^n \big\|\chi_{A_j}\big\|_{L^1(\nu)} \;=\; \Big\|\sum_{j=1}^n \chi_{A_j}\Big\|_{L^1(\nu)} \;=\; \big\|\chi_A\big\|_{L^1(\nu)},$$

which implies that $|\nu|(A) \le \|\chi_A\|_{L^1(\nu)}$.

Recalling that $|\nu|(A) \ge \|\nu\|(A) = \|\chi_A\|_{L^1(\nu)}$ (see (3.3)), the identity (3.35) follows.

Next we claim that

$$\|s\|_{L^1(|\nu|)} = \; \|s\|_{L^1(\nu)}, \qquad s \in \operatorname{sim}\Sigma. \tag{3.36}$$

Once we establish (3.36), we can deduce that $\|\cdot\|_{L^1(|\nu|)} = \|\cdot\|_{L^1(\nu)}$ because we know that $L^1(|\nu|) = L^1(\nu)$ with equivalent norms (see (iii)) and that $\operatorname{sim}\Sigma$ is dense in both $L^1(|\nu|)$ and $L^1(\nu)$. To verify (3.36), let $s = \sum_{j=1}^n a_j\chi_{A_j} \in \operatorname{sim}\Sigma$ for scalars $a_1,\dots,a_n \in \mathbb{C}$ and pairwise disjoint sets $A_1,\dots,A_n \in \Sigma$ (with $n \in \mathbb{N}$). Since both $L^1(|\nu|)$ and $L^1(\nu)$ are abstract L^1-spaces, it follows from (3.35), with $A := A_j$ for $j = 1,\dots,n$, that

$$\begin{aligned}\|s\|_{L^1(|\nu|)} &= \sum_{j=1}^n |a_j|\cdot|\nu|(A_j) \;=\; \sum_{j=1}^n |a_j|\cdot\big\|\chi_{A_j}\big\|_{L^1(\nu)}\\ &= \Big\|\sum_{j=1}^n |a_j|\cdot\chi_{A_j}\Big\|_{L^1(\nu)} \;=\; \|s\|_{L^1(\nu)}.\end{aligned}$$

So, (3.36) is established and hence, also part (iv). □

Example 3.15. Let $\{\psi_n\}_{n=1}^\infty$ be any unconditionally summable sequence in the abstract L^1-space ℓ^1 which is not absolutely summable. Such a sequence exists by the Dvoretzky–Rogers Theorem. Then $\{\psi_n\}_{n=1}^\infty$ is not a positive sequence; otherwise it would be absolutely summable. Define a vector measure $\nu : 2^{\mathbb{N}} \to \ell^1$ by $\nu(A) := \sum_{n\in A}\psi_n$ for $A \in 2^{\mathbb{N}}$, in which case $|\nu|(\mathbb{N}) = \sum_{n=1}^\infty \|\psi_n\|_{\ell^1} = \infty$. Then $\chi_{\mathbb{N}} \in L^1(\nu)\setminus L^1(|\nu|)$ and hence, $L^1(|\nu|) \ne L^1(\nu)$. So, the positivity assumption on ν in part (ii) of Lemma 3.14 is necessary. □

There exists a vector measure ν such that $L^1(\nu)$ is not an abstract L^1-space but, it is lattice isomorphic to an abstract L^1-space.

Example 3.16. Let $\Omega := \mathbb{N}$ and $\Sigma := 2^{\mathbb{N}}$. Select any scalar measure $\mu : \Sigma \to [0, \infty)$ such that $\mu(\{n\}) > 0$ for all $n \in \mathbb{N}$. Let $E := \ell^1(\mu)$ but, equip E with the lattice norm

$$f \mapsto \|f\|_E := \Big(\big|f(1)\big|^2 \mu(\{1\}) + \big|f(2)\big|^2 \mu(\{2\})\Big)^{1/2} + \sum_{n=3}^{\infty} \big|f(n)\big| \cdot \mu(\{n\}).$$

Then $(E, \|\cdot\|_E)$ is not an abstract L^1-space but, it is isomorphic to $\ell^1(\mu)$ endowed with its usual norm. Define a vector measure $\nu : \Sigma \to E$ by

$$\nu(A) := \chi_A, \qquad A \in \Sigma.$$

Then $L^1(\nu) = E$ with equal norms; this can be proved directly from the definition or via Corollary 3.66(ii) below. So, $L^1(\nu)$ is not an abstract L^1-space but, it is lattice isomorphic to the abstract L^1-space $\ell^1(\mu)$. □

It is useful to have available criteria, especially for examples, which reformulate the property $L^1(\nu) = L^1(|\nu|)$ in terms of certain scalar measures $\langle \nu, x^* \rangle$. So, let $\nu : \Sigma \to E$ be a Banach-space-valued-vector measure defined on a measurable space (Ω, Σ). Assume that there exist $n \in \mathbb{N}$ and $x_1^*, \ldots, x_n^* \in E^*$ satisfying

$$|\nu|(A) \le \sum_{j=1}^{n} \big|\langle \nu, x_j^* \rangle\big|(A), \qquad A \in \Sigma. \tag{3.37}$$

Then it follows from [125, Lemma 2.6] that $L^1(|\nu|) = L^1(\nu)$. Observe that the positive, finite measure $\eta_0 := \sum_{j=1}^{n} \big|\langle \nu, x_j^* \rangle\big|$ on Σ is a control measure for ν. Moreover, it is clear from (3.37) that $L^1(\eta_0) \subseteq L^1(|\nu|)$ and hence, that

$$L^1(\eta_0) \subseteq L^1(|\nu|) = L^1(\nu).$$

Of course, this particular control measure η_0 for ν satisfies $L^1(\eta_0) \subseteq L^1(\nu)$. It will be shown in Lemma 3.18(ii) below that $L^1(|\nu|)$ is the *largest* space within the class of all spaces $L^1(\eta)$ with η any control measure for ν satisfying $L^1(\eta) \subseteq L^1(\nu)$. Before proceeding to that lemma, let us make some comments.

Remark 3.17. Let us follow the notation in the discussion just prior to this remark.

(i) It is worth noting that [125, Remark 2.7(ii)] exhibits a c_0-valued vector measure ν satisfying $L^1(|\nu|) = L^1(\nu)$ although (3.37) fails to hold for *all* choices of $n \in \mathbb{N}$ and $x_1^*, \ldots, x_n^* \in \ell^1 = (c_0)^*$. In particular,

$$L^1(\nu_0) \neq L^1\big(|\langle \nu_0, x^* \rangle|\big), \qquad x^* \in \ell^1 = (c_0)^*.$$

(ii) As expected, if the codomain space E of ν is finite-dimensional, then (3.37) is always fulfilled. Indeed, let $n := \dim E$. Take a unit vector basis $\{e_j\}_{j=1}^n$

for E and its corresponding dual basis $\{e_j^*\}_{j=1}^n$ for E^*; that is, $\langle e_j, e_k^*\rangle = 0$ whenever $j \neq k$ and $\langle e_j, e_j^*\rangle = 1$ for $j = 1, \ldots, n$. Then

$$\nu(A) \;=\; \sum_{j=1}^{n} \langle \nu(A), e_j^* \rangle \, e_j, \qquad A \in \Sigma,$$

which implies that

$$\|\nu(A)\|_E \;\leq\; \sum_{j=1}^{n} \bigl|\langle \nu(A), e_j^*\rangle\bigr| \cdot \|e_j\|_E \;\leq\; \sum_{j=1}^{n} \bigl|\langle \nu, e_j^*\rangle\bigr|(A), \qquad A \in \Sigma.$$

This inequality, together with the definition of the variation measure $|\nu|$, easily imply (3.37). $\square$

Lemma 3.18. *Let $\nu : \Sigma \to E$ be a Banach-space-valued vector measure defined on a measurable space (Ω, Σ).*

(i) *There exists a control measure $\eta : \Sigma \to [0, \infty)$ for ν satisfying $L^1(\eta) \subseteq L^1(\nu)$ if and only if ν has finite variation, that is, $|\nu|(\Omega) < \infty$.*

(ii) *Suppose that $\eta : \Sigma \to [0, \infty)$ is any control measure for ν satisfying $L^1(\eta) \subseteq L^1(\nu)$. Then $L^1(\eta) \subseteq L^1(|\nu|)$.*

Proof. (i) If ν has finite variation, then the variation measure $|\nu| : \Sigma \to [0, \infty)$ is a control measure for ν with $L^1(|\nu|) \subseteq L^1(\nu)$; see Lemma 3.14(i).

Conversely, assume that ν admits a control measure η such that $L^1(\eta) \subseteq L^1(\nu)$. Observe that the corresponding inclusion map is continuous either via the Closed Graph Theorem or via Lemma 2.7. Therefore, there exists $C > 0$ such that $\|f\|_{L^1(\nu)} \leq C\|f\|_{L^1(\eta)}$ for every $f \in L^1(\eta)$. In particular,

$$\|\nu(A)\|_E \;\leq\; \|\nu\|(A) \;=\; \|\chi_A\|_{L^1(\nu)} \;\leq\; C\|\chi_A\|_{L^1(\eta)} \;=\; C\eta(A), \qquad A \in \Sigma.$$

Thus, the definition of the variation measure $|\nu|$ yields that $|\nu|(A) \leq C\eta(A)$ for all $A \in \Sigma$. In particular, $|\nu|(\Omega) < \infty$.

(ii) Let η be any control measure for ν satisfying $L^1(\eta) \subseteq L^1(\nu)$. By the proof of part (i), there is $C > 0$ such that $|\nu|(A) \leq C\eta(A)$ for $A \in \Sigma$. Accordingly,

$$\int_\Omega |f| \, d|\nu| \;\leq\; C\int_\Omega |f| \, d\eta \;=\; C\|f\|_{L^1(\eta)} \;<\; \infty, \qquad f \in L^1(\eta),$$

that is, $L^1(\eta) \subseteq L^1(|\nu|)$. $\square$

A typical control measure for a Banach-space-valued vector measure $\nu : \Sigma \to E$ is given by the scalar measure $|\langle \nu, x^*\rangle|$, for any Rybakov functional $x^* \in E^*$ (i.e., $x^* \in \mathbf{R}_\nu[E^*]$). From Lemma 3.14(i) and Theorem 3.7(iv), it follows that

$$L^1(|\nu|) \;\subseteq\; L^1(\nu) \;\subseteq\; L^1\bigl(|\langle \nu, x^*\rangle|\bigr). \tag{3.38}$$

Let us combine this with Lemma 3.18 in order to say something about the situation when the second inclusion in (3.38) is actually an equality.

Corollary 3.19. *Let $\nu : \Sigma \to E$ be any Banach-space-valued vector measure defined on a measurable space (Ω, Σ).*

(i) *The following conditions for $x^* \in \mathbf{R}_\nu[E^*]$ are equivalent.*

(a) $L^1\big(|\langle \nu, x^* \rangle|\big) \subseteq L^1(\nu)$ *with a continuous inclusion.*

(b) $L^1\big(|\langle \nu, x^* \rangle|\big) = L^1(\nu)$ *with their given norms being equivalent.*

(c) $L^1\big(|\langle \nu, x^* \rangle|\big) \subseteq L^1(|\nu|)$ *with a continuous inclusion.*

(d) $L^1\big(|\langle \nu, x^* \rangle|\big) = L^1(|\nu|)$ *with their given norms being equivalent.*

(e) $L^1\big(|\langle \nu, x^* \rangle|\big) = L^1(|\nu|) = L^1(\nu)$ *with their given norms being equivalent.*

(ii) *Suppose that $L^1(|\nu|) \neq L^1(\nu)$. Then*

$$L^1(\nu) \neq L^1\big(|\langle \nu, x^* \rangle|\big) \quad \textit{and} \quad L^1(|\nu|) \neq L^1\big(|\langle \nu, x^* \rangle|\big), \qquad x^* \in \mathbf{R}_\nu[E^*].$$

Proof. The stated properties regarding the norms are automatic, either via the Closed Graph Theorem or via Lemma 2.7, once we establish the validity of the corresponding inclusions or equalities as vector spaces. So, it suffices to consider statements (a) to (e) without reference to their norms.

Each of the equivalences (a) $\Leftrightarrow$ (b), (c) $\Leftrightarrow$ (d) and (d) $\Leftrightarrow$ (e) follows from (3.38). Moreover, the implication (a) $\Rightarrow$ (c) is a consequence of Lemma 3.18(ii) and the implication (c) $\Rightarrow$ (a) is clear because $L^1(|\nu|) \subseteq L^1(\nu)$. So, we have established the equivalence of (a) to (e).

(ii) This follows from part (i). □

Recall that, if E is an abstract L^1-space and ν is positive, then we can construct a particular functional $x_0^* \in \mathbf{R}_\nu[E^*]$ for which condition (e) of part (i) above holds with the given norms being equal; see Lemma 3.14(ii)(b) and its proof.

A set $A \in \Sigma$ is called $|\nu|$-*totally infinite* if $|\nu|(\Sigma \cap A) = \{0, \infty\}$. We say that $|\nu|$ is *totally infinite* on Σ if every $A \in \Sigma$ is $|\nu|$-totally infinite.

Lemma 3.20. *For the variation $|\nu| : \Sigma \to [0, \infty]$ of a vector measure $\nu : \Sigma \to E$, the following assertions hold.*

(i) *For every atom A of $|\nu|$ we have $0 < |\nu|(A) = \|\nu(A)\|_E < \infty$.*

(ii) *There are at most countably many atoms of $|\nu|$. Moreover, if $|\nu|$ is purely atomic and if $L^1\big(|\nu|\big)$ is infinite-dimensional, then the Banach lattice $L^1\big(|\nu|\big)$ is lattice isometric to the sequence space ℓ^1.*

(iii) *There exists a Σ-partition $\{\Omega_{\mathrm{a}}, \Omega_{\mathrm{na}}^{(1)}, \Omega_{\mathrm{na}}^{(2)}\}$ of Ω such that*

(a) $|\nu|$ *is purely atomic on $\Sigma \cap \Omega_{\mathrm{a}}$,*

(b) $|\nu|$ *is non-atomic and σ-finite on $\Sigma \cap \Omega_{\mathrm{na}}^{(1)}$,*

(c) $|\nu|$ *is totally infinite on $\Sigma \cap \Omega_{\mathrm{na}}^{(2)}$.*

(iv) *The Banach space $L^1(|\nu|)$ is infinite-dimensional if and only if either $|\nu|$ has countably infinitely many atoms or the set $\Omega_{\mathrm{na}}^{(1)}$ is not $|\nu|$-null.*

Proof. (i) Let μ be a control measure for ν. Any finite Σ-partition $\{A_1, \ldots, A_n\}$, with $n \in \mathbb{N}$, of A has the property that $|\nu|(A_j) = |\nu|(A)$ for exactly one $1 \leq j \leq n$ and all other sets A_k with $k \neq j$ are $|\nu|$-null. From this, (i) follows.

(ii) Since μ and $|\nu|$ have the same atoms and since the finite measure μ has at most countably many atoms, the first part of statement (ii) is clear.

Now assume that $|\nu|$ is purely atomic and that $L^1(|\nu|)$ is infinite-dimensional. Let $A(n)$, for $n \in \mathbb{N}$, be an enumeration of all the atoms of $|\nu|$. Then every member of $L^1(|\nu|)$ has a unique representation of the form

$$\sum_{n=1}^{\infty} a_n \chi_{A(n)}, \tag{3.39}$$

for appropriate $a_n \in \mathbb{C}$ $(n \in \mathbb{N})$ satisfying $\sum_{n=1}^{\infty} |a_n| \cdot |\nu|(A(n)) < \infty$. The map assigning to each function in $L^1(\nu)$ of the form (3.39) the function $\psi \in \ell^1$ defined by $\psi(n) := a_n \cdot |\nu|(A(n))$ for $n \in \mathbb{N}$ provides the desired lattice isometry.

(iii) Let Ω_{a} be the union of all the atoms of $|\nu|$, which is unique up to a $|\nu|$-null set, and let $\Omega_{\mathrm{na}} := \Omega \setminus \Omega_{\mathrm{a}}$. Then, $|\nu|$ is purely atomic on $\Sigma \cap \Omega_{\mathrm{a}}$ (which establishes part (a)) and non-atomic on $\Sigma \cap \Omega_{\mathrm{na}}$. Let $\{B_\alpha\}_\alpha$ be a maximal family of pairwise disjoint sets in $\Sigma \cap \Omega_{\mathrm{na}}$ such that $0 < |\nu|(B_\alpha) < \infty$ for each α. Such a family exists by Zorn´s Lemma. Now $0 < \mu(B_\alpha) < \infty$, for every α, with μ finite implies that $\{B_\alpha\}_\alpha$ is necessarily countable. So, the set $\Omega_{\mathrm{na}}^{(1)} := \bigcup_\alpha B_\alpha$ fulfills (b). Let $\Omega_{\mathrm{na}}^{(2)} := \Omega_{\mathrm{na}} \setminus \Omega_{\mathrm{na}}^{(1)}$. Then the definition of $\{B_\alpha\}_\alpha$ guarantees that (c) holds.

(iv) This is clear from (ii) and (iii). □

The equality $\mathcal{N}_0(\nu) = \mathcal{N}_0(|\nu|)$ ensures that the vector measure ν and the positive scalar measure $|\nu|$ have exactly the same atoms. Recall that a set $A \in \Sigma$ is said to be an *atom* for the vector measure ν if, for every $B \in \Sigma \cap A$, we have either $B \in \mathcal{N}_0(\nu)$ or $A \setminus B \in \mathcal{N}_0(\nu)$. So, ν is purely atomic (resp. non-atomic) if and only if $|\nu|$ is purely atomic (resp. non-atomic). In the notation of Lemma 3.20, we call Ω_{a} and $\Omega_{\mathrm{na}} := \Omega \setminus \Omega_{\mathrm{a}}$ the purely atomic and non-atomic parts of Ω *with respect to* ν, respectively. A detailed study of certain aspects of atomic vector measures occurs in [78].

Lemma 3.21. *Let $\nu : \Sigma \to E$ be a non-atomic vector measure and let $A \in \Sigma$ be a non-ν-null set. Then the subset $\{\chi_{A\cap B} : B \in \Sigma\}$ is not relatively compact in $L^1(\nu)$.*

Proof. Fix any Rybakov functional $x^* \in \mathbf{R}_\nu(E^*)$ and let μ denote the measure $|\langle \nu, x^* \rangle|$ restricted to $A \cap \Sigma$. Then μ is non-atomic and $0 < \mu(A) < \infty$. By the non-atomicity of μ there exists a sequence $\{\chi_{A(n)}\}_{n=1}^{\infty} \subseteq L^1(\mu)$ such that $\mu(A(n)) = \mu(A)/2$ for all $n \in \mathbb{N}$ and $\mu(A(n) \triangle A(m)) = \mu(A)/4$ for $m \neq n$ (with $\triangle$ denoting symmetric difference); see [42, Ch. III, Example 1.2], for instance. Hence, $\{\chi_B : B \in \Sigma \cap A\}$ cannot be relatively compact in $L^1(\mu)$. Since the inclusion

$L^1(\nu) \subseteq L^1(\mu)$ is continuous (see Theorem 3.7(iv), it follows that the subset $\{\chi_{A\cap B} : B \in \Sigma\}$ cannot be relatively compact in $L^1(\nu)$. $\square$

Recall that a Banach space E has the *Schur property* if every weakly convergent sequence in E is norm convergent. The best known example of such a space is ℓ^1. Note that the natural injection from $L^1(|\nu|)$ into $L^1(\nu)$ is continuous via (3.32).

In the following result let $\Omega_{\mathrm{a}}, \Omega_{\mathrm{na}}^{(1)}, \Omega_{\mathrm{na}}^{(2)}$ be as in Lemma 3.20.

Proposition 3.22. *Let $\nu : \Sigma \to E$ be a vector measure. The following assertions for the natural injection $j_1 : L^1(|\nu|) \to L^1(\nu)$ hold.*

(i) *The map j_1 is completely continuous if and only if $\Omega_{\mathrm{na}}^{(1)}$ is ν-null.*

(ii) *If $L^1(|\nu|)$ is infinite-dimensional, then j_1 is not compact.*

(iii) *If ν has infinitely many atoms or the set $\Omega_{\mathrm{na}}^{(1)}$ is non-ν-null, then j_1 is not compact.*

Proof. (i) Let j_1 be completely continuous. Assume that $\Omega_{\mathrm{na}}^{(1)}$ is not ν-null. We can select $A \in \Sigma \cap \Omega_{\mathrm{na}}^{(1)}$ with $0 < |\nu|(A) < \infty$ because $|\nu|$ is σ-finite and non-atomic on $\Sigma \cap \Omega_{\mathrm{na}}^{(1)}$. By the Lebesgue Dominated Convergence Theorem, the set function

$$B \longmapsto \chi_{B\cap A} \in L^1(|\nu|), \qquad B \in \Sigma, \tag{3.40}$$

is a vector measure. So, its range $\{\chi_{B\cap A} : B \in \Sigma\}$ is relatively weakly compact in $L^1(|\nu|)$ (see Lemma 3.3) and hence, is mapped by j_1 to the relatively compact set

$$\big\{j_1(\chi_{B\cap A}) : B \in \Sigma\big\} \;=\; \big\{\chi_{B\cap A} : B \in \Sigma\big\} \subseteq L^1(\nu).$$

However, this contradicts Lemma 3.21 because ν is non-atomic on $\Sigma \cap A$. Thus, $\Omega_{\mathrm{na}}^{(1)}$ must be ν-null.

To establish the converse, assume that $\Omega_{\mathrm{na}}^{(1)}$ is ν-null. Then every function $f \in L^1(|\nu|)$ vanishes ν-a.e. on $\Omega_{\mathrm{na}}^{(1)} \cup \Omega_{\mathrm{na}}^{(2)}$. Since $|\nu|$ is purely atomic on $\Omega_{\mathrm{a}} \cap \Sigma$, it follows from Lemma 3.20(ii)–(iii) that $L^1(|\nu|)$ is either finite-dimensional or isomorphic to ℓ^1. The Schur property of ℓ^1 then ensures that j_1 is completely continuous.

(ii) If $\Omega_{\mathrm{na}}^{(1)}$ is not ν-null, then part (i) implies that j_1 is not completely continuous and hence, is surely not compact. Since $L^1(|\nu|)$ is infinite-dimensional, the only other possibility (according to Lemma 3.20(iv)) is that $|\nu|$ admits infinitely many distinct atoms $A(n)$, for $n \in \mathbb{N}$. For all $k, n \in \mathbb{N}$ with $k \neq n$, we have by

(3.21) that

$$\begin{aligned}
&\Big\| j_1\Big(\frac{\chi_{A(n)}}{|\nu|(A(n))}\Big) - j_1\Big(\frac{\chi_{A(k)}}{|\nu|(A(k))}\Big)\Big\|_{L^1(\nu)} \\
&\qquad \geq \sup_{A\in\Sigma}\Big\| \int_A \Big(\frac{\chi_{A(n)}}{|\nu|(A(n))} - \frac{\chi_{A(k)}}{|\nu|(A(k))}\Big) d\nu \Big\|_E \\
&\qquad \geq \big\|\nu(A(n))\big\|_E \cdot \big[\,|\nu|(A(n))\,\big]^{-1} = 1,
\end{aligned}$$

where the last equality follows because the atom $A(n)$ of ν satisfies $\big\|\nu(A(n))\big\|_E = |\nu|(A(n))$ via Lemma 3.20(i) with $A(n)$ in place of A. So, the image of the bounded set

$$\Big\{ \big[\,|\nu|(A(n))\big]^{-1} \cdot \chi_{A(n)} :\ n \in \mathbb{N} \Big\} \ \subseteq\ L^1(|\nu|)$$

under j_1 is not relatively compact in $L^1(\nu)$. Accordingly, j_1 is not compact.

(iii) In either case, $L^1(|\nu|)$ is infinite-dimensional (see Lemma 3.20(iv)). So, (iii) follows from (ii). □

Again let $\Omega_{\mathrm{a}}, \Omega_{\mathrm{na}}^{(1)}, \Omega_{\mathrm{na}}^{(2)} \in \Sigma$ be as in Lemma 3.20.

Corollary 3.23. *Let $\nu : \Sigma \to E$ be a vector measure.*

(i) *If the Banach space $L^1(\nu)$ is reflexive, then $\Omega_{\mathrm{na}}^{(1)}$ is ν-null.*

(ii) *If $L^1(\nu)$ is reflexive and ν is non-atomic, then $|\nu|$ is totally infinite and hence, $L^1(|\nu|) = \{0\}$.*

Proof. (i) Assume that $\Omega_{\mathrm{na}}^{(1)}$ is not ν-null. Choose a set $A \in \Sigma \cap \Omega_{\mathrm{na}}^{(1)}$ such that $0 < |\nu|(A) < \infty$. Define a vector measure $\nu_A : \Sigma \cap A \to E$ by $\nu_A(B) = \nu(B)$ for $B \in \Sigma \cap A$. The Banach space $L^1(\nu_A)$, which is regarded as a closed subspace of the reflexive space $L^1(\nu)$, is also reflexive. Accordingly, the natural injection $j_1^{(A)} : L^1(|\nu_A|) \to L^1(\nu_A)$ is weakly compact. The Dunford–Pettis property of $L^1(|\nu_A|)$, [42, Ch. III, Corollary 2.14 and pp. 176–177], implies that $j_1^{(A)}$ is completely continuous. But, $j_1^{(A)}$ is *not* completely continuous by Proposition 3.22(i) because $\Omega_{\mathrm{na}}^{(1)}$ (computed relative to ν_A) is not ν_A-null. Therefore, $\Omega_{\mathrm{na}}^{(1)}$ must be ν-null.

(ii) Since $\Omega_{\mathrm{a}} = \emptyset$ for such a vector measure ν, the conclusion is clear from (i). □

Part (ii) of the previous result, for E a real Banach space, occurs in [20, Theorem 2.11], [23, Remark on pp. 1804–1805]; our proof is different and applies also in complex spaces E.

Let us give an example of a purely atomic vector measure.

Example 3.24. Let $1 \leq r < \infty$. Let $\varphi \in \ell^1 \subseteq \ell^r$ with $\varphi(n) > 0$ for every $n \in \Omega := \mathbb{N}$ and $E := \ell^r$. Define an ℓ^r-valued vector measure on $\Sigma := 2^{\mathbb{N}}$ by

$$\nu(A) :=\ \varphi\chi_A, \qquad A \in 2^{\mathbb{N}}. \tag{3.41}$$

Then, $L^1(\nu) = (1/\varphi) \cdot \ell^r$ and, in the notation of (3.22),

$$\|f\|_{L^1(\nu)} = \|\varphi f\|_{\ell^r} = \Big\| \int_{\mathbb{N}} f \, d\nu \Big\|_{\ell^r} = \|\!|\!| f |\!|\!|_{L^1(\nu)}, \qquad f \in L^1(\nu). \tag{3.42}$$

So, $L^1(\nu)$ is reflexive if and only if $1 < r < \infty$. Moreover,

$$|\nu|(A) = \sum_{n \in A} \varphi(n), \qquad A \in \Sigma.$$

It follows that $L^1(|\nu|) = (1/\varphi) \cdot \ell^1$ and $\|f\|_{L^1(|\nu|)} = \|\varphi f\|_{\ell^1}$ for $f \in L^1(|\nu|)$. Either by Proposition 3.22 or by direct computation we conclude that the natural inclusion $j_1 : L^1(|\nu|) \to L^1(\nu)$ is completely continuous but not compact. $\square$

Let us now consider the (non-atomic) Volterra measures; see Example 3.26 below. First we require the following preliminary result.

Lemma 3.25. *Let $1 \le r < \infty$. Suppose that $f : [0,1] \to \mathbb{R}^+$ is a Borel measurable function which is Lebesgue integrable over $[0,t]$ for each $t \in [0,1)$ and that the function*

$$t \mapsto \int_0^t f(u) \, du, \qquad t \in [0,1)$$

belongs to $L^r([0,1])$. Then f is necessarily integrable with respect to the Volterra measure $\nu_r : \mathcal{B}([0,1]) \to L^r([0,1])$ of order r.

Proof. Select a sequence $\{s_n\}_{n=1}^\infty \subseteq \operatorname{sim} \mathcal{B}([0,1])$ such that $0 \le s_n \uparrow f$ pointwise on $[0,1]$. Fix $A \in \mathcal{B}([0,1])$. By assumption $f\chi_A$ is Lebesgue integrable over $[0,t]$, whenever $0 \le t < 1$, and the function

$$g_A : t \mapsto \int_0^t f(u)\chi_A(u) \, du, \qquad t \in [0,1),$$

belongs to $L^r([0,1])$. Since $s_n\chi_A \uparrow f\chi_A$ pointwise, the Monotone Convergence Theorem guarantees that

$$I_{\nu_r}\big(s_n\chi_A\big)(t) = \int_0^t s_n(u)\chi_A(u) \, du \ \uparrow \ \int_0^t f(u)\chi_A(u) \, du = g_A(t)$$

for each $t \in [0,1)$ as $n \to \infty$. Namely, $I_{\nu_r}(s_n\chi_A) \uparrow g_A$ (relative to $n \in \mathbb{N}$) in the order of $L^r([0,1])$ which is σ-o.c. Therefore, $I_{\nu_r}(s_n\chi_A) \to g_A$ in the topology of $L^r([0,1])$ as $n \to \infty$. It follows from Theorem 3.5 that f is ν_r-integrable. $\square$

Let us apply this lemma when $1 < r < \infty$. Define a function g on $[0,1)$ by

$$g(t) := \frac{1}{(1-t)^{1/r}\big(1 - \ln(1-t)\big)}, \qquad t \in [0,1).$$

Then g is differentiable on $(0,1)$ and

$$g'(t) = \frac{\big(1-\ln(1-t)\big)-r}{r\big(1-t\big)^{1+(1/r)}\big(1-\ln(1-t)\big)^2}, \qquad t \in [0,1).$$

With $c := 1-\exp(1-r) < 1$, the non-negative, continuous function $f := g' \cdot \chi_{(c,1)}$ on $[0,1)$ satisfies both $\int_0^t f(u)\,du = 0$, for $0 \le t \le c$, and

$$\int_0^t f(u)\,du = \int_c^t f(u)\,du = g(t) - g(c) = \Big(g - g(c)\,\chi_{[0,1]}\Big)(t) < \infty, \qquad t \in (c,1).$$

So, we have

$$\int_0^t f(u)\,du = \Big(g\chi_{(c,1)} - g(c)\chi_{(c,1)}\Big)(t), \qquad t \in [0,1). \tag{3.43}$$

Now observe that $g \in L^r([0,1])$ via the following calculation:

$$\int_0^1 \big|g(t)\big|^r\,dt = \int_0^1 \frac{1}{(1-t)\big(1-\ln(1-t)\big)^r}\,dt = \int_1^\infty \frac{1}{u^r}\,du < \infty.$$

Therefore, $g\,\chi_{(c,1)} - g(c)\chi_{(c,1)} \in L^r([0,1])$. According to (3.43), also $t \mapsto \int_0^t f(u)\,du$ belongs to $L^r([0,1])$. Then Lemma 3.25 allows us to conclude that the function f is ν_r-integrable.

Example 3.26. Let $1 \le r \le \infty$. Consider the Volterra measure $\nu_r : \mathcal{B}([0,1]) \to L^r([0,1])$ as given by (3.26). Most results which we now present are from [119] and [129], except for the (strict) inclusion $L^1(\nu_r) \subseteq L^1(\nu_1)$. The variation $|\nu_r|$ of ν_r is weighted Lebesgue measure, namely

$$d|\nu_r|(t) = \begin{cases} (1-t)^{1/r}dt & \text{if} \quad 1 \le r < \infty, \\ dt & \text{if} \quad r = \infty. \end{cases} \tag{3.44}$$

(i) Let $r = 1$. Then $d|\nu_1|(t) = (1-t)\,dt$ and

$$L^1([0,1]) \subseteq L^1(|\nu_1|) = L^1(\nu_1);$$

see [119, Example 2].

(ii) Let $1 < r < \infty$. Then we have $d|\nu_r|(t) = (1-t)^{1/r}dt$ and

$$L^r([0,1]) \subseteq L^1([0,1]) \subseteq L^1(|\nu_r|) \subseteq L^1(\nu_r) \subseteq L^1(\nu_1) \tag{3.45}$$

with all inclusions being strict. That the first two inclusions are strict is clear from (3.44). So, let us discuss the last two inclusions separately.

(ii-a) Consider the function $f := g' \cdot \chi_{(c,1)}$ on $\Omega := [0,1]$ as given in the discussion prior to this example. Then $f \notin L^1(|\nu_r|) = L^1\big((1-t)^{1/r}dt\big)$ because

$$\begin{aligned}\int_\Omega |f|\, d|\nu_r| &= \int_c^1 f(t)\,(1-t)^{1/r}dt \\ &= \int_c^1 \frac{\big(1-\ln(1-t)\big)-r}{r\big(1-t\big)^{1+(1/r)}\big(1-\ln(1-t)\big)^2}\cdot(1-t)^{1/r}dt \\ &\geq \int_\alpha^1 \frac{1}{2r\,(1-t)\big(1-\ln(1-t)\big)}\,dt \\ &= \frac{1}{2r}\lim_{t\to1^-}\Big[\ln\big(1-\ln(1-t)\big)-\ln(2r)\Big] = \infty,\end{aligned}$$

where $\alpha := \big(1-\exp(1-2r)\big)$ has the property that $c < \alpha < 1$ and

$$\big(1-\ln(1-t)\big) - r \geq \frac{1}{2}\big(1-\ln(1-t)\big), \qquad t \in (\alpha, 1).$$

On the other hand, we have already shown immediately prior to this example that $f \in L^1(\nu_r)$ and hence, $f \in L^1(\nu_r) \setminus L^1(|\nu_r|)$. In other words, $L^1(\nu_r) \neq L^1(|\nu_r|)$.

(ii-b) The last inclusion $L^1(\nu_r) \subseteq L^1(\nu_1)$ is a special case of Lemma 3.27 below with T being the natural embedding from $E := L^r([0,1])$ into $Z := L^1([0,1])$ and with $\nu := \nu_r$. Now let us show that

$$L^1(\nu_r) \neq L^1(\nu_1). \tag{3.46}$$

The function $f : t \mapsto (1-t)^{-(3r+1)/(2r)}$, on $[0,1)$, belongs to $L^1(|\nu_1|) = L^1(\nu_1)$. Moreover, $I_{\nu_1}(f)(t) = (2r)/(r+1)\Big((1-t)^{-(r+1)/(2r)} - 1\Big)$, for $t \in [0,1)$, which satisfies $I_{\nu_1}(f) \notin L^r([0,1])$. Suppose that $f \in L^1(\nu_r)$. Since $\nu_1 = T \circ \nu_r$, it follows from the definition of T and (3.47) below, with $A = \Omega$, that

$$I_{\nu_1}(f) = \int_\Omega f\, d\nu_1 = \int_\Omega f\, d(T\circ\nu_r) = T\Big(\int_\Omega f\, d\nu_r\Big) = \int_\Omega f\, d\nu_r.$$

But, $\int_\Omega f\, d\nu_r \in L^r([0,1])$ and so $I_{\nu_1}(f) \in L^r([0,1])$, which is not the case. Accordingly, $f \notin L^1(\nu_r)$. This establishes (3.46).

(iii) For $r = \infty$, we have $d|\nu|_\infty = dt$ and

$$L^1([0,1]) = L^1(|\nu_\infty|) = L^1(\nu_\infty);$$

see [129, §4].

(iv) Let $1 < r < \infty$. Since ν_r has finite variation and is non-atomic, it follows from Corollary 3.23(ii) that $L^1(\nu_r)$ is not reflexive. However, $L^1(\nu_r)$ is weakly sequentially complete; see the discussion after Corollary 3.40 below. $\square$

Lemma 3.27. *Let $\nu : \Sigma \to E$ be a Banach-space-valued vector measure and T be a continuous linear operator from E into a Banach space Z.*

(i) *The set function $T\circ\nu : \Sigma \to Z$ is a vector measure. Moreover, every individual ν-integrable function f is also $(T\circ\nu)$-integrable and*

$$\int_A f\, d(T\circ\nu) \;=\; T\Big(\int_A f\, d\nu\Big), \qquad A \in \Sigma. \tag{3.47}$$

(ii) *Every ν-null set is $(T\circ\nu)$-null, in other words,*

$$\mathcal{N}_0(\nu) \;\subseteq\; \mathcal{N}_0(T\circ\nu).$$

(iii) *If, in addition, T is injective, then $\mathcal{N}(\nu) = \mathcal{N}(T\circ\nu)$, so that*

$$L^1(\nu) \;\subseteq\; L^1(T\circ\nu). \tag{3.48}$$

Proof. (i) The linearity of T ensures that $T\circ\nu$ is finitely additive. Moreover, if $\{A_n\}_{n=1}^\infty \subseteq \Sigma$ satisfies $A_n \downarrow \emptyset$, then $\nu(A_n) \to 0$ in E as $n \to \infty$ and hence, by continuity of T, also $(T\circ\nu)(A_n) \to 0$ in Z as $n \to \infty$. Accordingly, $T\circ\nu$ is σ-additive.

Suppose that $f : \Omega \to \mathbb{C}$ is ν-integrable. Given $A \in \Sigma$ there exists an element $\int_A f\, d\nu \in E$ such that $\big\langle \int_A f\, d\nu, x^*\big\rangle = \int_A f\, d\langle \nu, x^*\rangle$ for all $x^* \in E^*$. Then the element $T\big(\int_A f\, d\nu\big) \in Z$ satisfies

$$\Big\langle T\Big(\int_A f\, d\nu\Big),\, z^*\Big\rangle = \Big\langle \int_A f\, d\nu,\, T^*(z^*)\Big\rangle = \int_A f\, d\big\langle \nu,\, T^*(z^*)\big\rangle = \int_A f\, d\langle T\circ\nu,\, z^*\rangle$$

for all $z^* \in Z^*$. By definition, f is then $(T\circ\nu)$-integrable with $\int_A f\, d(T\circ\nu) = T\big(\int_A f\, d\nu\big)$; this is precisely (3.47).

(ii) Remark 3.4(i) and the formula $(T\circ\nu)(A) = T\big(\nu(A)\big)$, for $A \in \Sigma$, easily imply that every ν-null set is also $(T\circ\nu)$-null.

(iii) Let $A \in \mathcal{N}_0(T\circ\nu)$. Then $(T\circ\nu)(B) = T\big(\nu(B)\big) = 0$ for every $B \in \Sigma\cap A$ and so, by injectivity of T, also $\nu(B) = 0$ for every $B \in \Sigma\cap A$, that is, $A \in \mathcal{N}_0(\nu)$. This and part (ii) give $\mathcal{N}_0(\nu) = \mathcal{N}_0(T\circ\nu)$. Hence, Remark 3.4(ii) implies that $\mathcal{N}(\nu) = \mathcal{N}(T\circ\nu)$.

Since $L^1(\nu)$ and $L^1(T\circ\nu)$ are the quotient spaces of the individual ν-integrable and $(T\circ\nu)$-integrable functions with respect to $\mathcal{N}_0(\nu)$ and $\mathcal{N}_0(T\circ\nu)$, respectively, the inclusion (3.48) is clear from $\mathcal{N}(T\circ\nu) = \mathcal{N}(\nu)$ and part (i). □

According to Theorem 3.7, the space $L^1(\nu)$ is a B.f.s. over (Ω, Σ, μ) for any control measure μ for ν. Given $1 < p < \infty$, let

$$L^p(\nu) := \; L^1(\nu)_{[1/p]} \subseteq L^1(\nu). \tag{3.49}$$

Since $L^1(\nu)$ has σ-o.c. norm and the space $L^p(\nu)$ is the $(1/p)$-th power of $L^1(\nu)$ with $0 < 1/p < 1$, it follows that $L^p(\nu)$ is again a B.f.s. with σ-o.c. norm and is contained in $L^1(\nu)$; see Lemma 2.21 and Proposition 2.23(i). In particular, $\operatorname{sim}\Sigma$ is dense in $L^p(\nu)$. Let $\|\cdot\|_{L^p(\nu)}$ denote the corresponding norm on $L^p(\nu)$, i.e.,

$$\|f\|_{L^p(\nu)} := \|f\|_{L^1(\nu)_{[1/p]}} = \big\| |f|^p \big\|_{L^1(\nu)}^{1/p}, \qquad f \in L^p(\nu). \tag{3.50}$$

Via (3.49) we can conclude that

$$L^p(\nu) := \big\{ f \in L^1(\nu) : |f|^p \in L^1(\nu) \big\}, \tag{3.51}$$

which is consistent with the case when ν is a scalar measure. For E a real Banach space, $L^p(\nu)$ was originally defined via (3.51) in [146, Definition 1]; see also [57]. The original norm of $L^p(\nu)$ used in [146, p. 909], namely

$$\sup_{x^* \in \mathbf{B}[E^*]} \Big(\int_\Omega |f|^p \, d|\langle \nu, x^* \rangle| \Big)^{1/p},$$

turns out to be exactly the same as (3.50). The following identity, which is immediate from Lemma 2.20 and (3.49), will be useful later:

$$L^1(\nu) = L^p(\nu)_{[p]}. \tag{3.52}$$

A natural question is whether or not the inclusion $L^p(\nu) \subseteq L^1(\nu)$ is proper. Indeed, it will be seen that it is proper except for trivial cases. To be precise, we say that Σ is *σ-decomposable* relative to ν if Σ admits countably infinite, pairwise disjoint non-ν-null sets. The space $L^\infty(\nu)$ consists of all $\mathbb{C}$-valued, ν-essentially bounded Σ-measurable functions. It is equipped with the essential supremum norm $\|\cdot\|_{L^\infty(\nu)}$ and is a B.f.s. over (Ω, Σ, μ), with μ being any control measure for ν. Bounded Σ-measurable functions are ν-integrable. It follows that $L^\infty(\nu)$ is an order ideal of $L^1(\nu)$. Clearly $L^\infty(\nu) \subseteq L^p(\nu)$ for all $1 \le p \le \infty$; see (3.49).

Proposition 3.28. *Let $1 \le p \le \infty$. Let $\nu : \Sigma \to E$ be a Banach-space-valued measure and let $\mu : \Sigma \to [0, \infty)$ be any control measure for ν.*

- (i) *The space $L^p(\nu)$ is a B.f.s. based on (Ω, Σ, μ) with χ_Ω a weak order unit. Moreover, if $1 \le p < \infty$, then $L^p(\nu)$ is σ-o.c. and is a p-convex Banach lattice with p-convexity constant 1.*
- (ii) *Let Σ be σ-decomposable relative to ν. Then the inclusions $L^\infty(\nu) \subseteq L^p(\nu) \subseteq L^1(\nu)$ are proper for $1 < p < \infty$. Moreover, the inclusion $L^p(\nu) \subseteq L^q(\nu)$ is proper whenever $1 < q < p$.*
- (iii) *If $p \neq \infty$, then the B.f.s. $L^p(\nu)$ is weakly compactly generated.*

Proof. (i) First let $1 \le p < \infty$. That $L^p(\nu)$ is a σ-o.c. Banach lattice in which χ_Ω is a weak order unit has been given (over $\mathbb{R}$) in [146, Proposition 6]. For complex E we refer to the discussion immediately after (3.49).

In either case, $L^p(\nu)$ is then o.c. (see Remark 2.5). As observed in [57, p. 4], for real spaces E the definition of the norm $\|\cdot\|_{L^p(\nu)}$ implies directly that $L^p(\nu)$ is a p-convex Banach lattice with p-convexity constant 1. This can also be obtained from Proposition 2.23(iii) and (3.52), where E can also be over $\mathbb{C}$.

That $L^\infty(\nu)$ is a B.f.s over (Ω, Σ, μ) was indicated just prior to the proposition.

(ii) We first show that $L^p(\nu) \setminus L^\infty(\nu) \neq \emptyset$. Choose pairwise disjoint non-ν-null sets $A(n) \in \Sigma$, for $n \in \mathbb{N}$, and observe that $\|\nu\|(A(n)) \to 0$ as $n \to \infty$ because ν is strongly additive, [42, Ch. I, Corollary1.18]. So, by choosing a subsequence of $\{A(n)\}_{n=1}^\infty$ if necessary, we may assume that $\|\nu\|(A(n)) < 2^{-n}$ for each $n \in \mathbb{N}$. Then the pointwise sum $f := \sum_{n=1}^\infty n^{1/p} \chi_{A(n)}$ belongs to $L^p(\nu)$ because $\|f\|_{L^p(\nu)} \leq \sum_{i=1}^\infty n\|\nu\|(A(n)) < \infty$. Clearly $f \notin L^\infty(\nu)$.

Since $L^p(\nu) \neq L^\infty(\nu)$, we can apply Proposition 2.26 to the q-B.f.s. $X(\mu) := L^p(\nu)$ over (Ω, Σ, μ) to deduce that $L^p(\nu)_{[p]} \neq L^p(\nu)$. This and (3.52) imply that $L^1(\nu) \neq L^p(\nu)$.

Suppose that $1 < q < p$. If $L^p(\nu) = L^q(\nu)$, then also $L^p(\nu)_{[q]} = L^q(\nu)_{[q]}$, that is, $L^{p/q}(\nu) = L^1(\nu)$. Since $(p/q) > 1$, this is impossible.

(iii) This has formally been given in [27, Proposition 3], for real spaces E. Actually, all we need is the fact that the subset $\{\chi_A : A \in \Sigma\}$ is relatively weakly compact in the σ-order continuous B.f.s. $L^p(\nu)$, as argued in the proof of Theorem 3.7(ii) for the case $p = 1$; this argument also applies for E over $\mathbb{C}$. □

For a σ-decomposable scalar measure λ, if $1 \leq p < q < \infty$, then the Banach spaces $L^p(\lambda)$ and $L^q(\lambda)$ are not only distinct but are also non-isomorphic. Indeed, $L^p(\lambda)$ is not q-convex whereas $L^q(\lambda)$ is q-convex; see Example 2.73, for instance.

For σ-decomposable *vector* measures ν the situation can be different; even though $L^p(\nu)$ and $L^q(\nu)$ are distinct (see Proposition 3.28(ii)), they can be isomorphic (even as Banach lattices).

Example 3.29. Let $E := c_0$ and $\Omega := \mathbb{N}$ with $\Sigma := 2^{\mathbb{N}}$. Let $\psi \in c_0$ satisfy $\psi(n) > 0$ for every $n \in \Omega$ and define a finitely additive set function $\nu : \Sigma \to E$ by $\nu(A) = \chi_A \psi$ for each $A \in \Sigma$. For each $\xi \in E^* = \ell^1$ it is routine to check that

$$\langle \nu, \xi \rangle(A) = \sum_{n=1}^\infty \xi(n)\psi(n)\chi_A(n), \qquad A \in \Sigma,$$

and hence, $\langle \nu, \xi \rangle$ is σ-additive (as $\psi\xi \in \ell^1$) with variation measure

$$|\langle \nu, \xi \rangle|(A) = \sum_{n=1}^\infty |\xi(n)|\psi(n)\chi_A(n), \qquad A \in \Sigma.$$

In particular, ν is a vector measure. For each $f \in \operatorname{sim}\Sigma$ it can be calculated that

$$\int_A f\,d\nu = f\psi\chi_A, \qquad A \in \Sigma,$$

from which it follows that the same formula must hold for all $f \in L^1(\nu)$ and that

$$L^1(\nu) = \{f \in \mathbb{C}^{\mathbb{N}} : f\psi \in c_0\}, \qquad \text{i.e.,} \qquad L^1(\nu) = (1/\psi) \cdot c_0. \tag{3.53}$$

Using the fact that

$$\int_\Omega |f|\, d|\langle \nu, \xi\rangle| \;=\; \sum_{n=1}^{\infty} |f(n)| \cdot \psi(n) \cdot |\xi(n)|, \qquad \xi \in E^*,$$

for each $f \in L^1(\nu)$, we can conclude that

$$\|f\|_{L^1(\nu)} = \sup_{\|\xi\|_{\ell^1} \le 1} \big\langle |f|\psi,\, |\xi|\big\rangle \;=\; \|f\psi\|_{c_0}.$$

The multiplication operator $M_\psi : L^1(\nu) \to c_0$ defined by $f \mapsto \psi f$ for $f \in L^1(\nu)$ is a surjective linear isometry and (since $\psi > 0$) a lattice isomorphism. That is, $L^1(\nu)$ is lattice isometric to c_0.

Given $1 \le p < \infty$, it follows that

$$L^p(\nu) \;=\; \big\{f \in \mathbb{C}^{\mathbb{N}} : |f|^p \in L^1(\nu)\big\} \;=\; \big\{f \in \mathbb{C}^{\mathbb{N}} : |f|^p\psi \in c_0\big\} \tag{3.54}$$

with norm

$$\|f\|_{L^p(\nu)} \;=\; \big\|\, |f|^p \big\|_{L^1(\nu)}^{1/p} \;=\; \big\|\, |f|^p\psi \big\|_{c_0}^{1/p} \;=\; \big\|f\psi^{1/p}\big\|_{c_0}, \qquad f \in L^p(\nu);$$

we can then write $L^p(\nu) = \psi^{-1/p} \cdot c_0$. Then the multiplication operator $M_{\psi^{1/p}} : L^p(\nu) \to c_0$ defined by $f \mapsto \psi^{1/p} f$ for $f \in L^p(\nu)$ is an isometric lattice isomorphism of $L^p(\nu)$ onto c_0. In particular, every space $L^p(\nu)$, for $1 \le p < \infty$, is lattice isomorphic to c_0 and, none of them are reflexive or even weakly sequentially complete.

Note that ν has finite variation if and only if $\psi \in \ell^1$, in which case

$$|\nu|(A) \;=\; \sum_{n \in A} \psi(n), \qquad A \in \Sigma, \tag{3.55}$$

and hence,

$$L^p(|\nu|) \;=\; \ell^p(|\nu|) \;=\; \psi^{-1/p} \cdot \ell^p, \qquad 1 \le p < \infty. \tag{3.56}$$

$\square$

The L^p-counterpart of Proposition 3.9 (for real spaces) occurs in [57, Proposition 2.4]. We now provide a more detailed proof of this.

Proposition 3.30. *Let $1 < p < \infty$. If E is a p-convex (complex) Banach lattice with a weak order unit and o.c. norm, then there is an E-valued, positive vector measure ν such that $L^p(\nu)$ and E are lattice isomorphic.*

Proof. There exist a p-convex real Banach lattice $Z_{\mathbb{R}}$, with a weak order unit and p-convexity constant 1, and a real lattice isomorphism $S_{\mathbb{R}}$ from $Z_{\mathbb{R}}$ onto the real part $E_{\mathbb{R}}$ of E, [99, Proposition 1.d.8]. It follows from Lemma 3.8(iv), with $Z := Z_{\mathbb{R}} + iZ_{\mathbb{R}}$ and $W := E$, that the canonical extension $S : Z \to E$ of $S_{\mathbb{R}}$ is a surjective lattice isomorphism. Moreover, Lemma 3.8(v) implies that the Banach lattice Z is itself lattice isometric to a B.f.s. $X(\mu)$ over (Ω, Σ, μ) for some probability measure μ, as shown in the proof of Proposition 3.9 via [99, Theorem 1.b.14]. Consequently, $X(\mu)$ is a p-convex B.f.s. with p-convexity constant 1. Then the p-th power $X(\mu)_{[p]}$ is a B.f.s. whose norm $\|\cdot\|_{X(\mu)_{[p]}}$ is σ-o.c.; see Lemma 2.21(iii) and Proposition 2.23(ii). So, the set function $\nu : A \mapsto \chi_A \in X(\mu)_{[p]}$, defined on Σ, is a vector measure satisfying $X(\mu)_{[p]} = L^1(\nu)$ with equal lattice norms (adapt the proof of Theorem 8 in [21] or Corollary 3.66(ii) below). Consequently,

$$X(\mu) \;=\; \big(X(\mu)_{[p]}\big)_{[1/p]} \;=\; L^1(\nu)_{[1/p]} \;=\; L^p(\nu)$$

with equal lattice norms (see Proposition 2.23(i) and (3.52)). Hence, E and $L^p(\nu)$ are lattice isomorphic. $\square$

Given $1 \le p \le \infty$, let p' stand for its adjoint index, that is,

$$p' := \begin{cases} \infty & \text{if} \quad p = 1, \\ p/(p-1) & \text{if} \quad 1 < p < \infty, \\ 1 & \text{if} \quad p = \infty. \end{cases} \tag{3.57}$$

Let ν be a Banach-space-valued measure and μ be a control measure for ν. We proceed to establish some Hölder type inequalities for ν and to investigate the natural inclusion map

$$\alpha_p : L^p(\nu) \to L^1(\nu). \tag{3.58}$$

First consider the case when $p = 1$. Given $f \in L^1(\nu)$, we have from (3.7) that

$$\|fg\|_{L^1(\nu)} \;\le\; \|f\|_{L^1(\nu)}\, \|g\|_{L^\infty(\nu)}, \qquad g \in L^\infty(\nu). \tag{3.59}$$

Apply this, (3.21) and Lemma 3.11 to deduce that

$$\begin{aligned} \sup_{s \in \mathbf{B}[L^\infty(\nu)] \cap \operatorname{sim}\Sigma} \Big\| \int_\Omega f s \, d\nu \Big\|_E &= \sup_{g \in \mathbf{B}[L^\infty(\nu)]} \Big\| \int_\Omega f g \, d\nu \Big\|_E \\ &= \sup_{g \in \mathbf{B}[L^\infty(\nu)]} \|fg\|_{L^1(\nu)} = \|f\|_{L^1(\nu)}. \end{aligned} \tag{3.60}$$

Next let $1 < p < \infty$. Lemma 2.21(i) implies that $fg \in L^1(\nu)$ for $f \in L^p(\nu)$ and $g \in L^{p'}(\nu)$, that is,

$$L^p(\nu) \cdot L^{p'}(\nu) \;\subseteq\; L^1(\nu). \tag{3.61}$$

Here, of course, $L^p(\nu) \cdot L^{p'}(\nu) := \{fg : f \in L^p(\nu),\ g \in L^{p'}(\nu)\}$. We point out that actually $L^p(\nu) \cdot L^{p'}(\nu) = L^1(\nu)$ in (3.61) holds. Indeed, given $f \in L^1(\nu)$ we have $f = (\varphi|f|^{1/p}) \cdot (|f|^{1/p'})$ with $(\varphi|f|^{1/p}) \in L^p(\nu)$ and $|f|^{1/p'} \in L^{p'}(\nu)$, where $\varphi := \big(f/|f|\big) \cdot \chi_{\Omega \setminus f^{-1}(\{0\})}$ belongs to $L^\infty(\nu)$. The reverse inclusion to (3.61) is clear. Lemma 2.21(i) also yields the Hölder type inequality

$$\|fg\|_{L^1(\nu)} \leq \|f\|_{L^p(\nu)} \|g\|_{L^{p'}(\nu)}, \qquad f \in L^p(\nu), \quad g \in L^{p'}(\nu). \tag{3.62}$$

The above-mentioned fact, that every $f \in L^1(\nu)$ can be factorized as $f = gh$, with $g \in L^p(\nu)$ and $h \in L^{p'}(\nu)$, is an analogue of a similar result for general B.f.s.' of the form L_ρ (relative to some finite measure $\mu \geq 0$); see Remark 2.3(ii) for the definition. Namely, if $\chi_\Omega \in L_\rho$ and ρ is a σ-o.c. norm with the Fatou property, then every $f \in L^1(\mu)$ can be written as $f = gh$ for some $g \in L_\rho$ and $h \in L'_\rho$ (the associate space of L_ρ), [65, Theorem 3.5], [66], [100]

In Chapter 2 we defined uniformly μ-absolutely continuous subsets of a q-B.f.s. over (Ω, Σ, μ). For the particular B.f.s. $L^1(\nu)$, it is easy to see that a subset W of $L^1(\nu)$ is uniformly μ-absolutely continuous if and only if

$$\lim_{\|\nu\|(A) \to \infty} \sup_{f \in W} \|f\chi_A\|_{L^1(\nu)} = 0. \tag{3.63}$$

In this setting it is therefore more natural to say simply that such a set W satisfying (3.63) is *uniformly ν-integrable* because μ can be *any* control measure for ν.

Proposition 3.31. *Let $\nu : \Sigma \to E$ be a Banach-space-valued vector measure and let $1 \leq p \leq \infty$.*

(i) *We have the identities*

$$\sup_{g \in \mathbf{B}[L^{p'}(\nu)]} \Big\| \int_\Omega fg\, d\nu \Big\|_E = \|f\|_{L^p(\nu)} = \sup_{g \in \mathbf{B}[L^{p'}(\nu)]} \|fg\|_{L^1(\nu)}, \qquad f \in L^p(\nu). \tag{3.64}$$

(ii) *The natural embedding $\alpha_p : L^p(\nu) \to L^1(\nu)$ is continuous and its operator norm satisfies*

$$\|\alpha_p\| = \|\chi_\Omega\|_{L^{p'}(\nu)} = \Big(\|\nu\|(\Omega)\Big)^{1/p'}, \tag{3.65}$$

with the understanding that $1/p' = 0$ when $p' = \infty$.

(iii) *Let $1 < p \leq \infty$. Then α_p maps the unit ball of $L^p(\nu)$ to a bounded, uniformly ν-integrable subset of $L^1(\nu)$. In particular, α_p is weakly compact.*

(iv) *Let $1 < p \leq \infty$. Then α_p is compact if and only if ν is purely atomic.*

Proof. (i) If $p = 1$, then (3.64) reduces to (3.60).

Let $1 < p < \infty$. Given $f \in L^p(\nu)$ with $\|f\|_{L^p(\nu)} = 1$ and $\varepsilon > 0$, it follows from (3.60) with $|f|^p \in L^1(\nu)$ in place of f, and the fact that $|f|^{p-1} \in L^{p'}(\nu)$ has norm 1, that there exists a Σ-simple function $s \in \mathbf{B}[L^\infty(\nu)]$ such that

$$\begin{aligned}\big\| \, |f|^p \big\|_{L^1(\nu)} &\le \varepsilon + \left\| \int_\Omega s\,|f|^p \, d\nu \right\|_E \\ &= \varepsilon + \left\| \int_\Omega |f| \cdot \big(s\,|f|^{p-1} \big) \, d\nu \right\|_E \\ &\le \varepsilon + \sup_{g \in \mathbf{B}[L^{p'}(\nu)]} \left\| \int_\Omega |f|\, g \, d\nu \right\|_E .\end{aligned}$$

We have used the fact that $s\,|f|^{p-1} \in \mathbf{B}[L^{p'}(\nu)]$. By writing

$$|f| = f \cdot \big(\chi_{\Omega \setminus f^{-1}(\{0\})} |f|/f \big) \qquad \text{with} \qquad \chi_{\Omega \setminus f^{-1}(\{0\})} |f|/f \in \mathbf{B}[L^\infty(\nu)], \tag{3.66}$$

it follows, since $\varepsilon > 0$ is arbitrary, that

$$\big\| \, |f|^p \big\|_{L^1(\nu)} \le \sup_{g \in \mathbf{B}[L^{p'}(\nu)]} \left\| \int_\Omega f g \, d\nu \right\|_E . \tag{3.67}$$

For $g \in \mathbf{B}[L^{p'}(\nu)]$, it follows from (3.21) and (3.59) that

$$\left\| \int_\Omega f g \, d\nu \right\|_E \le \; \|fg\|_{L^1(\nu)} \le \; \|f\|_{L^p(\nu)}. \tag{3.68}$$

This inequality implies that

$$\sup_{g \in \mathbf{B}[L^{p'}(\nu)]} \left\| \int_\Omega f g \, d\nu \right\|_E \le \; \|f\|_{L^p(\nu)} \; = \; \big\| \, |f|^p \big\|_{L^1(\nu)}^{1/p}. \tag{3.69}$$

The two inequalities (3.67) and (3.69) imply that

$$\|f\|_{L^p(\nu)} \; = \sup_{g \in \mathbf{B}[L^{p'}(\nu)]} \left\| \int_\Omega f g \, d\nu \right\|_E$$

whenever $f \in L^p(\nu)$ satisfies $\|f\|_{L^p(\nu)} = 1$ and hence, for arbitrary $f \in L^p(\nu)$. This establishes the first equality in (3.64).

To deduce the second equality in (3.64), fix $f \in L^p(\nu)$. From (3.68), which does not require the condition that $\|f\|_{L^p(\nu)} = 1$, it follows that

$$\sup_{g \in \mathbf{B}[L^{p'}(\nu)]} \left\| \int_\Omega f g \, d\nu \right\|_E \le \sup_{g \in \mathbf{B}[L^{p'}(\nu)]} \|fg\|_{L^1(\nu)} \le \|f\|_{L^p(\nu)}. \tag{3.70}$$

We have already established that the left-hand side and right-hand side of (3.70) coincide, and so (3.64) holds.

Let $p := \infty$. Fix $f \in L^\infty(\nu)$. We shall first establish the second equality

$$\|f\|_{L^\infty(\nu)} = \sup_{g \in \mathbf{B}[L^1(\nu)]} \|fg\|_{L^1(\nu)} \tag{3.71}$$

in (3.64) with $p := \infty$. Since $|f|$ can be expressed as in (3.66), we have that

$$\sup_{g \in \mathbf{B}[L^1(\nu)]} \|fg\|_{L^1(\nu)} = \sup_{g \in \mathbf{B}[L^1(\nu)]} \big\| |f|g \big\|_{L^1(\nu)},$$

which enables us to assume that $f \geq 0$. Choose non-negative Σ-simple functions $s_n \uparrow f$ such that $\lim_{n\to\infty} \|s_n - f\|_{L^\infty(\nu)} = 0$. In particular, $\|s_n\|_{L^\infty(\nu)} \uparrow \|f\|_{L^\infty(\nu)}$.

Fix $n \in \mathbb{N}$. Let $a := \|s_n\|_{L^\infty(\nu)}$ and $A := s_n^{-1}(\{a\})$, in which case $\|\nu\|(A) > 0$. The function $h := \big(\|\nu\|(A)\big)^{-1}\chi_A$ satisfies

$$\|s_n h\|_{L^1(\nu)} \;=\; a \;=\; \|s_n\|_{L^\infty(\nu)} \qquad \text{and} \qquad \|h\|_{L^1(\nu)} = 1$$

and we have

$$\sup_{g \in \mathbf{B}[L^1(\nu)]} \|fg\|_{L^1(\nu)} \;\geq\; \|s_n h\|_{L^1(\nu)} \;=\; \|s_n\|_{L^\infty(\nu)}. \tag{3.72}$$

Letting $n \to \infty$ in (3.72) yields that

$$\sup_{g \in \mathbf{B}[L^1(\nu)]} \|fg\|_{L^1(\nu)} \;\geq\; \|f\|_{L^\infty(\nu)}. \tag{3.73}$$

On the other hand, the definition of the norm $\|\cdot\|_{L^1(\nu)}$ (see (3.7)) gives

$$\|fg\|_{L^1(\nu)} \;\leq\; \|f\|_{L^\infty(\nu)} \|g\|_{L^1(\nu)}, \qquad g \in L^1(\nu). \tag{3.74}$$

This and (3.73) establish (3.71).

Now let us prove the first equality in (3.64) for $p := \infty$, namely, for any fixed $f \in L^\infty(\nu)$,

$$\sup_{g \in \mathbf{B}[L^1(\nu)]} \left\| \int_\Omega fg \, d\nu \right\|_E \;=\; \|f\|_{L^\infty(\nu)}. \tag{3.75}$$

Let $\varepsilon > 0$. According to (3.71), select $h \in \mathbf{B}[L^1(\nu)]$ satisfying

$$\|f\|_{L^\infty(\nu)} \;\leq\; \varepsilon + \|fh\|_{L^1(\nu)}. \tag{3.76}$$

Appealing to (3.60) with fh in place of f, choose a function $s \in \mathbf{B}[L^\infty(\nu)] \cap \operatorname{sim}\Sigma$ such that

$$\|fh\|_{L^1(\nu)} \;<\; \varepsilon + \left\| \int_\Omega fhs \, d\nu \right\|_E. \tag{3.77}$$

It now follows from (3.76) and (3.77) that

$$\|f\|_{L^\infty(\nu)} \;<\; 2\varepsilon + \left\| \int_\Omega f(hs) \, d\nu \right\|_E. \tag{3.78}$$

Since $\varepsilon > 0$ is arbitrary and $hs \in \mathbf{B}[L^1(\nu)]$ in (3.78), we have the following inequality $\|f\|_{L^\infty(\nu)} \le \sup_{g\in\mathbf{B}[L^1(\nu)]} \big\| \int_\Omega fg\,d\nu\big\|_E$. The reverse inequality is a consequence of the inequality

$$\|fg\|_{L^1(\nu)} \;\le\; \|f\|_{L^\infty(\nu)}\|g\|_{L^1(\nu)},$$

which is immediate from (3.21). This establishes (3.75).

(ii) We can interchange p with p' in (3.64) to derive

$$\sup_{f\in\mathbf{B}[L^p(\nu)]} \left\| \int_\Omega fg\,d\nu\right\|_E \;=\; \sup_{f\in\mathbf{B}[L^p(\nu)]} \big\|fg\big\|_{L^1(\nu)} \;=\; \big\|g\big\|_{L^{p'}(\nu)}, \qquad g \in L^{p'}(\nu). \tag{3.79}$$

By the definition of the operator norm,

$$\|\alpha_p\| := \sup_{f\in\mathbf{B}[L^p(\nu)]} \|f\|_{L^1(\nu)} \;=\; \sup_{f\in\mathbf{B}[L^p(\nu)]} \|f\cdot\chi_\Omega\|_{L^1(\nu)}.$$

It follows from (3.79), with $g := \chi_\Omega$, that $\|\alpha_p\| = \|\chi_\Omega\|_{L^{p'}(\nu)} = \big(\|\nu\|(\Omega)\big)^{1/p'}$; this is precisely (3.65).

(iii) By (3.79), with $g := \chi_A$ and $A \in \Sigma$, we have

$$\sup_{f\in\mathbf{B}[L^p(\nu)]} \|f\chi_A\|_{L^1(\nu)} \;=\; \Big(\|\nu\|(A)\Big)^{1/p'}.$$

Since $1 \le p' < \infty$, it is clear that $\big(\|\nu\|(A)\big)^{1/p'} \to 0$ as $\|\nu\|(A) \to 0$. According to (3.63), the subset $\alpha_p\big(\mathbf{B}[L^p(\nu)]\big)$ is bounded and uniformly ν-integrable in $L^1(\nu)$. Now apply Proposition 2.39(ii) to conclude that α_p is weakly compact.

(iv) If ν is not purely atomic, then its non-atomic part Ω_{na} is non-ν-null and so the set $\big\{\alpha_p(\chi_B) : B \in \Sigma\cap\Omega_{\mathrm{na}}\big\} = \big\{\chi_{B\cap\Omega_{\mathrm{na}}} : B \in \Sigma\big\}$ is not relatively compact in $L^1(\nu)$ via Lemma 3.21. Hence, α_p is not compact.

So, assume that ν is purely atomic. The vector measure $[\nu] : \Sigma \to L^1(\nu)$ given by (3.10) is also purely atomic and hence, has compact range, [78, Theorem 19]. Let $T : L^1(\nu) \to L^1(\nu)$ denote the identity map. Then $T(\chi_A) = [\nu](A)$ for $A \in \Sigma$ and so $\big\{T(\chi_A) : A \in \Sigma\big\}$ is relatively compact in $L^1(\nu)$. Apply Proposition 2.41 with $X(\mu) := L^1(\nu)$ and $E := L^1(\nu)$ to deduce that $\alpha_p\big(\mathbf{B}[L^p(\nu)]\big) = T\big(\alpha_p(\mathbf{B}[L^p(\nu)])\big)$ is relatively compact in $L^1(\nu)$ because $\alpha_p\big(\mathbf{B}[L^p(\nu)]\big)$ is bounded and uniformly ν-integrable in $L^1(\nu)$ via part (iii). Thus, α_p is compact. □

When the scalar field is real and $1 < p < \infty$, part (iii) of Proposition 3.31 follows from [57, Proposition 3.3]. For further details, see Remark 3.42(i) below.

Remark 3.32. We give an alternate proof of Proposition 3.31(iv) which may be of some interest. Let $\tilde{E}$ denote the (complex) space E interpreted as a vector space over $\mathbb{R}$ and $\tilde{\nu}$ denote ν considered as an $\tilde{E}$-valued vector measure. Then a Σ-measurable function $f : \Omega \to \mathbb{R}$ is ν-integrable if and only if f is $\tilde{\nu}$-integrable and then $\int_A f\,d\tilde{\nu} = \int_A f\,d\nu$ for all $A \in \Sigma$, [58, Lemma 3]. Moreover, $f : \Omega \to \mathbb{C}$ is ν-integrable if and only if $|f|$ is $\tilde{\nu}$-integrable if and only if both $\mathrm{Re}(f)$, $\mathrm{Im}(f)$ are $\tilde{\nu}$-integrable, [58, Lemma 2]. It follows from [58, Lemma 4] that $L^1(\tilde{\nu})$ is isomorphic to $L^1_{\mathbb{R}}(\nu) := \{f \in L^1(\nu) : \mathrm{Im}(f) = 0\}$ equipped with the relative topology from $L^1(\nu)$. According to [58, Lemma 1], there exist positive constants C_1, C_2 such that

$$C_1\|\tilde{\nu}\|(A) \le \|\nu\|(A) \le C_2\|\tilde{\nu}\|(A), \qquad A \in \Sigma.$$

It is then clear from (3.63) and the fact that both $L^1(\tilde{\nu})$ and $L^1(\nu)$ have lattice norms that a subset $W \subseteq L^1(\nu)$ is bounded and uniformly ν-integrable if and only if $|W| := \{|f| : f \in W\} \subseteq L^1(\tilde{\nu}) \simeq L^1_{\mathbb{R}}(\nu)$ is bounded and uniformly $\tilde{\nu}$-integrable. In particular, both $\mathrm{Re}(W) := \{\mathrm{Re}(f) : f \in W\}$ and $\mathrm{Im}(W) := \{\mathrm{Im}(f) : f \in W\}$ are bounded and uniformly $\tilde{\nu}$-integrable in $L^1(\tilde{\nu})$.

Now, suppose that ν is purely atomic, in which case $L^1(\nu) = \ell^1(\nu) \subseteq \mathbb{C}^{\mathbb{N}}$ (see Lemma 3.20(ii)) is an atomic Banach lattice with o.c. norm. Let $W \subseteq L^1(\nu)$ be any bounded, uniformly ν-integrable set. According to Remark 2.38(b) and the previous paragraph, both $\mathrm{Re}(W)$, $\mathrm{Im}(W)$ are L-weakly compact sets in $L^1(\tilde{\nu})$. Since $L^1(\tilde{\nu}) = \ell^1(\tilde{\nu})$ is an atomic real Banach lattice with o.c. norm, it follows that both $\mathrm{Re}(W)$, $\mathrm{Im}(W)$ are actually relatively compact in $L^1(\tilde{\nu}) \simeq L^1_{\mathbb{R}}(\nu)$, [107, Beispiel II.7(ii)] (see also [6, Satz 1.1] and [21, Lemma 2]). Accordingly, $W \subseteq \mathrm{Re}(W) + i\mathrm{Im}(W)$ is relatively compact in $L^1(\nu)$. Since α_p maps $\mathbf{B}[L^p(\nu)]$ to a bounded, uniformly ν-integrable subset of $L^1(\nu)$ (see Proposition 3.31(iii)), it follows that α_p is compact. $\square$

Remark 3.33. Let $1 < p < \infty$ and the notation be as in Proposition 3.31. Then the following three conditions are mutually equivalent:

(a) α_p is compact,
(b) α_p is completely continuous, and
(c) ν is purely atomic.

In other words, under the assumption that $1 < p < \infty$ (i.e., $p = \infty$ is excluded), we can improve part (iv) of Proposition 3.31. Since (a) $\Rightarrow$ (b) is clear, to verify the equivalences of (a), (b) and (c) it suffices to show that (b) $\Rightarrow$ (c). Now, the subset $\{\chi_A : A \in \Sigma\}$ is relatively weakly compact in $L^p(\nu)$ as it is the range of the vector measure $A \mapsto \chi_A \in L^p(\nu)$ defined on Σ (see Lemma 3.3). So, (b) implies that $\{\chi_A : A \in \Sigma\}$ is relatively compact in $L^1(\nu)$. However, via Lemma 3.21, this holds only if ν is purely atomic. So, (c) holds.

Concerning the case when $p = \infty$, it turns out that $\alpha_\infty : L^\infty(\nu) \to L^1(\nu)$ is always completely continuous. Indeed, α_∞ is weakly compact by Proposition 3.31(iii). Since $L^\infty(\nu)$ is isomorphic to a $C(K)$-space, it has the Dunford–Pettis property, [42, Ch. VI, Corollary 2.6], and hence, α_∞ is completely continuous. $\square$

We now consider the space of scalarly integrable functions with respect to a Banach-space-valued measure $\nu : \Sigma \to E$. A function $f \in \mathcal{L}^0(\Sigma)$ satisfying condition (I-1) is called *scalarly ν-integrable.* By identifying scalarly ν-integrable functions which are ν-a.e. equal, we denote by $L^1_{\mathrm{w}}(\nu)$ the vector space of (equivalence classes of) all scalarly ν-integrable functions. Clearly $L^1(\nu) \subseteq L^1_{\mathrm{w}}(\nu)$. If E does not contain an isomorphic copy of c_0, then $L^1(\nu) = L^1_{\mathrm{w}}(\nu)$; see [98, Theorem 5.1] for E over $\mathbb{C}$, [86, Ch. II, Theorem 5.1] for E over $\mathbb{R}$. The first systematic study of $L^1_{\mathrm{w}}(\nu)$, for E over $\mathbb{R}$, was carried out by G.F. Stefansson [151] who showed that $L^1_{\mathrm{w}}(\nu)$ is a Banach space containing $L^1(\nu)$ as a closed subspace. To be precise, let

$$\|f\|_{L^1_{\mathrm{w}}(\nu)} := \sup\left\{\int_\Omega |f|\, d|\langle \nu, x^*\rangle| : x^* \in \mathbf{B}[E^*]\right\}, \qquad f \in L^1_{\mathrm{w}}(\nu). \tag{3.80}$$

It follows from [151, Theorem 9] that $\|f\|_{L^1_{\mathrm{w}}(\nu)} < \infty$ for every $f \in L^1_{\mathrm{w}}(\nu)$ and that $\|\cdot\|_{L^1_{\mathrm{w}}(\nu)}$ is a norm for which $L^1_{\mathrm{w}}(\nu)$ becomes a Banach space. In particular, $L^1(\nu)$ is a closed subspace of $L^1_{\mathrm{w}}(\nu)$. An examination of the proofs given in [151] shows that the same conclusions hold for E over $\mathbb{C}$; the "general results" from integration with respect to ν having values in E (over $\mathbb{R}$) used in [151] also apply to complex spaces E and can be found in [58], [97], [98], for example.

It is clear that $L^1_{\mathrm{w}}(\nu)$ is a vector lattice, with respect to the ν-a.e. order, and that $\|\cdot\|_{L^1_{\mathrm{w}}(\nu)}$ is a lattice norm. Indeed, using the notation of Remark 3.32 and recalling, for a complex measure λ, that $f \in L^1(\lambda)$ means $\|f\|_1 := \int |f|\, d|\lambda| < \infty$, it follows that $L^1_{\mathrm{w}}(\nu)$ is the complexification of the real vector lattice $L^1_{\mathrm{w}}(\tilde{\nu}) := \{f \in L^1_{\mathrm{w}}(\nu) : f \text{ is } \mathbb{R}\text{-valued}\}$. To investigate $L^1_{\mathrm{w}}(\nu)$ in the context of B.f.s.' let $\mu : \Sigma \to [0,\infty)$ be a control measure for ν. Since $\operatorname{sim}\Sigma \subseteq L^1(\nu) \subseteq L^1_{\mathrm{w}}(\nu)$, it follows that $(L^1_{\mathrm{w}}(\nu), \|\cdot\|_{L^1_{\mathrm{w}}(\nu)})$ is a B.f.s. based on the measure space (Ω, Σ, μ) and that $L^1(\nu)$ is a closed sublattice of $L^1_{\mathrm{w}}(\nu)$.

A general Banach lattice $(Z, \|\cdot\|_Z)$ has the *weak Fatou property* if every norm bounded, increasing sequence $\{z_n\}_{n=1}^\infty \subseteq Z^+$ has a lattice supremum $z := \bigvee_{n\in\mathbb{N}} z_n$ in Z. If, in addition, $\|z_n\|_Z \uparrow \|z\|_Z$, then Z is said to have the *σ-Fatou property.* Since these definitions are in terms of Z^+, there is no distinction between Z being a Banach lattice over $\mathbb{R}$ or $\mathbb{C}$.

It follows from [57, Lemma 3.8] that the B.f.s. $L^1_{\mathrm{w}}(\nu)$ has the weak Fatou property. Moreover, according to [26, Proposition 2.1], the space $L^1_{\mathrm{w}}(\nu)$ has the σ-Fatou property. Indeed, assume that $L^1_{\mathrm{w}}(\nu)^+ \ni f_n \uparrow$ with $\sup_{n\in\mathbb{N}} \|f_n\|_{L^1_{\mathrm{w}}(\nu)} < \infty$. Let $f := \sup_{n\in\mathbb{N}} f_n$ (defined μ-a.e. pointwise). Then

$$\begin{aligned}\sup_{x^*\in\mathbf{B}[E^*]} \int_\Omega f\, d|\langle \nu, x^*\rangle| &= \sup_{x^*\in\mathbf{B}[E^*]} \sup_{n\in\mathbb{N}} \int_\Omega f_n\, d|\langle \nu, x^*\rangle| \\ &= \sup_{n\in\mathbb{N}} \sup_{x^*\in\mathbf{B}[E^*]} \int_\Omega f_n\, d|\langle \nu, x^*\rangle| \;=\; \sup_{n\in\mathbb{N}} \|f_n\|_{L^1_{\mathrm{w}}(\nu)} < \infty.\end{aligned}$$

So, $f \in L^1_{\mathrm{w}}(\nu)$ and clearly $f = \bigvee_{n\in\mathbb{N}} f_n$. Hence, $L^1_{\mathrm{w}}(\nu)$ has the σ-Fatou property. On the other hand, $L^1(\nu)$ may not have the σ-Fatou or the weak Fatou property; see Example 3.34 below.

Given $1 \le p < \infty$, let $L^p_{\mathrm{w}}(\nu)$ denote the $(1/p)$-th power of the B.f.s. $L^1_{\mathrm{w}}(\nu)$, that is,

$$L^p_{\mathrm{w}}(\nu) := L^1_{\mathrm{w}}(\nu)_{[1/p]} \subseteq L^1_{\mathrm{w}}(\nu); \tag{3.81}$$

see Lemma 2.21(iv). As for $p = 1$, we point out that the vector lattice $L^p_{\mathrm{w}}(\nu)$ is the complexification of the real vector lattice $\{f \in L^p_{\mathrm{w}}(\nu) : f \text{ is } \mathbb{R}\text{-valued}\}$. According to Proposition 2.23(i), the vector lattice $L^p_{\mathrm{w}}(\nu)$ is again a B.f.s. (over (Ω, Σ, μ)) with respect to the corresponding norm

$$\|\cdot\|_{L^p_{\mathrm{w}}(\nu)} := \|\cdot\|_{L^1_{\mathrm{w}}(\nu)_{[1/p]}}. \tag{3.82}$$

The following identity is clear:

$$\|f\|_{L^p_{\mathrm{w}}(\nu)} = \sup_{x^* \in \mathbf{B}[E^*]} \left(\int_\Omega |f|^p \, d|\langle \nu, x^* \rangle| \right)^{1/p}, \qquad f \in L^p_{\mathrm{w}}(\nu). \tag{3.83}$$

The space $L^p_{\mathrm{w}}(\nu)$, for real space E, was defined and studied for the first time in [57], where the norm is defined by the right-hand side of (3.83). Clearly, $L^p(\nu)$ is a closed sublattice of $L^p_{\mathrm{w}}(\nu)$. Furthermore, $L^p_{\mathrm{w}}(\nu)$ is a p-convex B.f.s. with p-convexity constant equal to 1; this can be seen by direct computation or by appealing to Proposition 2.23(ii) and the identity $L^1_{\mathrm{w}}(\nu) = L^p_{\mathrm{w}}(\nu)_{[p]}$.

Convention. In order to apply various results from Chapter 2, we have so far been treating $L^p(\nu)$ $(1 \le p \le \infty)$ and $L^p_{\mathrm{w}}(\nu)$ $(1 \le p < \infty)$ as B.f.s.' over a measure space (Ω, Σ, μ), where μ is any control measure for ν and, of course, the same μ is used simultaneously for both $L^p(\nu)$ and $L^p_{\mathrm{w}}(\nu)$. Subsequently, we may sometimes speak of the B.f.s.' $L^p(\nu)$ and $L^p_{\mathrm{w}}(\nu)$ without referring to such a control measure μ. In this case, it is to be understood that such spaces are B.f.s.' relative to some control measure for ν.

Example 3.34. Let the setting be as in Example 3.29. Then, it is easy to check that

$$L^1_{\mathrm{w}}(\nu) = (1/\psi) \cdot \ell^\infty,$$

in contrast with the identity $L^1(\nu) = (1/\psi) \cdot c_0$ in (3.53). The corresponding norm on $L^1_{\mathrm{w}}(\nu)$ is the weighted sup-norm $\|f\|_{L^1_{\mathrm{w}}(\nu)} = \sup_{n\in\mathbb{N}} \big|\psi(n) f(n)\big|$ for $f \in L^1_{\mathrm{w}}(\nu)$. Consequently, $L^p_{\mathrm{w}}(\nu) = \psi^{-1/p} \cdot \ell^\infty$, for each $1 \le p < \infty$, with $\|f\|_{L^p_{\mathrm{w}}(\nu)} = \sup_{n\in\mathbb{N}} \big|\psi(n)^{1/p} f(n)\big|$. So, both the inclusions

$$L^p(|\nu|) \subseteq L^p(\nu) \subseteq L^p_{\mathrm{w}}(\nu), \qquad 1 \le p < \infty,$$

are proper because $L^p(|\nu|) = \psi^{-1/p} \cdot \ell^p$ via (3.56) and $L^p(\nu) = \psi^{-1/p} \cdot c_0$ via (3.54).

Note that the norm bounded, increasing sequence $\big\{\psi^{-1/p} \chi_{\{1,\dots,n\}}\big\}_{n=1}^\infty$ in $L^p(\nu)^+$ does not have a supremum in the Banach lattice $L^p(\nu)$. It follows that $L^p(\nu)$ fails to have the weak Fatou property. As for the spaces $L^p(\nu)$ in Example

3.29, we see that the spaces $L^p_{\mathrm{w}}(\nu)$, for $1 \le p < \infty$, are all distinct yet, each one is lattice isomorphic to ℓ^∞.

On the other hand, the *complete* normed function space $L^p(\nu)$ has the Riesz–Fischer property. This serves as a counterexample for the converse of Lemma 2.33. □

The above example is standard; see [98, §5] and [86, Ch. II, Example 5.1]. Further classical examples of vector measures ν for which $L^1(\nu)$ fails to have the weak Fatou property (hence, $L^1(\nu) \neq L^1_{\mathrm{w}}(\nu)$) arise in the theory of kernel operators, [26, §3]. Note that there also exist c_0-valued vector measures ν for which we do have the equality $L^1(\nu) = L^1_{\mathrm{w}}(\nu)$, [21, Example on pp. 320–321].

Let $1 \le p < \infty$. Just like $L^1_{\mathrm{w}}(\nu)$ itself, its $(1/p)$-th power $L^1_{\mathrm{w}}(\nu)_{[1/p]} = L^p_{\mathrm{w}}(\nu)$ always has the σ-Fatou property. Indeed, for ν with values in a real Banach space E this is Proposition 1 of [26]. Since the Fatou property of E over $\mathbb{C}$ is defined in terms of $\{f \in L^p_{\mathrm{w}}(\nu) : f \text{ is } \mathbb{R}\text{-valued}\}$, the same conclusion holds for complex spaces E. However, the B.f.s. $L^p(\nu)$ may not have the σ-Fatou property. Actually, $L^p_{\mathrm{w}}(\nu)$ is the minimal B.f.s. which possesses the σ-Fatou property and contains $L^p(\nu)$ in such a way that the natural embedding has operator norm at most 1. We refer to [27] for the details.

For E a real Banach space, the weak sequential completeness of $L^p(\nu)$ and $L^p_{\mathrm{w}}(\nu)$ has already been investigated; see [26], [151] for $p = 1$, and [27] for $1 < p < \infty$. For complex E, a basic tool needed in this regard is Lemma 3.37 below. For its proof, we shall apply the following criterion.

Lemma 3.35. *Let Z be a complex Banach lattice.*

(i) *Z is weakly sequentially complete if and only if its real part $Z_{\mathbb{R}}$ is weakly sequentially complete.*

(ii) *Z is weakly sequentially complete if and only if every norm bounded, increasing sequence in Z^+ is convergent in Z.*

Proof. (i) Suppose that Z is weakly sequentially complete. Let $\{x_n\}_{n=1}^\infty$ be a weak Cauchy sequence in $Z_{\mathbb{R}}$ and $\eta + i\zeta \in Z^* = Z^*_{\mathbb{R}} + iZ^*_{\mathbb{R}}$ with $\eta, \zeta \in Z^*_{\mathbb{R}}$. Then the inequalities

$$\big|\langle x_n - x_k,\ \eta + i\zeta\rangle\big| \ \le\ \big|\langle x_n - x_k,\ \eta\rangle\big| + \big|\langle x_n - x_k,\ \zeta\rangle\big|, \qquad n, k \in \mathbb{N},$$

show that $\{x_n\}_{n=1}^\infty$ is weakly Cauchy in Z and hence, has a weak limit $z \in Z$. Write $z = x + iy$ with $x, y \in Z_{\mathbb{R}}$. Given $\xi \in Z^*_{\mathbb{R}}$ we have $\xi = \xi + i0 \in Z^*$ and so (in $\mathbb{C}$)

$$\langle x_n, \xi\rangle \to \langle x + iy,\ \xi\rangle = \langle x, \xi\rangle + i\langle y, \xi\rangle \qquad \text{as} \quad n \to \infty.$$

Since $\big\{\langle x_n, \xi\rangle\big\}_{n=1}^\infty \subseteq \mathbb{R}$, it follows that $\langle y, \xi\rangle = 0$ and $\langle x_n - x,\ \xi\rangle \to 0$ (in $\mathbb{R}$) as $n \to \infty$. Accordingly, $x_n \to x$ weakly in $Z_{\mathbb{R}}$, that is, $Z_{\mathbb{R}}$ is weakly sequentially complete.

Conversely, assume that $Z_{\mathbb{R}}$ is weakly sequentially complete and $\{z_n\}_{n=1}^{\infty}$ is a weak Cauchy sequence in Z. Write $z_n = x_n + iy_n$ with $x_n, y_n \in Z_{\mathbb{R}}$, for $n \in \mathbb{N}$. Then, given $\rho \in Z_{\mathbb{R}}^* \subseteq Z^*$ the sequence $\big\{\langle z_n, \rho\rangle\big\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{C}$, that is, $\big\{\langle x_n, \rho\rangle\big\}_{n=1}^{\infty}$ and $\big\{\langle y_n, \rho\rangle\big\}_{n=1}^{\infty}$ are both Cauchy sequences in $\mathbb{R}$. Hence, both $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are weak Cauchy sequences in $Z_{\mathbb{R}}$; denote their weak limits (in $Z_{\mathbb{R}}$) by x and y, respectively. It is routine to check that

$$\big|\langle z_n - (x+iy),\ \eta + i\zeta\rangle\big| \le \big|\langle x_n - x,\ \eta\rangle\big| + \big|\langle y_n - y,\ \zeta\rangle\big| + \big|\langle x_n - x, \zeta\rangle\big| + \big|\langle y_n - y, \eta\rangle\big|$$

for all $n \in \mathbb{N}$ and $\eta + i\zeta \in Z^*$ (with $\eta, \zeta \in Z_{\mathbb{R}}^*$), from which it follows that $z_n \to (x + iy)$ weakly in Z as $n \to \infty$. So, Z is weakly sequentially complete.

(ii) According to part (i), $Z = Z_{\mathbb{R}} + iZ_{\mathbb{R}}$ is weakly sequentially complete if and only if this is the case for $Z_{\mathbb{R}}$. Since $Z^+ = Z_{\mathbb{R}}^+ \subseteq Z_{\mathbb{R}}$ and the condition:

$$\text{Every norm bounded, increasing sequence in } Z^+ \text{ is convergent in } Z \qquad (3.84)$$

is invariant under exchanging Z with $Z_{\mathbb{R}}$, the stated result for Z follows from that for $Z_{\mathbb{R}}$ (which can be found in [99, Theorem 1.c.4], for example). □

Banach lattices satisfying condition (3.84) are usually called KB-*spaces*, [2, Definition 14.10]. So, Lemma 3.35 states that a Banach lattice is weakly sequentially complete if and only if it is a KB-space; for real spaces Z see [2, Theorem 14.12].

For real spaces E, the following result is known, [57, Proposition 3.9].

Lemma 3.36. *Let* $1 < p < \infty$ *and* $\nu : \Sigma \to E$ *be a Banach-space-valued measure. Then the following conditions are equivalent.*

(i) $L^p_{\mathrm{w}}(\nu)$ *has o.c. norm.*

(ii) $L^p_{\mathrm{w}}(\nu)$ *is a KB-space.*

(iii) $L^p_{\mathrm{w}}(\nu)$ *is reflexive.*

Proof. (i) ⇒ (ii). This can be argued as in the proof of (3) ⇒ (2) in Proposition 3.9 of [57], after noting that Lemma 3.8 of [57] is also valid for complex spaces E (with the same proof).

(ii) ⇒ (iii). According to the comment immediately after (3.84), it follows from (ii) that the real Banach lattice $\{f \in L^p_{\mathrm{w}}(\nu) : f \text{ is } \mathbb{R}\text{-valued}\}$ is also a KB-space. According to [57, Proposition 3.9], this real Banach lattice is reflexive and hence, so is its complexification $L^p_{\mathrm{w}}(\nu)$.

(iii) ⇒ (i). Since the real part $\{f \in L^p_{\mathrm{w}}(\nu) : f \text{ is } \mathbb{R}\text{-valued}\}$ of $L^p_{\mathrm{w}}(\nu)$ is also reflexive, it follows from [57, Proposition 3.9] that this real part has o.c.-norm. But, order continuity of the norm is completely determined by the positive cone and hence, $L^p_{\mathrm{w}}(\nu)$ has o.c.-norm. □

Lemma 3.37. *Let* $(Z, \|\cdot\|_Z)$ *be a Banach lattice. The following assertions are equivalent.*

(i) *Z is weakly sequentially complete.*

(ii) *Z has the σ-Fatou property and σ-o.c. norm.*

(iii) *Z has the weak Fatou property and σ-o.c. norm.*

Proof. We sketch the proof even though this lemma is known; for example, the equivalence (i) $\Leftrightarrow$(ii) is in [26, Lemma 1.2].

(i) $\Rightarrow$ (ii). Let $\{z_n\}_{n=1}^\infty \subseteq Z^+$ be a norm bounded, increasing sequence. By Lemma 3.35(ii), $\{z_n\}_{n=1}^\infty$ has a norm limit z in Z. Therefore, given $n \in \mathbb{N}$, we have

$$\left\|(z_n \wedge z) - z_n\right\|_Z = \lim_{k\to\infty} \left\|(z_n \wedge z_k) - z_n\right\|_Z = 0,$$

since $z_n \wedge z_k = z_n$ for all $k \geq n$. Hence, $z_n \wedge z = z_n$, that is, $z \geq z_n$. Since Z is Dedekind σ-complete, [165, pp. 421–422], the supremum $w := \bigvee_{n\in\mathbb{N}} z_n \in Z$ exists, and then $z \geq w$ is clear. Hence, $w - z_n \leq z - z_n$ and so $\|w - z_n\|_Z \leq \|z - z_n\|_Z$ for all $n \in \mathbb{N}$. Accordingly, $z = w$ and it follows that Z has the σ-Fatou property. Proving that $\|\cdot\|_Z$ is σ-o.c. is now routine.

(ii) $\Rightarrow$ (iii). Clear.

(iii) $\Rightarrow$ (i). Let $\{z_n\}_{n=1}^\infty$ be a norm bounded, increasing sequence in Z^+. The weak Fatou property of Z ensures that $z := \bigvee_{n\in\mathbb{N}} z_n \in Z$ exists. Since $(z - z_n) \downarrow 0$ and Z has σ-o.c. norm, it follows that $\|z - z_n\|_Z \to 0$, that is, $\{z_n\}_{n=1}^\infty$ is convergent in Z. So, (i) holds via Lemma 3.35(ii). $\square$

An immediate consequence of Lemma 3.37 is that any Banach lattice with the weak Fatou property but, failing the σ-Fatou property, does not admit any σ-o.c. norm. An example of such a Banach lattice can be found in [164, Exercise 65.1]. Such a Banach space *cannot* be represented as $L^1(\nu)$ for any vector measure ν. Nevertheless, the above lemma turns out to be useful when applied to $L^p(\nu)$ and $L^p_{\mathrm{w}}(\nu)$, as demonstrated in [26], [27] for real spaces E. Indeed, the B.f.s. $L^p(\nu)$, which always has o.c. norm (see Proposition 3.28), is weakly sequentially complete if and only if it has the σ-Fatou property; see Lemma 3.37. Moreover, the B.f.s. $L^p_{\mathrm{w}}(\nu)$, which always has the σ-Fatou property (as established above), is weakly sequentially complete if and only if it has o.c. norm; again see Lemma 3.37. Let us formally present some basic facts along these lines.

Proposition 3.38. *Let $\nu : \Sigma \to E$ be a Banach-space-valued measure and let $p \in [1, \infty)$.*

(I) *The following assertions are equivalent.*

  (i) *$L^p(\nu)$ is weakly sequentially complete.*

  (ii) *$L^p_{\mathrm{w}}(\nu)$ is weakly sequentially complete.*

  (iii) *$L^p(\nu) = L^p_{\mathrm{w}}(\nu)$.*

  (iv) *$L^p_{\mathrm{w}}(\nu)$ has o.c. norm.*

  (v) *$L^p_{\mathrm{w}}(\nu)$ is weakly compactly generated.*

(vi) $L^p(\nu)$ *has the* σ*-Fatou property.*

(vii) $L^p(\nu)$ *has the weak Fatou property.*

(viii) $\operatorname{sim}\Sigma$ *is dense in* $L^p_{\mathrm{w}}(\nu)$.

(II) *If* $1 < p < \infty$, *then* (i)–(viii) *are equivalent to each of the following assertions.*

(ix) $L^p(\nu)$ *is reflexive.*

(x) $L^p_{\mathrm{w}}(\nu)$ *is reflexive.*

Proof. (I) For $p = 1$ and a real space E, the equivalences of (i)–(iv) have been obtained in [151, Theorem 10] while the equivalences of these with (v)–(vi) occur in [26, Proposition 2.3]. A careful examination of the proofs given there, together with Lemma 3.35, Lemma 3.37 and the discussion concerning these lemmas, show that the proofs given for real E carry over to complex spaces E.

When $1 < p < \infty$ and E is a real space, the equivalences of (i)–(vi) have been established in [27, Proposition 3]. The "same arguments" can be adapted to complex spaces E.

For $1 \le p < \infty$, the equivalence (i) $\Leftrightarrow$ (vii) follows from Lemma 3.37 because $L^p(\nu)$ has o.c. norm. The equivalence (iii) $\Leftrightarrow$ (viii) is clear from the fact that $\operatorname{sim}\Sigma$ is dense in the B.f.s. $L^p(\nu)$ which has o.c. norm.

(II) See [57, Corollary 3.10] for E over $\mathbb{R}$. For general E, since $L^p(\nu)$ is a closed subspace of $L^p_{\mathrm{w}}(\nu)$, the implication (x) $\Rightarrow$ (ix) is clear. On the other hand, (ix) implies (i) from which (x) follows via Lemma 3.36 together with (i) $\Leftrightarrow$ (iv). Finally, (iv) $\Leftrightarrow$ (x) is part of Lemma 3.36. $\square$

Remark 3.39. In part (I) of the previous proposition, if (i)–(viii) hold for *some* $1 \le p < \infty$, then, for *every* $1 \le r < \infty$, conditions (i)–(iii) hold with r in place of p. Indeed, by part (iii) we have $L^p(\nu) = L^p_{\mathrm{w}}(\nu)$ and hence, $L^p(\nu)_{[p/r]} = L^p_{\mathrm{w}}(\nu)_{[p/r]}$ for all $1 \le r < \infty$. But, $L^p(\nu)_{[p/r]} = L^r(\nu)$ and $L^p_{\mathrm{w}}(\nu)_{[p/r]} = L^r_{\mathrm{w}}(\nu)$.

A similar remark applies to part (II). $\square$

A real Banach lattice is weakly sequentially complete if and only if it does not contain a (lattice) isomorphic copy of c_0, [108, Theorems 2.4.12 and 2.5.6]. Recall that a closed subspace V of a complex Banach lattice $Z = Z_{\mathbb{R}} + iZ_{\mathbb{R}}$ is said to be a *sublattice* of Z if there exists a closed (real) sublattice U of $Z_{\mathbb{R}}$ such that $V = U + iU$ is the complexification of U and $|v| \in U$ for every $v \in V$. This later condition is automatic whenever U is Dedekind complete. Note that V is conjugate closed. If U is lattice isomorphic to the real (Dedekind complete) sequence space $(c_0)_{\mathbb{R}}$, then we say that V is lattice isomorphic to complex c_0, because $V \simeq (c_0)_{\mathbb{R}} + i(c_0)_{\mathbb{R}}$. According to Lemma 3.35, a complex Banach lattice Z is weakly sequentially complete if and only if its real part $Z_{\mathbb{R}}$ is weakly sequentially complete. These observations, together with the above fact for real Banach lattices, imply that a complex Banach lattice is weakly sequentially complete if and only if it does not contain a sublattice which is isomorphic to complex c_0. This fact and Proposition

3.38 imply the following result, first obtained for real spaces and $p = 1$ in [21, Theorem 3 and Corollary].

Corollary 3.40. *Let $1 \le p < \infty$. If the codomain of a vector measure ν is a complex Banach lattice which does not contain a sublattice isomorphic to complex c_0, then $L^1(\nu)$ is weakly sequentially complete or, equivalently, $L^1(\nu)$ does not contain a lattice isomorphic copy of complex c_0.*

Proof. If E does not contain an isomorphic copy of complex c_0, then $L^1(\nu) = L^1_{\mathrm{w}}(\nu)$; see the discussion after Remark 3.33. Now apply Proposition 3.38. □

The converse of the previous corollary is not always valid; see [21, Example on pp. 320–321].

Corollary 3.40 shows, for the Volterra measure ν_r of order r with $1 \le r < \infty$, that $L^1(\nu_r)$ is weakly sequentially complete because the codomain $L^r([0,1])$ of ν_r does not contain a sublattice isomorphic to c_0. For $r = \infty$, the weak sequential completeness of $L^1(\nu_\infty)$ is clear from the fact that $L^1(\nu_\infty) = L^1([0,1])$ (see Example 3.26(iii)).

Let $1 \le p < \infty$, and let μ be a control measure for a Banach-space-valued vector measure $\nu : \Sigma \to E$. The Köthe dual $L^p(\nu)'$ of the order continuous B.f.s. $L^p(\nu)$ over (Ω, Σ, μ) can be identified with the topological dual $L^p(\nu)^*$, that is,

$$L^p(\nu)' = L^p(\nu)^*;$$

see Proposition 2.16 and Remark 2.18. When $p = 1$, a description of the dual space $L^1(\nu)^*$ is available in [117, Theorem 8], which then also identifies $L^1(\nu)'$. A corresponding description of $L^p(\nu)^*$ for $1 < p < \infty$ is yet to be discovered. Concerning the Köthe bidual $L^p(\nu)'' := (L^p(\nu)')'$ it turns out, for E a real space, that

$$L^p(\nu)'' = L^p_{\mathrm{w}}(\nu); \tag{3.85}$$

see [26, Proposition 2.1] for $p = 1$ and [27, Proposition 2] for $1 < p < \infty$. An examination of these proofs (based on results from [165, Ch. 15], which has function spaces over $\mathbb{C}$ as its setting) shows that these proofs can be adapted to the case when E is a complex Banach space.

As seen earlier, $L^p_{\mathrm{w}}(\nu)$ may not be order continuous. It turns out that its closed sublattice $L^p(\nu)$ is the order continuous part of $L^p_{\mathrm{w}}(\nu)$. To be precise, given a Banach lattice $(Z, \|\cdot\|_Z)$, the *order continuous part* Z_{a} of Z is defined by

$$Z_{\mathrm{a}} := \Big\{ z \in Z \ : \ |z| \ge |z_n| \downarrow 0 \text{ with } z_n \in Z \text{ implies } \|z_n\|_Z \downarrow 0 \Big\};$$

it is a closed order ideal of Z, [165, pp. 317–318]. The order continuous part is also called the absolutely continuous part of Z. Observe that Z_{a} is the largest order ideal of Z to which the restriction of the norm $\|\cdot\|_Z$ is σ-o.c. It may happen that $Z_{\mathrm{a}} = \{0\}$. For example, $(L^\infty([0,1]))_{\mathrm{a}} = \{0\}$. On the other hand $(\ell^\infty)_{\mathrm{a}} = c_0$. Therefore, $L^\infty([0,1])$ and ℓ^∞ are not isomorphic as Banach lattices whereas they are isomorphic as Banach spaces, [79, p. 18]. With the above notation we have, for ν with values in a real Banach space E, that

$$L^p_{\mathrm{w}}(\nu)_{\mathrm{a}} = L^p(\nu). \tag{3.86}$$

This has been verified in [26, pp. 191–192], for $p = 1$, via the facts that $\operatorname{sim}\Sigma$ is dense in $L^p(\nu)$ and $L^p(\nu)$ has o.c. norm and hence, the proof carries over to complex E. The "same" argument applies to $p > 1$. Now, (3.86) and the fact that $L^p_{\mathrm{w}}(\nu)$ is a p-convex B.f.s. whose weak order unit χ_Ω belongs to $(L^p_{\mathrm{w}}(\nu))_{\mathrm{a}}$ lead us to the following realization of a large class of Banach lattices, [27, Theorem 4].

Proposition 3.41. *Let $1 \le p < \infty$ and Z be a p-convex Banach lattice which has the σ-Fatou property and admits a weak order unit which belongs to its order continuous part Z_{a}. Then there is a Z_{a}-valued vector measure ν such that $L^p(\nu)$ is lattice isomorphic to Z_{a} and $L^p_{\mathrm{w}}(\nu)$ is lattice isomorphic to Z.*

Proof. The identity $Z_{\mathrm{a}} = (Z_{\mathbb{R}})_{\mathrm{a}} + i(Z_{\mathbb{R}})_{\mathrm{a}}$ follows from Theorems 91.3, 91.4 and 91.6 of [165]. Since $Z_{\mathbb{R}}$ is also p-convex, there exists a $(Z_{\mathbb{R}})_{\mathrm{a}}$-valued vector measure $\tilde{\nu}$ such that $(Z_{\mathbb{R}})_{\mathrm{a}}$ is lattice isomorphic to $L^p(\tilde{\nu})$ and $Z_{\mathbb{R}}$ is lattice isomorphic to $L^p_{\mathrm{w}}(\tilde{\nu})$; see [26, Theorem 2.5] for $p = 1$ and [27, Theorem 4] for $p > 1$. Consider the Z_{a}-valued vector measure $\nu : A \mapsto \tilde{\nu}(A) + i0$. Then a $\mathbb{C}$-valued function f is ν-integrable (resp. scalarly ν-integrable) if and only if both of the $\mathbb{R}$-valued functions $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are $\tilde{\nu}$-integrable (resp. scalarly $\tilde{\nu}$-integrable) in the real space $(Z_{\mathbb{R}})_{\mathrm{a}}$ (resp. $Z_{\mathbb{R}}$), in which case $\int_A f\,d\nu = \int_A \operatorname{Re}(f)\,d\tilde{\nu} + i\int_A \operatorname{Im}(f)\,d\tilde{\nu}$ for each measurable set A. It follows that $L^p(\nu)$ (resp. $L^p_{\mathrm{w}}(\nu)$) is the complexification of $L^p(\tilde{\nu})$ (resp. $L^p_{\mathrm{w}}(\tilde{\nu})$) and that $L^p(\nu)$ (resp. $L^p_{\mathrm{w}}(\nu)$) is lattice isomorphic to Z_{a} (resp. Z). $\square$

Let $1 < p < \infty$. Whereas the definition of $L^p_{\mathrm{w}}(\nu)$ ensures that $L^p_{\mathrm{w}}(\nu) \subseteq L^1_{\mathrm{w}}(\nu)$, the inclusion

$$L^p_{\mathrm{w}}(\nu) \ \subseteq\ L^1(\nu) \tag{3.87}$$

is not so obvious. This inclusion has been proved in [57, Proposition 3.1] when the scalar field is real; a similar proof is valid in the complex setting. We shall now establish both the inclusion

$$L^p_{\mathrm{w}}(\nu) \cdot L^{p'}(\nu) \ \subseteq\ L^1(\nu), \tag{3.88}$$

which is an improvement of (3.87), and the Hölder type inequality

$$\|fg\|_{L^1(\nu)} \ \le\ \|f\|_{L^p_{\mathrm{w}}(\nu)} \|g\|_{L^{p'}(\nu)}, \qquad f \in L^p_{\mathrm{w}}(\nu), \quad g \in L^{p'}(\nu). \tag{3.89}$$

To this end, let $f \in L^p_{\mathrm{w}}(\nu)$ and $g \in L^{p'}(\nu)$. Choose $s_n \in \operatorname{sim}\Sigma$ for $n \in \mathbb{N}$ such that $s_n \to g$ as $n \to \infty$ both pointwise and in the norm $\|\cdot\|_{L^{p'}(\nu)}$. The inclusion (3.87) yields $fs_n \in L^1(\nu)$ for $n \in \mathbb{N}$. Now, with $X(\mu) := L^1_{\mathrm{w}}(\nu)$, apply Lemma 2.21(i) (where $K = 1$ for our current setting) to obtain

$$\|fs_n - fs_k\|_{L^1(\nu)} \ \le\ \|f\|_{L^p_{\mathrm{w}}(\nu)} \|s_n - s_k\|_{L^{p'}_{\mathrm{w}}(\nu)} \ =\ \|f\|_{L^p_{\mathrm{w}}(\nu)} \|s_n - s_k\|_{L^{p'}(\nu)}, \quad n, k \in \mathbb{N},$$

because $L^p_{\mathrm{w}}(\nu) = L^1_{\mathrm{w}}(\nu)_{[1/p]}$ and $L^{p'}_{\mathrm{w}}(\nu) = L^1_{\mathrm{w}}(\nu)_{[1/p']}$, and $\|h\|_{L^r(\nu)} = \|h\|_{L^r_{\mathrm{w}}(\nu)}$ for every $h \in L^r(\nu)$ and $1 \le r < \infty$. So, $\{fs_n\}_{n=1}^\infty$ is a Cauchy sequence in the B.f.s. $L^1(\nu)$ and hence, its limit is necessarily ν-a.e. equal to fg. Thus, $fg \in L^1(\nu)$ and

$$\|fg\|_{L^1(\nu)} = \lim_{n\to\infty} \|fs_n\|_{L^1(\nu)} \le \lim_{n\to\infty} \big(\|f\|_{L^p_{\mathrm{w}}(\nu)}\|s_n\|_{L^{p'}(\nu)}\big) = \|f\|_{L^p_{\mathrm{w}}(\nu)}\|g\|_{L^{p'}(\nu)}$$

again via Lemma 2.21. This establishes both (3.88) and (3.89).

Let us now compute the operator norm of the natural inclusion map

$$\alpha_p^{(\mathrm{w})} : L^p_{\mathrm{w}}(\nu) \to L^1(\nu); \tag{3.90}$$

see (3.87). Since $L^p(\nu)$ is a closed subspace of $L^p_{\mathrm{w}}(\nu)$, it follows from (3.79) and (3.89)

$$\sup_{f\in\mathbf{B}[L^p(\nu)]} \|fg\|_{L^1(\nu)} = \sup_{f\in\mathbf{B}[L^p_{\mathrm{w}}(\nu)]} \|fg\|_{L^1(\nu)} = \|g\|_{L^{p'}(\nu)}, \qquad g \in L^{p'}(\nu). \tag{3.91}$$

Substituting $g := \chi_\Omega$ into (3.91) yields

$$\|\alpha_p^{(\mathrm{w})}\| := \sup_{f\in\mathbf{B}[L^p_{\mathrm{w}}(\nu)]} \|f\chi_\Omega\|_{L^1(\nu)} = \|\chi_\Omega\|_{L^{p'}(\nu)} = \Big(\|\nu\|(\Omega)\Big)^{1/p'}. \tag{3.92}$$

Remark 3.42. Let $1 < p < \infty$.

(i) When the scalar field is real, the inclusion map $\alpha_p^{(\mathrm{w})} : L^p_{\mathrm{w}}(\nu) \to L^1(\nu)$ is known to be weakly compact, [57, Proposition 3.3]. It is possible to extend this to the complex case via a complexification argument. However, we instead adopt a direct approach. Let $1 < r < p$. By Proposition 3.31(iii), with r in place of p, the inclusion map $\alpha_r : L^r(\nu) \to L^1(\nu)$ is weakly compact. We claim that $L^p_{\mathrm{w}}(\nu) \subseteq L^r(\nu)$ with a continuous inclusion; this has been presented in [57, Corollary 3.2] for the real case. So, consider the complex case. An adaptation of the proof of (3.88) yields that

$$L^p_{\mathrm{w}}(\nu) \cdot L^q(\nu) \ \subseteq\ L^r(\nu), \tag{3.93}$$

provided $p^{-1} + q^{-1} = r^{-1}$. In particular, since $\chi_\Omega \in L^q(\nu)$, we have $L^p_{\mathrm{w}}(\nu) \subseteq L^r(\nu)$ with a continuous inclusion. So, the map $\alpha_p^{(\mathrm{w})} : L^p_{\mathrm{w}}(\nu) \to L^1(\nu)$, which is the composition of α_r with the inclusion map from $L^p_{\mathrm{w}}(\nu)$ into $L^r(\nu)$, is weakly compact.

(ii) The map $\alpha_p^{(\mathrm{w})}$ is compact if and only if ν is purely atomic. Indeed, if ν is not purely atomic, then $\alpha_p^{(\mathrm{w})}$ is not compact by the fact that $\{\chi_A : A \in \Sigma\}$ is not relatively compact in $L^1(\nu)$; see the proof of Proposition 3.31(iv). Conversely, let ν be purely atomic and $1 < r < p$. By Proposition 3.31(iv), with r in place of p, the inclusion map $\alpha_r : L^r(\nu) \to L^1(\nu)$ is compact and hence, so is $\alpha_p^{(\mathrm{w})}$ because $L^p_{\mathrm{w}}(\nu) \subseteq L^r(\nu)$ continuously. □

Let $1 < p < \infty$. Recalling the notation of multiplication operators (see Chapter 2), it follows from (3.88) that

$$L^{p'}(\nu) \subseteq \mathcal{M}\big(L^p_{\mathrm{w}}(\nu),\, L^1(\nu)\big). \tag{3.94}$$

Moreover, (3.91) implies, for every $g \in L^{p'}(\nu)$, that the norm $\|M_g\|$ of the corresponding multiplication operator $M_g : f \mapsto gf$, from $L^p_{\mathrm{w}}(\nu)$ into $L^1(\nu)$, equals $\|g\|_{L^{p'}(\nu)}$. Similarly,

$$L^{p'}_{\mathrm{w}}(\nu) \subseteq \mathcal{M}\big(L^p(\nu),\, L^1(\nu)\big) \tag{3.95}$$

because of the inclusion

$$L^{p'}_{\mathrm{w}}(\nu) \cdot L^p(\nu) \subseteq L^1(\nu), \tag{3.96}$$

which can be derived from (3.88) by exchanging p' with p. The question arises of whether the inclusions in (3.94) and (3.95) are actually equalities. The answer is affirmative.

Proposition 3.43. *Let $1 < p < \infty$ and let $\nu : \Sigma \to E$ be a Banach-space-valued measure. Then the following assertions hold.*

(i) $\mathcal{M}\big(L^p(\nu), L^1(\nu)\big) = L^{p'}_{\mathrm{w}}(\nu)$ *and* $\|M_g\| = \|g\|_{L^{p'}_{\mathrm{w}}(\nu)}$ *for each* $g \in L^{p'}_{\mathrm{w}}(\nu)$.

(ii) $\mathcal{M}\big(L^p_{\mathrm{w}}(\nu),\, L^1(\nu)\big) = L^{p'}(\nu)$ *and* $\|M_g\| = \|g\|_{L^{p'}(\nu)}$ *for each* $g \in L^{p'}(\nu)$.

Proof. (i) Let $g \in \mathcal{M}\big(L^p(\nu), L^1(\nu)\big)$. Given $n \in \mathbb{N}$, we have that

$$g_n := |g| \wedge n \in L^\infty(\nu)^+ \subseteq L^{p'}(\nu) \subseteq L^{p'}_{\mathrm{w}}(\nu) \subseteq \mathcal{M}\big(L^p(\nu), L^1(\nu)\big)$$

via (3.95) and that

$$\|g_n\|_{L^{p'}_{\mathrm{w}}(\nu)} = \|g_n\|_{L^{p'}(\nu)} = \|M_{g_n}\| \le \|M_g\| < \infty$$

via (3.91) with g_n in place of g. Since $g_n \uparrow |g|$, the σ-Fatou property of $L^{p'}_{\mathrm{w}}(\nu)$ yields that

$$|g| = \bigvee_{n\in\mathbb{N}} g_n \in L^{p'}_{\mathrm{w}}(\nu)$$

and

$$\|g\|_{L^{p'}_{\mathrm{w}}(\nu)} = \big\|\,|g|\,\big\|_{L^{p'}_{\mathrm{w}}(\nu)} = \sup_{n\in\mathbb{N}} \|g_n\|_{L^{p'}_{\mathrm{w}}(\nu)} = \sup_{n\in\mathbb{N}} \|M_{g_n}\|.$$

In particular, $g \in L^{p'}_{\mathrm{w}}(\nu)$ and so $\mathcal{M}\big(L^p(\nu),\, L^1(\nu)\big) \subseteq L^{p'}_{\mathrm{w}}(\nu)$. Actually, because of (3.95), we have equality. Concerning the norms, it follows from Theorem 3.7(i) that

$$\begin{aligned}\|M_g\| = \sup_{f\in\mathbf{B}[L^p(\nu)]} \|gf\|_{L^1(\nu)} &= \sup_{f\in\mathbf{B}[L^p(\nu)]} \sup_{n\in\mathbb{N}} \|g_n f\|_{L^1(\nu)} \\ &= \sup_{n\in\mathbb{N}} \sup_{f\in\mathbf{B}[L^p(\nu)]} \|g_n f\|_{L^1(\nu)} = \sup_{n\in\mathbb{N}} \|M_{g_n}\|.\end{aligned}$$

Therefore, we have $\|g\|_{L^{p'}_{\mathrm{w}}(\nu)} = \|M_g\|$.

(ii) Let $g \in \mathcal{M}\big(L^p_{\mathrm{w}}(\nu),\, L^1(\nu)\big)$. Then

$$g \cdot L^p(\nu) \subseteq g \cdot L^p_{\mathrm{w}}(\nu) \subseteq L^1(\nu)$$

and so $g \in \mathcal{M}\big(L^p(\nu),\, L^1(\nu)\big) = L^{p'}_{\mathrm{w}}(\nu)$ via part (i). That is, $|g|^{p'} \in L^1_{\mathrm{w}}(\nu)$ which implies that $|g|^{p'-1} \in L^p_{\mathrm{w}}(\nu)$ because

$$\big(|g|^{p'-1}\big)^p = |g|^{pp'-p} = |g|^{p'} \in L^1_{\mathrm{w}}(\nu).$$

Therefore,

$$|g|^{p'} = |g| \cdot |g|^{p'-1} \in |g| \cdot L^p_{\mathrm{w}}(\nu) = g \cdot L^p_{\mathrm{w}}(\nu) \subseteq L^1(\nu).$$

In other words, $g \in L^1(\nu)_{[1/p']} = L^{p'}(\nu)$. The identity $\mathcal{M}\big(L^p_{\mathrm{w}}(\nu), L^1(\nu)\big) = L^{p'}(\nu)$ is then a consequence of this fact and (3.94).

Given $g \in L^{p'}(\nu)$, the (operator) norm $\|M_g\|$ of the corresponding operator $M_g \in \mathcal{L}\big(L^p_{\mathrm{w}}(\nu),\, L^1(\nu)\big)$ equals $\|g\|_{L^{p'}(\nu)}$; see (3.91). $\square$

Remark 3.44. Adapting the proof of Proposition 3.43 we can obtain

$$\mathcal{M}\big(L^p_{\mathrm{w}}(\nu), L^1_{\mathrm{w}}(\nu)\big) = L^{p'}_{\mathrm{w}}(\nu) \quad \text{and} \quad \|M_g\| = \|g\|_{L^{p'}_{\mathrm{w}}(\nu)}$$

for each $g \in L^{p'}_{\mathrm{w}}(\nu)$. $\square$

3.2 Bochner and Pettis integrals

Important classes of vector measures arise from Bochner and Pettis integrals. Such vector measures will occur in Chapter 7, and elsewhere, where they will play an important role. Throughout this section let (Ω, Σ, μ) be a finite (positive) measure space and E be a complex Banach space. A function $F : \Omega \to E$ is called *strongly μ-measurable* if there exists a sequence of E-valued, Σ-measurable simple functions $\{\Phi_n\}_{n=1}^\infty$ such that $\Phi_n(\omega) \to F(\omega)$ as $n \to \infty$ (in the norm of E) for μ-a.e. $\omega \in \Omega$. For the definition of the *Bochner integral* we refer to [42, Ch. II, Definition 2.1]. It is known that a strongly μ-measurable function $F : \Omega \to E$ is *Bochner μ-integrable* if and only if $\int_\Omega \|F(\omega)\|_E \, d\mu(\omega) < \infty$. In this case, the *Bochner indefinite integral* $\mu_F : A \longmapsto \text{(B)-}\!\int_A F \, d\mu$, for $A \in \Sigma$, is an E-valued vector measure of finite variation given by

$$|\mu_F|(A) = \int_A \|F(\omega)\|_E \, d\mu(\omega), \qquad A \in \Sigma,$$

[42, Ch. II, Theorem 2.4]. Here, (B)-$\int_A F \, d\mu$ is an element of E and denotes the Bochner μ-integral of F over a set $A \in \Sigma$. The range $\mathcal{R}(\mu_F) := \{\mu_F(A) : A \in \Sigma\}$

of μ_F is always a relatively compact subset of E, [42, Ch. II, Corollary 3.9]. The space of all E-valued, Bochner μ-integrable functions on Ω is denoted by $\mathbb{B}(\mu, E)$. The Bochner integral is preserved under continuous linear operators. To be precise, let $S \in \mathcal{L}(E, Z)$ for some Banach space Z. If $F \in \mathbb{B}(\mu, E)$, then the function $S \circ F \in \mathbb{B}(\mu, Z)$ and

$$\text{(B)-}\int_A S \circ F \, d\mu = S\Big(\text{(B)-}\int_A F \, d\mu\Big), \qquad A \in \Sigma,$$

[42, Ch. II, Theorem 2.6].

A function $H : \Omega \to E$ is called *Pettis μ-integrable* if, for each $x^* \in E^*$, the scalar function

$$\big\langle H(\cdot),\ x^* \big\rangle : \omega \longmapsto \big\langle H(\omega),\ x^* \big\rangle, \qquad \omega \in \Omega,$$

is μ-integrable and, for every $A \in \Sigma$, there is a unique element $\text{(P)-}\int_A H d\mu \in E$ satisfying

$$\Big\langle \text{(P)-}\int_A H \, d\mu,\ x^* \Big\rangle = \int_A \big\langle H(\cdot), x^* \big\rangle \, d\mu, \qquad x^* \in E^*.$$

In this case, we call $\text{(P)-}\int_A H \, d\mu$ the Pettis μ-integral over A. The *Pettis indefinite integral* $A \mapsto \text{(P)-}\int_A H \, d\mu$, for $A \in \Sigma$, is an E-valued vector measure, [42, Ch. II, Theorem 3.5], and has σ-finite variation, [162, Proposition 5.6(iv)]. Let $\mathbb{P}(\mu, E)$ denote the space of all E-valued, Pettis μ-integrable functions on Ω. A function $H \in \mathbb{P}(\mu, E)$ is called *μ-scalarly bounded* if, for each $x^* \in E^*$, the function $\langle H(\cdot), x^* \rangle$ is μ-essentially bounded (the exceptional set depends on x^*). If $F \in \mathbb{B}(\mu, E)$, then clearly $F \in \mathbb{P}(\mu, E)$ and

$$\text{(B)-}\int_A F \, d\mu = \text{(P)-}\int_A F \, d\mu, \qquad A \in \Sigma.$$

Example 3.45. Let $\mu : \mathcal{B}([0,1]) \to [0,1]$ be Lebesgue measure. Let $1 \le r < \infty$. The function $F_{(r)} : [0,1] \to E := L^r([0,1])$ defined by

$$F_{(r)}(t) := \chi_{[t,1]} \in L^r([0,1]), \qquad t \in [0,1],$$

is continuous and hence, strongly μ-measurable via the Pettis Measurability Theorem, [42, Ch. II, Theorem 1.2]. The function $F_{(r)}$ is bounded because we have $\big\|F_{(r)}(t)\big\|_{L^r([0,1])} = (1-t)^{1/r}$ for $t \in [0,1]$, and hence, $F_{(r)} \in \mathbb{B}\big(\mu, L^r([0,1])\big)$. Actually, $F_{(r)}$ is the Radon–Nikodým derivative $d\nu_r/d\mu$ of the Volterra measure ν_r of order r (see Section 3.1) with respect to μ; that is,

$$\nu_r(A) \;=\; \text{(B)-}\int_A F_{(r)} \, d\mu, \qquad A \in \mathcal{B}([0,1]),$$

[129, pp. 136–137]. In particular,

$$|\nu_r|(A) = \int_A \big\|F_{(r)}(t)\big\|_{L^r([0,1])}\,d\mu(t) = \int_A (1-t)^{1/r}d\mu(t), \qquad A \in \mathcal{B}([0,1]).$$

(i) Let $r = 1$. By [119, Example 2], we have

$$\begin{aligned} L^1(\nu_1) &= \Big\{f \in L^0(\mu) : fF_{(1)} \in \mathbb{B}\big(\mu, L^1([0,1])\big)\Big\} \\ &= \Big\{f \in L^0(\mu) : fF_{(1)} \in \mathbb{P}\big(\mu, L^1([0,1])\big)\Big\}. \end{aligned}$$

(ii) Let $1 < r < \infty$. By [129, §5], we have that

$$L^1(\nu_r) = \Big\{f \in L^0(\mu) : fF_{(r)} \in \mathbb{P}\big(\mu, L^r([0,1])\big)\Big\}.$$

We have already noted in Example 3.26(ii-a) that $L^1(|\nu_r|) \neq L^1(\nu_r)$. □

A continuous linear operator $T : L^1(\mu) \to E$ is called *Bochner representable* if there is a bounded function $F \in \mathbb{B}(\mu, E)$ such that

$$T(f) \;=\; (\mathrm{B})\text{-}\!\int_\Omega fF\,d\mu, \qquad f \in L^1(\mu),$$

[42, Ch. III, Definition 1.3]. Similarly, $S \in \mathcal{L}(L^1(\mu), E)$ is called *Pettis representable* if there is a μ-scalarly bounded function $H \in \mathbb{P}(\mu, E)$ such that, for every $f \in L^1(\mu)$, we have $fH \in \mathbb{P}(\mu, E)$ and

$$S(f) \;=\; (\mathrm{P})\text{-}\!\int_\Omega fH\,d\mu, \qquad f \in L^1(\mu).$$

Clearly, Bochner representable operators are Pettis representable. Sometimes, the converse is also valid, a result which will be needed in Chapter 7.

Proposition 3.46. *Let (Ω, Σ, μ) be a positive, finite measure space and E be a weakly compactly generated Banach space. Let $T \in \mathcal{L}(L^1(\mu), E)$ be a Pettis representable operator such that, for every $f \in L^1(\mu)$, the E-valued vector measure*

$$m_{f,T} : A \mapsto T(f\chi_A), \qquad A \in \Sigma,$$

has finite variation. Then T is also Bochner representable.

Proof. Choose a μ-scalarly bounded function $H \in \mathbb{P}(\mu, E)$ such that $T(f) = (\mathrm{P})\text{-}\!\int_\Omega fH d\mu$ for $f \in L^1(\mu)$. The weakly compactly generated Banach space E is a Lindelöf space for its weak topology $\sigma(E, E^*)$ and hence, is measure-compact; see for example, [158, Theorem (2.7.2)]. So, by [49, Proposition 5.4], there is a strongly μ-measurable function $F : \Omega \to E$ such that, for every $x^* \in E^*$, the scalar functions $\langle F(\cdot), x^*\rangle$ and $\langle H(\cdot), x^*\rangle$ are μ-a.e. equal. Thus, $F \in \mathbb{P}(\mu, E)$ and

$$T(f) = (\mathrm{P})\text{-}\int_\Omega fH\,d\mu = (\mathrm{P})\text{-}\int_\Omega fF d\mu, \qquad f \in L^1(\mu).$$

Fix $f \in L^1(\mu)$. We claim that $fF \in \mathbb{B}(\mu, E)$. In fact, since $m_{f,T}(A) = (\mathrm{P})\text{-}\int_A fF d\mu$ for each $A \in \Sigma$, it follows that

$$|m_{f,T}|(A) = \int_A |f(\omega)| \cdot \|F(\omega)\|_E \, d\mu(\omega), \qquad A \in \Sigma. \tag{3.97}$$

Now, the restriction of fF to $A_n := \{\omega \in \Omega : |f(\omega)| \cdot \|F(\omega)\|_E < n\}$ for each $n \in \mathbb{N}$ is Bochner μ-integrable (being strongly μ-measurable and bounded). Then (3.97) can be proved by applying [42, Ch. II, Theorem 2.4(iv)] to the restriction of the measure $m_{f,T}$ to A_n, which yields

$$\left|m_{f,T}\right|(A_n \cap A) = \int_{A_n \cap A} |f(\omega)| \cdot \|F(\omega)\|_E \, d\mu(\omega),$$

and then letting $n \to \infty$. By assumption $\left|m_{f,T}\right|(\Omega) < \infty$ and so $fF \in \mathbb{B}(\mu, E)$. Since this is valid for *every* $f \in L^1(\mu)$, the function F is μ-essentially bounded. This proves that $T(f) = (\mathrm{B})\text{-}\int_\Omega fF\,d\mu$ for every $f \in L^1(\mu)$. $\square$

The following Dunford–Pettis Integral Representation Theorem, [42, Ch. III, Theorem 2.2], characterizes Banach-space-valued compact operators on $L^1(\mu)$.

Proposition 3.47. *Let (Ω, Σ, μ) be a positive, finite measure space and E be a Banach space. Then an operator $T \in \mathcal{L}(L^1(\mu), E)$ is compact if and only if T is Bochner representable and the representing Bochner μ-integrable function has μ-essentially relatively compact range.*

As an interesting application of the previous result let us give an alternate proof of the compactness of the Volterra operators $V_r : L^r([0,1]) \to L^r([0,1])$ for $1 \le r < \infty$; see (3.27) and the discussion following it. So, fix $r \in [1, \infty)$ and let $i_r : L^r([0,1]) \to L^1([0,1])$ denote the natural inclusion. Define a linear map $V_{1,r} : L^1([0,1]) \to L^r([0,1])$ by

$$V_{1,r}(f) : t \mapsto \int_0^t f(s)\,ds, \qquad t \in [0,1],$$

for $f \in L^1([0,1])$. That $V_{1,r}$ is continuous with $\|V_{1,r}\| \le 1$ follows from

$$\begin{aligned}\left\|V_{1,r}(f)\right\|_{L^r([0,1])} &= \Big(\int_0^1 \Big|\int_0^t f(s)\,ds\Big|^r dt\Big)^{1/r} \\ &\le \Big(\int_0^1 \Big(\int_0^1 |f(s)|\,ds\Big)^r dt\Big)^{1/r} = \left\|f\right\|_{L^1([0,1])},\end{aligned}$$

valid for each $f \in L^1([0,1])$. Since $V_r = i_r \circ V_{1,r}$, the compactness of V_r will follow from that of $V_{1,r}$. Let $E := L^r([0,1])$ and define $F : [0,1] \to E$ by $F(t) := \chi_{[t,1]}$ for $t \in [0,1]$. Direct calculation shows that

$$\big\|F(t) - F(u)\big\|_E = |t-u|^{1/r}, \qquad t, u \in [0,1],$$

from which the continuity of F follows. This then implies (note that E is separable) that F is Bochner integrable with respect to Lebesgue measure μ on $[0,1]$ and that F has compact range in E. According to Proposition 3.47 the operator $V_{1,r}$, which can be expressed as

$$V_{1,r}(f) = (\mathrm{B})\text{-}\int_{[0,1]} F(t)\, f(t) d\mu(t), \quad f \in L^1([0,1]),$$

is compact and hence, so is V_r.

3.3 Compactness properties of integration operators

Throughout this section let (Ω, Σ) be a measurable space and E be a complex Banach space. Let $\nu : \Sigma \to E$ be a vector measure. Associated with ν is the *integration operator* $I_\nu : L^1(\nu) \to E$ defined by

$$I_\nu(f) := \int_\Omega f\, d\nu, \qquad f \in L^1(\nu); \tag{3.98}$$

it is clearly linear and continuous. Moreover, its operator norm $\|I_\nu\| = 1$. Indeed, fix $f \in L^1(\nu) \setminus \{0\}$. Since $\|\cdot\|_{L^1(\nu)}$ is a lattice norm, it follows from (3.60) that

$$\|f\|_{L^1(\nu)} = \sup_{g \in \mathbf{B}[L^\infty(\nu)]} \big\|I_\nu(fg)\big\|_E \le \|I_\nu\| \cdot \sup_{g \in \mathbf{B}[L^\infty(\nu)]} \|fg\|_{L^1(\nu)} = \|I_\nu\| \cdot \|f\|_{L^1(\nu)},$$

which implies that $\|I_\nu\| \ge 1$ because $\|f\|_{L^1(\nu)} \ne 0$. On the other hand, (3.21) yields that $\|I_\nu\| \le 1$ and hence,

$$\|I_\nu\| = 1. \tag{3.99}$$

When E is a Banach lattice, the operator $I_\nu : L^1(\nu) \to E$ is positive if and only if ν is a positive vector measure. Indeed, since $\{\chi_A : A \in \Sigma\} \subseteq L^1(\nu)^+$, it is clear that the positivity of ν follows from that of I_ν. Conversely, if ν is positive, then it is clear from the definition of the integral that $I_\nu(s) = \int_\Omega s\, d\nu \in E^+$ for every non-negative $s \in \operatorname{sim} \Sigma$. Given $f \in L_1(\nu)^+$ there exists a sequence of simple functions $\{s_n\}_{n=1}^\infty \subseteq L^1(\nu)^+$ such that $s_n \uparrow f$ pointwise ν-a.e. Then Theorem 3.7(i) implies that $\big\{I_\nu(s_n)\big\}_{n=1}^\infty \subseteq E^+$ converges to $I_\nu(f)$ in E and hence, $I_\nu(f) \in E^+$. Accordingly, I_ν is a positive operator.

Let us consider the restriction (for E a Banach space again)

$$I_{|\nu|} : L^1(|\nu|) \to E$$

of I_ν to $L^1(|\nu|)$, which is continuous because $I_{|\nu|} = I_\nu \circ j_1$ with j_1 denoting the natural embedding $L^1(|\nu|) \to L^1(\nu)$. The following result characterizes the compactness of I_ν. Its proof (see [125, Theorems 1 and 4]) requires arguments which reduce the compactness of I_ν to that of $I_{|\nu|}$ at which stage Proposition 3.47 can be applied.

Proposition 3.48. *Let $\nu : \Sigma \to E$ be a Banach-space-valued measure. Then the integration operator $I_\nu : L^1(\nu) \to E$ is compact if and only if ν has finite variation and admits a Radon–Nikodým derivative $F = d\nu/d|\nu| \in \mathbb{B}(|\nu|, E)$ which has $|\nu|$-essentially relatively compact range in E. In this case, $L^1(\nu) = L^1(|\nu|)$ holds and*

$$I_\nu(f) = (\mathrm{B})\text{-}\int_\Omega fF\, d|\nu|, \qquad f \in L^1(\nu).$$

Compact operators between Banach spaces are always completely continuous (because weakly compact subsets of Banach spaces are necessarily bounded). So, if $I_\nu : L^1(\nu) \to E$ is compact, then it is also completely continuous. Next, if I_ν is completely continuous, then the range $\mathcal{R}(\nu)$ of ν is relatively compact. To see this, first note that the range $\mathcal{R}([\nu])$ of the $L^1(\nu)$-valued vector measure $[\nu] : A \mapsto \chi_A$, for $A \in \Sigma$, is relatively weakly compact; see Lemma 3.3. By complete continuity of I_ν, the range $\mathcal{R}(\nu) = I_\nu\big(\mathcal{R}([\nu])\big)$ is necessarily relatively compact in E. So we have:

$$I_\nu \text{ compact} \Longrightarrow I_\nu \text{ completely continuous} \Longrightarrow \mathcal{R}(\nu) \text{ relatively compact.}$$

These two implications are not reversible, in general. There are many ℓ^1-valued measures ν for which I_ν is not compact. However, the Schur property of ℓ^1 implies that I_ν is completely continuous. We refer to [122, §5] for a systematic way of constructing ℓ^1-valued measures ν, defined on $\mathcal{B}([0,1])$ and having finite variation, such that I_ν is not compact. We now present some different types of examples.

Example 3.49. (i) Let the notation be as in Example 3.15. By Proposition 3.48, the integration operator $I_\nu : L^1(\nu) \to \ell^1$ cannot be compact because the variation $|\nu|(\mathbb{N}) = \infty$. Again the Schur property of ℓ^1 ensures the complete continuity of I_ν.

(ii) In the notation of Example 3.24 let $r := 1$. Then $I_\nu : L^1(\nu) \to \ell^1$ is a surjective linear isometry and hence, is surely not compact. Of course, I_ν is completely continuous by the Schur property of ℓ^1.

(iii) Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the circle group. Let λ be any non-zero, complex measure defined on the Borel σ-algebra $\mathcal{B}(\mathbb{T})$. Define a vector measure $\nu_\lambda : \mathcal{B}(\mathbb{T}) \to L^1(\mathbb{T})$ by

$$\nu_\lambda(A) := \chi_A * \lambda, \qquad A \in \mathcal{B}(\mathbb{T}),$$

where $*$ denotes convolution. Then $L^1(\nu_\lambda) = L^1(|\nu_\lambda|) = L^1(\mathbb{T})$ and

$$I_{\nu_\lambda}(f) = f * \lambda, \qquad f \in L^1(\mathbb{T}).$$

It turns out that the integration operator $I_{\nu_\lambda} : L^1(\nu_\lambda) \to L^1(\mathbb{T})$ is compact if and only if I_{ν_λ} is weakly compact if and only if λ is absolutely continuous with respect to Haar (arc length) measure on $\mathbb{T}$. However, I_{ν_λ} is completely continuous if and only if the Fourier–Stieltjes transform $\widehat{\lambda} : \mathbb{Z} \to \mathbb{C}$ of λ belongs to $c_0(\mathbb{Z})$. Since there exist Borel measures λ which are not absolutely continuous but satisfy $\widehat{\lambda} \in c_0(\mathbb{Z})$, the corresponding integration operator I_{ν_λ} fails to be weakly compact, yet it is completely continuous.

The above claims are verified in Chapter 7 in full generality, that is, for vector measures derived from convolution operators on $L^1(G)$ for any compact abelian group G.

(iv) Let $r := 1$, or ∞, and consider the respective Volterra measure given by $\nu_r : \mathcal{B}([0,1]) \to L^r([0,1])$. Then $\mathcal{R}(\nu_r) = \{V_r(\chi_A) : A \in \mathcal{B}([0,1])\}$ is relatively compact because the Volterra operator V_r is compact. We also know that $L^1(\nu_r) = L^1(|\nu_r|)$; see Example 3.26. So, it follows from Corollary 2.42 applied to $L^1(|\nu_r|)$ that I_{ν_r} is completely continuous. But, I_{ν_r} is not even weakly compact; see [119, Example 2] for $r = 1$ and [129, Proposition 5.2] for $r = \infty$. $\square$

Let us return to the Volterra measure $\nu_r : \mathcal{B}([0,1]) \to L^r([0,1])$ of order r for $1 < r < \infty$. Since the corresponding Volterra operator $V_r : L^r([0,1]) \to L^r([0,1])$ is compact, the range $\mathcal{R}(\nu_r)$ of ν_r is relatively compact. However, the integration operator $I_{\nu_r} : L^1(\nu_r) \to L^r([0,1])$ is not compact, [129, Proposition 4.2]. Actually, I_{ν_r} also fails to be completely continuous. The following proof of this is due to L. Rodríguez-Piazza, for which we first require some preliminary results. Given $1 < q < \infty$, in the following lemma we shall express elements of ℓ^q as complex sequences in the traditional way, whereas for most cases in this monograph we consider elements of ℓ^q as $\mathbb{C}$-valued functions on $\mathbb{N}$. To emphasize this we write $\ell^q_{\mathbb{C}} := \ell^q$. Similarly, the real part $\ell^q_{\mathbb{R}}$ of the complex Banach lattice ℓ^q is also considered to be a space of real sequences.

Lemma 3.50. *Let E be a Banach space and $\{x_n\}_{n=1}^\infty \subseteq E$ be a sequence for which there exist a constant $C > 0$ and $1 < r < \infty$ such that*

$$\Big\| \sum_{n\in A} x_n \Big\|_E \le C|A|^{1/r} \qquad \text{for any finite subset} \quad A \subseteq \mathbb{N}, \tag{3.100}$$

where $|A|$ denotes the cardinality of A. Then, for any $1 < q < r$, there exists a constant $K_q > 0$ such that

$$\Big\| \sum_{i=1}^N a_n x_n \Big\|_E \le K_q \Big(\sum_{n=1}^N |a_n|^q \Big)^{1/q}, \qquad N \in \mathbb{N}, \quad (a_n)_{n=1}^\infty \in \ell^q_{\mathbb{C}}. \tag{3.101}$$

Proof. Since each one of $\big(\mathrm{Re}(a_n)^+\big)_{n=1}^\infty$, $\big(\mathrm{Re}(a_n)^-\big)_{n=1}^\infty$, $\big(\mathrm{Im}(a_n)^+\big)_{n=1}^\infty$ and $\big(\mathrm{Im}(a_n)^-\big)_{n=1}^\infty$ belongs to $\big(\ell^q_{\mathbb{R}}\big)^+$, whenever $(a_n)_{n=1}^\infty \in \ell^q_{\mathbb{C}}$, and E can also be considered as a vector space over $\mathbb{R}$, it suffices to establish (3.101) for E a real Banach space and each $(a_n)_{n=1}^\infty \in \big(\ell^q_{\mathbb{R}}\big)^+$. So, let $1 < q < r$ and choose any $(a_n)_{n=1}^\infty \in \big(\ell^q_{\mathbb{R}}\big)^+$. Fix $N \ge 2$. Since (3.100) does not depend on the order of the elements in A, we may relabel $\{a_j\}_{j=1}^N$, if necessary, so that $a_1 \geq a_2 \geq \cdots \geq a_N$. Define

$$y_j := \sum_{k=1}^{j} x_k, \qquad j = 1, 2, \ldots, N,$$

and note (via (3.100)) that $\|y_j\|_E \leq Cj^{1/r}$ for $1 \leq j \leq N$. It follows that

$$\begin{aligned}\Big\| \sum_{n=1}^{N} a_n x_n \Big\|_E &= \Big\| a_N y_N + \sum_{k=1}^{N-1} (a_k - a_{k+1}) y_k \Big\|_E \\ &\leq C \Big(a_N N^{1/r} + \sum_{k=1}^{N-1} (a_k - a_{k+1}) k^{1/r} \Big) = C \Big(a_1 + \sum_{k=2}^{N} a_k \big(k^{1/r} - (k-1)^{1/r} \big) \Big).\end{aligned}$$

By the Mean Value Theorem there exist numbers $(k-1) < \xi_k < k$ such that

$$k^{1/r} - (k-1)^{1/r} = \xi_k^{(1/r)-1}/r \leq \xi_k^{(1/r)-1} < (k-1)^{(1/r)-1},$$

for $2 \leq k \leq N$. Accordingly, by Hölder's inequality and with $1/q' + 1/q = 1$, we have

$$\begin{aligned}\Big\| \sum_{n=1}^{N} a_n x_n \Big\|_E &\leq C \Big(a_1 + \sum_{k=2}^{N} a_k (k-1)^{(1/r)-1} \Big) \\ &\leq C \Big(\sum_{k=1}^{N} a_k^q \Big)^{1/q} \Big(1 + \sum_{k=2}^{N} (k-1)^{q'((1/r)-1)} \Big)^{1/q'} \leq K_q \Big(\sum_{k=1}^{N} a_k^q \Big)^{1/q},\end{aligned}$$

where $K_q := C\big(1 + \sum_{k=2}^{\infty} (k-1)^{q'((1/r)-1)}\big)^{1/q'} < \infty$ since $q'(1 - (1/r)) > 1$. □

Remark 3.51. Let E and $\{x_n\}_{n=1}^{\infty} \subseteq E$ satisfy condition (3.100) in Lemma 3.50 for some $C > 0$ and $1 < r < \infty$. Fix any $q \in (1, r)$. Then it follows from that lemma that there exists a unique operator $T \in \mathcal{L}(\ell^q, E)$ such that $T(e_n) = x_n$, for $n \in \mathbb{N}$, where $\{e_n\}_{n=1}^{\infty} \subseteq \ell^q_{\mathbb{C}}$ is the standard unconditional basis. In particular, $\{x_n\}_{n=1}^{\infty}$ is necessarily a weakly null sequence in E.

Now the promised result about Volterra operators.

Proposition 3.52. *Let $1 < r < \infty$. Consider the Volterra measure $\nu_r : \mathcal{B}([0,1]) \to L^r([0,1])$ of order r as given by formula (3.26). Then the associated integration operator $I_{\nu_r} : L^1(\nu_r) \to L^r([0,1])$ is not completely continuous.*

Proof. We proceed via several steps. Let $E := L^r([0,1])$.

Step 1. *For each $u \in (0,1)$, let $J_u := \big[1-u,\, 1-(u/2)\big] \subseteq [0,1]$. Then*

$$\Big(\frac{u}{2}\Big)^{1+(1/r)} \leq \big\| \chi_{J_u} \big\|_{L^1(\nu_r)} = \big\| \nu_r(J_u) \big\|_E \leq \frac{1}{2} u^{1+(1/r)}. \tag{3.102}$$

Because ν_r is a positive vector measure in E, the equality in (3.102) follows from Lemma 3.13.

To establish the inequalities, first note (by direct calculation) that

$$\nu_r(J_u)(t) = (t+u-1)\,\chi_{J_u}(t) + \frac{u}{2}\chi_{[1-(u/2),\,1]}(t), \qquad t \in [0,1]. \tag{3.103}$$

In particular, for each $u \in (0,1)$, we have

$$0 \le \nu_r(J_u)(t) \le \frac{u}{2}, \qquad t \in [0,1]. \tag{3.104}$$

Since $t \mapsto t^r$ is increasing in $[0,1]$ and $\nu_r(J_u)(\cdot) \le \frac{u}{2}\,\chi_{[1-u,1]}(\cdot)$ pointwise on $[0,1]$, it follows that

$$\big\|\nu_r(J_u)\big\|_E = \Big(\int_0^1 \big|\nu_r(J_u)(t)\big|^r\,dt\Big)^{1/r} \le \Big(\int_{1-u}^1 \Big(\frac{u}{2}\Big)^r\,dt\Big)^{1/r} = \frac{1}{2}u^{1+(1/r)},$$

which is the right-hand inequality in (3.102). Similarly, since the inequality $\frac{u}{2}\chi_{[1-(u/2),1]}(\cdot) \le \nu_r(J_u)(\cdot)$ holds pointwise on $[0,1]$ it also follows that

$$\big\|\nu_r(J_u)\big\|_E = \Big(\int_0^1 \big|\nu_r(J_u)(t)\big|^r\,dt\Big)^{1/r} \ge \Big(\int_{1-\frac{u}{2}}^1 \Big(\frac{u}{2}\Big)^r\,dt\Big)^{1/r} = \Big(\frac{u}{2}\Big)^{1+(1/r)},$$

which is the left-hand inequality in (3.102). This establishes Step 1.

For each $n \in \mathbb{N}$, let $u(n) := 2^{-n}$ and define $f_n := (2^n)^{1+(1/r)}\chi_{J_{u(n)}}$. Since each f_n is bounded, it is clear that $\{f_n\}_{n=1}^\infty \subseteq L^1(\nu_r)$.

Step 2. For each $n \in \mathbb{N}$, we have $\big\|I_{\nu_r}(f_n)\big\|_E \ge \big(\frac{1}{2}\big)^{1+(1/r)}$.

To see this, observe that

$$I_{\nu_r}(f_n) = (2^n)^{1+(1/r)}\nu_r(J_{u(n)}), \qquad n \in \mathbb{N}, \tag{3.105}$$

and so, by (3.102) and the definition of $u(n)$, it follows that

$$\big\|I_{\nu_r}(f_n)\big\|_E \ge (2^n)^{1+(1/r)}\Big(\frac{u(n)}{2}\Big)^{1+(1/r)} = \Big(\frac{1}{2}\Big)^{1+(1/r)}, \qquad n \in \mathbb{N},$$

which is the desired inequality.

Step 3. The sequence $\{f_n\}_{n=1}^\infty \subseteq L^1(\nu_r)$ is weakly null.

The proof of this fact is based on Lemma 3.50. Let $A \subseteq \mathbb{N}$ be any (fixed) finite set. Let $t \in [1/2,1)$ and fix $k \in \mathbb{N}$ (which depends on t) so that $t \in J_{u(k)}$.

Suppose that $n \in A$. If $n > k$, then (3.103) and (3.105) imply that $I_{\nu_r}(f_n)(t) = 0$. On the other hand, if $n \le k$, then it follows from (3.104) and (3.105) that

$$I_{\nu_r}(f_n)(t) = (2^n)^{1+(1/r)}\nu_r(J_{u(n)})(t) \le (2^n)^{1+(1/r)}\frac{u(n)}{2} = \frac{1}{2}\cdot 2^{n/r}.$$

Since $t \in J_{u(k)} \subseteq [1-u(k),\, 1]$, it follows that

$$\begin{aligned}\sum_{n\in A} I_{\nu_r}(f_n)(t) &= \sum_{n\in A\cap[1,k]} I_{\nu_r}(f_n)(t)\\ &\le \Big(\sum_{n\in A\cap[1,k]} 2^{n/r}\Big)\cdot\frac{1}{2}\chi_{[1-u(k),1]}(t) \le \Big(\sum_{1\le j\le n(k)} 2^{j/r}\Big)\cdot\frac{1}{2}\chi_{[1-u(k),1]}(t),\end{aligned}$$

where $n(k) := \max\{n\in A : n\le k\}$. But, with $a := 2^{1/r}$, we have that

$$\sum_{1\le j\le n(k)} 2^{j/r} = a\sum_{j=0}^{n(k)-1} a^j = \frac{a(a^{n(k)}-1)}{(a-1)} < \frac{a}{(a-1)}\cdot a^{n(k)}.$$

With $C_r := \frac{1}{2}\cdot\frac{a}{(a-1)}$, it follows that

$$\sum_{n\in A} I_{\nu_r}(f_n)(t) \le C_r 2^{n(k)/r}\chi_{[1-u(k),1]}(t)$$

and hence, that

$$\Big|\sum_{n\in A} I_{\nu_r}(f_n)(t)\Big|^r \le (C_r)^r 2^{n(k)}\chi_{[1-u(k),1]}(t) \le (C_r)^r\sum_{n\in A} 2^n\chi_{[1-u(n),1]}(t) \quad (3.106)$$

because $n(k)\in A$. Since all terms in the sum on the left-hand side of (3.106) are 0 whenever $t\in[0,\,1/2)$, it follows that (3.106) actually holds for all $t\in[0,1)$. But, $\sum_{n\in A} f_n \ge 0$ and so, by (3.106) and Lemma 3.13, it follows that

$$\begin{aligned}\Big\|\sum_{n\in A} f_n\Big\|_{L^1(\nu_r)} &= \Big\|\sum_{n\in A} I_{\nu_r}(f_n)\Big\|_E\\ &\le C_r\Big(\int_0^1 \sum_{n\in A} 2^n\chi_{[1-u(n),1]}(t)dt\Big)^{1/r} = C_r|A|^{1/r}.\end{aligned}$$

Since $A\subseteq\mathbb{N}$ is an arbitrary finite set, it follows from Remark 3.51 (with $E := L^1(\nu_r)$ and $x_n := f_n$ for $n\in\mathbb{N}$) that the sequence $\{f_n\}_{n=1}^\infty \subseteq L^1(\nu_r)$ is weakly null. This establishes Step 3.

Finally, by Step 3 and the continuity (hence, also weak continuity) of the map $I_{\nu_r} : L^1(\nu_r)\to E$ it follows that $I_{\nu_r}(f_n)\to 0$ weakly in E as $n\to\infty$. If I_{ν_r} were completely continuous, then (again by Step 3) the sequence $\{I_{\nu_r}(f_n)\}_{n=1}^\infty$ would be norm convergent in E, necessarily to 0. However, this contradicts Step 2. Accordingly, I_{ν_r} cannot be completely continuous. □

Under certain conditions, vector measures always have relatively compact range.

Lemma 3.53. *Let $\nu : \Sigma \to E$ be a Banach-space-valued measure.*

(i) *If ν is a Bochner indefinite integral on a positive, finite measure space (Ω, Σ, μ), that is,*

$$\nu(A) = \text{(B)-}\int_A F\, d\mu, \qquad A \in \Sigma,$$

for some $F \in \mathbb{B}(\mu, E)$, then $\mathcal{R}(\nu)$ is relatively compact in E.

(ii) *If ν is a Pettis indefinite integral on a positive, finite, perfect measure space (Ω, Σ, μ), that is,*

$$\nu(A) = \text{(P)-}\int_A H\, d\mu, \qquad A \in \Sigma,$$

for some $H \in \mathbb{P}(\mu, E)$, then $\mathcal{R}(\nu)$ is relatively compact.

(iii) *If ν is purely atomic, then $\mathcal{R}(\nu)$ is compact.*

(iv) *If $E = W^*$, for some Banach space W which does not contain an isomorphic copy of ℓ^1, and if ν has σ-finite variation, then $\mathcal{R}(\nu)$ is relatively compact.*

(v) *Let $1 \leq r < 2$. Every ℓ^r-valued vector measure has relatively compact range.*

Proof. (i) We have already noted this in Section 3.2.

(ii) This is a result due to C. Stegall; see [60, Proposition on p. 135].

(iii) See [78, Theorem 10].

(iv) This is a result due to V.I. Rybakov; see, for example, [115, Corollary 10].

(v) Let $r = 1$. Since the range of every ℓ^1-valued measure is relatively weakly compact (see Lemma 3.3), it is also relatively compact by the Schur property of ℓ^1.

Now let $1 < r < 2$ and $\nu : \Sigma \to \ell^r$ be a vector measure. It follows from [136, Remark 2, p. 211] that the restriction $I_\nu^{(\infty)}$ of I_ν to $L^\infty(\nu) \subseteq L^1(\nu)$ is a compact operator. So, $\mathcal{R}(\nu) = \{I_\nu^{(\infty)}(\chi_A) : A \in \Sigma\}$ is relatively compact in ℓ^r. $\square$

Part (v) of Lemma 3.53 cannot be extended to $2 \leq r < \infty$. Indeed, S. Banach proved that the unit ball of ℓ^2 coincides with the range of some vector measure; see [81, p. 250] and also [85]. A concrete example of a vector measure $\nu : \Sigma \to \ell^2_{\mathbb{R}}$ whose range is precisely the closed unit ball of $\ell^2_{\mathbb{R}}$ is given in [134, p. 210]. In particular, $\nu(\Sigma)$ cannot be relatively compact in $\ell^2_{\mathbb{R}}$. Let $\widetilde{\nu}$ denote the vector measure ν considered as taking its values in $\ell^2 = \ell^2_{\mathbb{R}} + i\ell^2_{\mathbb{R}}$, that is, $\tilde{\nu}(A) := \nu(A) + i0$ for each $A \in \Sigma$. Since the function Φ from the metric space ℓ^2 into the metric space $\ell^2_{\mathbb{R}}$ which maps each $\varphi \in \ell^2_{\mathbb{C}}$ to $\mathrm{Re}(\varphi) \in \ell^2_{\mathbb{R}}$ is continuous and satisfies $\Phi\big(\widetilde{\nu}(\Sigma)\big) = \nu(\Sigma)$, it follows that $\tilde{\nu}(\Sigma)$ cannot be a relatively compact set in $\ell^2_{\mathbb{C}}$. It is also known, for every $2 \leq r < \infty$, that the unit ball of ℓ^r is the range of a vector measure, [134]. Obviously, the range of such a vector measure cannot be relatively compact in ℓ^r. For further relevant information see [3] and [42, p. 275].

We now exhibit examples of vector measures with relatively compact range whose associated integration operator is not completely continuous.

Example 3.54. (i) Let E be a Banach space which does *not* have the Schur property. Let ν be an E-valued, purely atomic measure whose associated integration operator $I_\nu : L^1(\nu) \to E$ is a surjective isomorphism. Since E (hence, also $L^1(\nu)$) must be infinite-dimensional, I_ν cannot be completely continuous. However, ν has compact range; see Lemma 3.53(iii).

(a) Let the notation be as in Example 3.24 with $1 < r < \infty$. The integration map $I_\nu : L^1(\nu) \to \ell^r$ is a surjective, linear isometry and hence, is not completely continuous. However, I_ν is weakly compact because ℓ^r is reflexive.

(b) Let the notation be as in Example 3.29. Then $I_\nu : L^1(\nu) \to c_0$ is also a surjective, linear isometry. This I_ν is not completely continuous or weakly compact.

(c) Further examples follow from Proposition 3.64(i) below by choosing there $\nu = Px$, where P is an atomic spectral measure and $x \in E$ is chosen so that $L^1(\nu)$ is infinite-dimensional.

(ii) The Volterra measure of order r, for $1 < r < \infty$, provides an example of a non-atomic vector measure with relatively compact range such that the associated integration operator is not completely continuous; see the discussion immediately after Example 3.49 and Proposition 3.52. □

Problem 3.55. Characterize those Banach-space-valued vector measures whose associated integration operators are completely continuous.

We now investigate vector measures $\nu : \Sigma \to E$ with relatively compact range. Given $1 < p \le \infty$, let $I_\nu^{(p)} : L^p(\nu) \to E$ denote the restriction of $I_\nu : L^1(\nu) \to E$ to $L^p(\nu) \subseteq L^1(\nu)$, that is,

$$I_\nu^{(p)} = I_\nu \circ \alpha_p, \tag{3.107}$$

where $\alpha_p : L^p(\nu) \to L^1(\nu)$ is the natural inclusion map; see (3.58). Similarly, for $1 < p < \infty$, let $I_{\nu,\mathrm{w}}^{(p)} : L^p_{\mathrm{w}}(\nu) \to E$ denote the restriction of I_ν to $L^p_{\mathrm{w}}(\nu) \subseteq L^1(\nu)$, that is, $I_{\nu,\mathrm{w}}^{(p)} = I_\nu \circ \alpha_p^{(\mathrm{w})}$ (see (3.90)). The following result asserts that compactness of the operators $I_\nu^{(p)}$ and $I_{\nu,\mathrm{w}}^{(p)}$ turns out to be *equivalent* to relative compactness of $\mathcal{R}(\nu)$. This shows that $I_\nu^{(p)}$ exhibits additional features not possessed by I_ν (in general).

Proposition 3.56. *Let $\nu : \Sigma \to E$ be a Banach-space-valued measure.*

(I) *The following assertions are equivalent.*

  (i) *The range $\mathcal{R}(\nu)$ of ν is relatively compact.*

  (ii) *$I_\nu : L^1(\nu) \to E$ maps every bounded, uniformly ν-integrable subset of $L^1(\nu)$ to a relatively compact set in E,*

  (iii) *$I_\nu^{(p)} : L^p(\nu) \to E$ is compact for some/every $1 < p \le \infty$.*

  (iv) *$I_{\nu,\mathrm{w}}^{(p)} : L^p_{\mathrm{w}}(\nu) \to E$ is compact for some/every $1 < p < \infty$.*

(II) *If the variation* $|\nu| : \Sigma \to [0,\infty]$ *is* σ*-finite, then* (i)–(iv) *above are also equivalent to each of the following assertions.*

(v) $I_{|\nu|} : L^1(|\nu|) \to E$ *is completely continuous.*

(vi) *For every set* $A \in \Sigma$ *with* $|\nu|(A) < \infty$*, the subset* $\nu(\Sigma \cap A) \subseteq E$ *is relatively compact.*

(vii) *The subset* $\{\nu(A) : A \in \Sigma, |\nu|(A) < \infty\}$ *is relatively compact in* E.

Proof. (I) The equivalence (i) $\Leftrightarrow$ (ii) follows from Proposition 2.41 with $X(\mu) := L^1(\nu)$. The implication (ii) $\Rightarrow$ (iii) follows from $I_\nu^{(p)} = I_\nu \circ \alpha_p$ and Proposition 3.31(iii). Moreover, the implication (iii) $\Rightarrow$ (i) is straightforward as $\{\chi_A : A \in \Sigma\}$ is contained in a multiple of $\mathbf{B}[L^p(\nu)]$. Similarly, the implication (iv) $\Rightarrow$ (i) follows as $\{\chi_A : A \in \Sigma\}$ is contained in a multiple of $\mathbf{B}[L^p_{\mathrm{w}}(\nu)]$. Now let $1 < p < \infty$ be fixed and assume (i). If $1 < r < p$, then $I_\nu^{(r)}$ is compact because (i) implies (iii) with r in place of p. Since $L^p_{\mathrm{w}}(\nu)$ is continuously contained in $L^r(\nu)$ (see Remark 3.42(i)), we have (iv). So, part (I) is established.

(II) Assume that $|\nu|$ is σ-finite. Choose pairwise disjoint sets $A_n \in \Sigma$ with $0 < |\nu|(A_n) < \infty$, for $n \in \mathbb{N}$, such that $\Omega = \bigcup_{n=1}^\infty A_n$. Since the sequence $\{\nu(B_n)\}_{n=1}^\infty$ is unconditionally summable in E whenever $B_n \in \Sigma \cap A_n$ for $n \in \mathbb{N}$, we can define the sum

$$\sum_{n=1}^\infty \nu(\Sigma \cap A_n) := \left\{ \sum_{n=1}^\infty x_n : x_n \in \nu(\Sigma \cap A_n) \ \text{ for } \ n \in \mathbb{N} \right\}$$

as in [86, p. 3].

(i) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii). The implications (i) $\Rightarrow$ (vii) $\Rightarrow$ (vi) are clear. Conversely, by [86, Ch. I, Lemma 1.3], condition (vi) implies that the sum $\sum_{n=1}^\infty \nu(\Sigma \cap A_n)$ is relatively compact. But this sum is exactly $\nu(\Sigma)$. So, (i) holds.

(vi) $\Leftrightarrow$ (v). Apply Corollary 2.43 with $T := I_{|\nu|}$. $\square$

Remark 3.57. (i) When the scalar field is real, the equivalence (i) $\Leftrightarrow$ (iv) occurs in [57, Theorem 3.6].

(ii) Let $1 < r < \infty$ and $\nu_r : \mathcal{B}([0,1]) \to L^r([0,1])$ be the Volterra measure of order r. Then ν_r has finite variation with $L^1(|\nu_r|) = L^1\big((1-t)^{1/r}dt\big)$; see Example 3.26. According to the discussion immediately after Example 3.49, the range $\mathcal{R}(\nu_r)$ of ν_r is relatively compact. Hence, Proposition 3.56 implies that $I_{|\nu_r|} : L^1(|\nu_r|) \to L^r([0,1])$ is completely continuous. If it were the case that $L^1(\nu_r) = L^1(|\nu_r|)$, then $I_{\nu_r} = I_{|\nu_r|}$ and so I_{ν_r} would be completely continuous, in contradiction to Proposition 3.52. This provides an alternative proof of the fact that each inclusion

$$L^1(|\nu_r|) \subseteq L^1(\nu_r), \qquad 1 < r < \infty,$$

is *proper* (which was established in Example 3.26(ii-a) above). $\square$

Now let us consider integration operators which are weakly compact. If the codomain space of a vector measure is reflexive then, of course, the associated integration operator is weakly compact. In a sense, this is the only way that weakly compact integration operators can arise. A precise statement is the following one: we refer to [120, Proposition 2.1] for the proof.

Proposition 3.58. *Let $\nu : \Sigma \to E$ be a Banach-space-valued measure. The integration operator $I_\nu : L^1(\nu) \to E$ is weakly compact if and only if there exist a reflexive Banach space Z, a Z-valued vector measure $\eta : \Sigma \to Z$ and a continuous linear injection $S : Z \to E$ such that*

(i) *the vector measures ν and η have the same null sets,*
(ii) *$L^1(\nu) = L^1(\eta)$ as isomorphic Banach spaces, and*
(iii) *$I_\nu = S \circ I_\eta$.*

Now we present an integration operator which is weakly compact but not compact.

Example 3.59. Let $A(1) := [0,1]$ and $A(n) := [n^{-1}, (n-1)^{-1})$ for $n = 2, 3, \ldots$. With μ denoting Lebesgue measure on $[0,1]$, define vector measures

$$\nu^{(1)} : \mathcal{B}([0,1]) \to \ell^1 \quad \text{and} \quad \nu^{(2)} : \mathcal{B}([0,1]) \to \ell^2$$

by

$$\nu^{(1)}(A) := \sum_{n=1}^{\infty} n^{-2} \mu\big(A \cap A(n)\big) \chi_{\{n\}} \quad \text{and} \quad \nu^{(2)}(A) := \sum_{n=1}^{\infty} \mu\big(A \cap A(n)\big) \chi_{\{n\}}$$

for each $A \in \mathcal{B}([0,1])$. As before, we treat ℓ^1 and ℓ^2 as linear subspaces of $\mathbb{C}^{\mathbb{N}}$ (which is the space of $\mathbb{C}$-valued functions on $\mathbb{N}$). We have

$$L^1([0,1]) \;=\; L^1(\nu^{(1)}) \;=\; L^1(\nu^{(2)}).$$

The integration operator $I_{\nu^{(1)}} : L^1(\nu^{(1)}) \to \ell^1$ is compact by [120, Proposition 3.6]. On the other hand, $I_{\nu^{(2)}} : L^1(\nu^{(2)}) \to \ell^2$ is not compact because $I_{\nu^{(2)}}$ maps the bounded set $\big\{n(n+1)\chi_{A(n)} : n = 2, 3, \ldots\big\} \subseteq L^1(\nu^{(2)})$ to $\big\{\chi_{\{1\}} + \chi_{\{n\}} : n \in \mathbb{N}\big\}$ which is not relatively compact in ℓ^2.

Define a non-atomic vector measure $\nu : \mathcal{B}([0,1]) \to \ell^1 \times \ell^2$ by the formula $\nu(A) := (\nu^{(1)}(A),\ \nu^{(2)}(A))$ for $A \in \mathcal{B}([0,1])$. Then

$$L^1(\nu) = L^1([0,1]) \quad \text{and} \quad I_\nu(f) = (I_{\nu^{(1)}}(f),\ I_{\nu^{(2)}}(f)) \quad \text{for } f \in L^1(\nu).$$

Since both $I_{\nu^{(1)}}$ and $I_{\nu^{(2)}}$ are weakly compact, so is I_ν. Nevertheless, I_ν is not compact because $I_{\nu^{(2)}}$ is not compact. For the details, we refer to [120, Example 3.13]. □

Remark 3.60. In general, weak compactness and complete continuity of integration operators are not related; in other words, one does not imply the other. However, if $L^1(\nu) = L^1(|\nu|)$, that is, $L^1(\nu)$ is an abstract L^1-space, then $I_\nu : L^1(\nu) \to E$ is necessarily completely continuous whenever it is weakly compact. This is the Dunford–Pettis property of $L^1(|\nu|)$, [42, Ch. III, Corollary 2.14].

The converse is not valid, in general. For instance, see parts (iii) and (iv) of Example 3.49, where the vector measures ν there satisfy $L^1(\nu) = L^1(|\nu|)$ with I_ν completely continuous but not weakly compact. On the other hand, the vector measure ν in Example 3.24 (with $1 < r < \infty$) has compact range and its integration map $I_\nu : L^1(\nu) \to \ell^r$ is weakly compact but not completely continuous. □

The following example provides a non-atomic vector measure ν without relatively compact range for which I_ν is weakly compact but not completely continuous.

Example 3.61. Let $1 \leq r < \infty$. Then $\nu : \mathcal{B}([0,1]) \to E := L^r([0,1])$ defined by

$$\nu(A) := \chi_A, \qquad A \in \mathcal{B}([0,1]), \tag{3.108}$$

is a vector measure. Moreover, $L^1(\nu) = L^r([0,1])$ with equal norms and the associated integration operator I_ν turns out to be the identity map; this can be obtained by direct computation or follows from Corollary 3.66(ii) below with $x := \chi_{[0,1]}$. So, I_ν is not completely continuous because $L^r([0,1])$ does not have the Schur property. If we assume further the restriction that $1 < r < \infty$, then I_ν is weakly compact. Moreover, it follows from Lemma 3.21 that the range of the non-atomic measure ν is not relatively compact in $L^r([0,1])$ for $1 \leq r < \infty$. □

Remark 3.62. (i) Let $\nu : \Sigma \to E$ be a Banach-space-valued vector measure. As seen earlier, the associated integration operator $I_\nu : L^1(\nu) \to E$ need not be weakly compact. In contrast to this, whenever $1 < p \leq \infty$, the integration operator $I_\nu^{(p)} : L^p(\nu) \to E$ *is always* weakly compact because $I_\nu^{(p)} = I_\nu \circ \alpha_p$ and because the inclusion map $\alpha_p : L^p(\nu) \to L^1(\nu)$ is weakly compact by Proposition 3.31(iii).

(ii) Similarly, if $1 < p < \infty$, then the integration operator $I_{\nu,\mathrm{w}}^{(p)} : L_{\mathrm{w}}^p(\nu) \to E$ is also weakly compact because $I_{\nu,\mathrm{w}}^{(p)} = I_\nu \circ \alpha_p^{(\mathrm{w})}$ and because the inclusion map $\alpha_p^{(\mathrm{w})} : L_{\mathrm{w}}^p(\nu) \to L^1(\nu)$ is weakly compact (see Remark 3.42(i)). This occurs in [57, Corollary 3.4] when the scalar field is real. □

In the final part of this section we concentrate on a class of vector measures whose associated integration operator is an isomorphism onto its range. By an isomorphism we mean a bicontinuous linear map, as usual.

Definition 3.63. A finitely additive set function $P : \Sigma \to \mathcal{L}(E)$ is called a *spectral measure* if P satisfies the following conditions:

(i) $P(\Omega)$ is the identity operator $\mathrm{id}_E : E \to E$.

(ii) $P(A \cap B) = P(A)P(B)$ for all $A, B \in \Sigma$ (i.e., P is *multiplicative*), and

(iii) for every $x \in E$, the E-valued set function

$$Px : A \mapsto P(A)x \in E, \qquad A \in \Sigma, \tag{3.109}$$

called the *evaluation* of P at x, is a vector measure.

Spectral measures are Banach space analogues of resolutions of the identity for self-adjoint and normal operators in a Hilbert space. We refer to the detailed study of spectral measures and spectral operators in the monograph [47]. For more recent results, see also [131]. We now show that the class of all Banach-space-valued vector measures with the property that their associated integration operator is an isomorphism onto its range coincides with all the evaluations of spectral measures.

Proposition 3.64. *Let E be a Banach space.*

(i) *Given a spectral measure $P : \Sigma \to \mathcal{L}(E)$ and $x \in E$, the integration operator*

$$I_{Px} : L^1(Px) \to E$$

associated with the vector measure Px (see (3.109)) is an isomorphism onto its range.

(ii) *Conversely, let $\nu : \Sigma \to E$ be a vector measure such that $I_\nu : L^1(\nu) \to E$ is an isomorphism onto its range. Let $Z := I_\nu\big(L^1(\nu)\big)$. Then there exist a spectral measure $Q : \Sigma \to \mathcal{L}(Z)$ and a vector $z \in Z$ such that $\nu = J_Z \circ Qz$ with $J_Z : Z \to E$ denoting the natural inclusion map.*

Proof. (i) By the Uniform Boundedness Principle we have $c := \sup_{A\in\Sigma} \|P(A)\| < \infty$ because, given any $y \in E$, the set $\{P(A)y : A \in \Sigma\} = \mathcal{R}(Py)$ is bounded; see Lemma 3.3. Let $f \in L^1(Px)$. Since f is the limit of Σ-simple functions in the norm of $L^1(Px)$, the multiplicativity of P gives that $\int_A f\, d(Px) = P(A)\big(\int_\Omega f\, d(Px)\big)$ for $A \in \Sigma$. Apply (3.21) with $\nu := Px$ to obtain

$$\begin{aligned}\big\|I_{Px}(f)\big\|_E \le \big\|f\big\|_{L^1(Px)} &\le 4 \sup_{A\in\Sigma} \Big\| \int_A f\, d(Px)\Big\|_E \\ &= 4 \sup_{A\in\Sigma} \Big\|P(A)\Big(\int_\Omega f\, d(Px)\Big)\Big\|_E \le 4c\big\|I_{Px}(f)\big\|_E.\end{aligned}$$

This means precisely that I_{Px} is an isomorphism onto its range.

(ii) For each $A \in \Sigma$, let $M_{\chi_A} \in \mathcal{L}\big(L^1(\nu)\big)$ denote the multiplication operator by χ_A (see (2.76)). The closed linear subspace $Z = I_\nu(L^1(\nu))$ of E is a Banach space for the induced norm from E. Since I_ν is bicontinuous and $L^1(\nu)$ has σ-o.c. norm, the set function

$$Q : A \longmapsto I_\nu \circ M_{\chi_A} \circ I_\nu^{-1} \in \mathcal{L}(Z), \qquad A \in \Sigma,$$

is a spectral measure. Finally, with $z := \nu(\Omega) \in Z$, we have

$$Qz(A) \;=\; Q(A)z \;=\; \nu(A) \;\in Z, \qquad A \in \Sigma.$$

So, part (ii) holds. $\square$

Part (i) of Proposition 3.64 is a special case of a more general result in the locally convex setting, which is formally stated in [121, Proposition 2.5]. It was originally given in [43, Proposition 2.1] with the extra assumption that $\mathcal{R}(P)$ is closed in the strong operator topology of $\mathcal{L}(E)$. However, this assumption was not needed in the original proof. Our proof above is exactly that given in [43], specialized to the Banach space setting.

Part (ii) of Proposition 3.64 is essentially in Theorem 10 of [117], which provides further equivalent conditions to I_ν being an isomorphism onto its range. $\square$

We have already presented various examples which are evaluations of spectral measures. Let us review these in the light of Proposition 3.64.

Example 3.65. (i) Let $\nu : \Sigma \to E$ be a Banach-space-valued measure. For every $A \in \Sigma$, let $P(A) \in \mathcal{L}\big(L^1(\nu)\big)$ denote the multiplication operator by χ_A, that is, $P(A)f := f\chi_A$ for $f \in L^1(\nu)$. Then $P : \Sigma \to \mathcal{L}\big(L^1(\nu)\big)$ is a spectral measure and its evaluation at χ_Ω is exactly the $L^1(\nu)$-valued vector measure $[\nu] : A \mapsto \chi_A$ on Σ as given in (3.10). In this case, $I_{[\nu]}$ is a positive operator (as $[\nu]$ is clearly a positive vector measure).

(ii) Let E be either c_0 or ℓ^r with $1 \le r < \infty$. Given $A \in 2^{\mathbb{N}}$, define $P(A) \in \mathcal{L}(E)$ as the multiplication operator by χ_A, that is, $P(A)f := f\chi_A$ for $f \in E$. The so-defined set function $P : \Sigma \to \mathcal{L}(E)$ is a spectral measure because E is an o.c. Banach lattice. Then the vector measures in Examples 3.24 and 3.34 are evaluations of P at suitable elements of E. Note that for both examples the corresponding vector measure is positive.

(iii) Let $1 \le r < \infty$. Given $A \in \mathcal{B}([0,1])$, let $P(A) \in \mathcal{L}\big(L^r([0,1])\big)$ denote the multiplication operator by χ_A. Then the vector measure (3.108) in Example 3.61 is the evaluation of P at $\chi_{[0,1]}$. $\square$

Corollary 3.66. *Let $X(\mu)$ be a σ-order continuous B.f.s. based on a positive, finite measure space (Ω, Σ, μ).*

(i) *The following set function is a spectral measure:*

$$P : A \mapsto M_{\chi_A}, \qquad A \in \Sigma. \tag{3.110}$$

(ii) *The $X(\mu)$-valued set function*

$$\nu : A \mapsto \chi_A, \qquad A \in \Sigma, \tag{3.111}$$

is the evaluation $P\chi_\Omega$ of the spectral measure P at $\chi_\Omega \in X(\mu)$ and hence, is a positive vector measure. Moreover, the following assertions hold for ν.

(a) *$L^1(\nu) = X(\mu)$ with their given norms being equal.*

(b) $I_\nu = \mathrm{id}_{X(\mu)}$.

(c) *Let $g \in X(\mu)$. If a sequence $\{f_n\}_{n=1}^{\infty} \subseteq X(\mu)$ converges pointwise μ-a.e. to a function $f \in X(\mu)$ and if $|f_n| \le |g|$ (μ-a.e.) for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} f_n = f$ in the norm of $X(\mu)$.*

(iii) *Let $g \in X(\mu)$. Consider the evaluation $Pg : \Sigma \to X(\mu)$ of the spectral measure P at g, that is,*

$$(Pg)(A) \;=\; \chi_A g, \qquad A \in \Sigma.$$

The vector measure Pg has the following properties.

(a) $L^1(Pg) \;=\; \{f \in L^0(\mu) : fg \in X(\mu)\}$.

(b) *$I_{Pg}(f) = fg$ for every $f \in L^1(Pg)$ and $I_{Pg} : L^1(Pg) \to X(\mu)$ is a linear isometry onto its range. If, in addition, $g(\omega) \ne 0$ for μ-a.e. ω, then the integration operator $I_{Pg} : L^1(Pg) \to X(\mu)$ is a surjective linear isometry.*

(c) *Pg is a positive vector measure if and only if $g \in X(\mu)^+$.*

Proof. (i) Given any $g \in X(\mu)$, the $X(\mu)$-valued set function $A \mapsto P(A)g = M_{\chi_A}(g)$ for $A \in \Sigma$ is σ-additive because $X(\mu)$ is σ-o.c. Since $P(\Omega) = M_{\chi_\Omega} = \mathrm{id}_{X(\mu)}$ and

$$P(A \cap B) \;=\; M_{\chi_{A\cap B}} \;=\; M_{\chi_A} M_{\chi_B} \;=\; P(A)P(B), \qquad A,\, B \in \Sigma,$$

the set function P is a spectral measure.

(ii) That ν is a vector measure follows from part (i). Clearly $I_\nu(s) = s$ for every $s \in \operatorname{sim}\Sigma$. Since $I_\nu : L^1(\nu) \to X(\mu)$ is an isomorphism onto its range (by Proposition 3.64(i) with $x := \chi_\Omega$) and since $\operatorname{sim}\Sigma$ is dense in both $L^1(\nu)$ and $X(\mu)$, it follows that $I_\nu(f) = f$ for every $f \in L^1(\nu)$; in other words, $L^1(\nu) = X(\mu)$ and $I_\nu = \mathrm{id}_{X(\mu)}$. To establish that $L^1(\nu)$ and $X(\mu)$ have equal norms, fix $f \in L^1(\nu)$. Then Lemma 3.11 and the identity $I_\nu = \mathrm{id}_{X(\mu)}$ imply that

$$\|f\|_{L^1(\nu)} = \sup_s \big\|I_\nu(sf)\big\|_{X(\mu)} = \sup_s \|sf\|_{X(\mu)}, \tag{3.112}$$

where the supremum is taken over all $s \in \operatorname{sim}\Sigma$ with $\sup_{\omega\in\Omega}|s(\omega)| \le 1$. Since $\|\cdot\|_{X(\mu)}$ is a lattice norm, the right-hand side of (3.112) equals $\|f\|_{X(\mu)}$ and hence, $\|f\|_{L^1(\nu)} = \|f\|_{X(\mu)}$. So, we have established (a) and (b).

Since $X(\mu) = L^1(\nu)$ with their given norms being equal, we can apply the Lebesgue Dominated Convergence Theorem for the vector measure ν (see Theorem 3.7(i)) to deduce (c).

(iii) To verify (a), first observe that $\int_\Omega s\, d(Pg) = sg \in X(\mu)$ for every $s \in \operatorname{sim}\Sigma$. Fix $f \in L^1(Pg)$. By appealing to Theorem 3.5 with Pg in place of ν, select $s_n \in \operatorname{sim}\Sigma$ for $n \in \mathbb{N}$ such that $s_n \to f$ pointwise and $s_n g = \int_\Omega s_n\, d(Pg) \to \int_\Omega f\, d(Pg)$ in the norm of $X(\mu)$ as $n \to \infty$. By Proposition 2.2(ii), the norm convergent sequence $\{s_n g\}_{n=1}^{\infty}$ in $X(\mu)$ admits a subsequence $\{s_{n(k)} g\}_{k=1}^{\infty}$ which converges pointwise μ-a.e. to the *function* $\int_\Omega f\, d(Pg) \in X(\mu)$. Therefore, $fg = \int_\Omega f\, d(Pg) \in X(\mu)$, which establishes the inclusion

$$L^1(Pg) \subseteq \{f \in L^0(\mu) : fg \in X(\mu)\}.$$

To prove the reverse inclusion, let $f \in L^0(\mu)$ satisfy $fg \in X(\mu)$. Choose a sequence $\{s_n\}_{n=1}^\infty \subseteq \operatorname{sim}\Sigma$ such that $|s_n| \le |f|$ for $n \in \mathbb{N}$ and $s_n \to f$ pointwise as $n \to \infty$. It follows from part (ii)(c) that $s_n g \to fg$ in the norm of $X(\mu)$ as $n \to \infty$ because $|s_n g| \le |fg| \in X(\mu)$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n g = fg$ pointwise. Since $\|\cdot\|_{X(\mu)}$ is a lattice norm, for every $A \in \Sigma$ we have that

$$\Big\| \int_A s_n \, d(Pg) \; - \; fg\chi_A \Big\|_{X(\mu)} = \; \big\| s_n g\chi_A \; - \; fg\chi_A \big\|_{X(\mu)} \le \; \big\| s_n g \; - \; fg \big\|_{X(\mu)} \to 0$$

as $n \to \infty$. So, Theorem 3.5 again yields that $f \in L^1(Pg)$ and $\int_A f \, d(Pg) = fg\chi_A$ for $A \in \Sigma$. This establishes (a).

Concerning (b), fix $f \in L^1(Pg)$. The identity $I_{Pg}(f) = fg$ has already been verified in the proof of (a). Now, from Lemma 3.11 with Pg in place of ν, we have that

$$\begin{aligned} \|f\|_{L^1(Pg)} &= \sup \Big\| \int_\Omega sf \, d(Pg) \Big\|_{X(\mu)} = \sup \big\| sfg \big\|_{X(\mu)} \\ &= \big\| fg \big\|_{X(\mu)} = \big\| I_{Pg}(f) \big\|_{X(\mu)}, \end{aligned}$$

where the supremum is taken over all $s \in \operatorname{sim}\Sigma$ with $\sup_{\omega\in\Omega} |s(\omega)| \le 1$. Hence, I_{Pg} is a linear isometry.

Finally, assume that $g(\omega) \ne 0$ for μ-a.e. $\omega \in \Omega$. Then, every $f \in X(\mu)$ can be written as $f = (f/g) \cdot g$ (μ-a.e.) with $(f/g) \in L^0(\mu)$, which implies that I_{Pg} is surjective.

Statement (c) follows immediately from the definition of Pg. $\square$

In view of the previous corollary, it seems interesting to identify the class of all those spectral measures $P : \Sigma \to \mathcal{L}(E)$ such that the integration operator $I_{Px} : L^1(Px) \to E$ is a linear isometry onto the range of I_{Px}, for *every* $x \in E$. The spectral measure P given in Corollary 3.66(i) belongs to this class, as verified in part (iii).

We now exhibit an example of a vector measure whose associated integration operator is injective but not an isomorphism.

Example 3.67. Let $F_{1,0} : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ be the Fourier transform map, that is, $F_{1,0}$ assigns to each $f \in L^1(\mathbb{T})$ its Fourier transform $\widehat{f} : \mathbb{Z} \to \mathbb{C}$ in $c_0(\mathbb{Z})$. Since $F_{1,0}$ is linear and continuous and since $L^1(\mathbb{T})$ is σ-o.c., the $c_0(\mathbb{Z})$-valued set function $\nu : A \mapsto F_{1,0}(\chi_A)$, for $A \in \Sigma$, is a vector measure. Then $L^1(\nu) = L^1(\mathbb{T}) = L^1(\langle \nu, \xi_0 \rangle)$, where $\xi_0 \in c_0(\mathbb{Z})^* = \ell^1(\mathbb{Z})$ is the coordinate functional $\psi \mapsto \psi(0)$ for $\psi \in c_0(\mathbb{Z})$ (see [119, Example 1]). So, I_ν is injective, [140, p. 29], and its range is a proper dense subspace of $c_0(\mathbb{Z})$, [140, Theorem 1.2.4]. In particular, I_ν is not an isomorphism. Moreover, $\big\| F_{1,0}(e^{in(\cdot)}) \big\|_{c_0(\mathbb{Z})} = 1$ for all $n \in \mathbb{N}$ with $e^{in(\cdot)} \to 0$ weakly in $L^1(\nu)$. So $F_{1,0}$ is not completely continuous and hence, is not weakly compact via the Dunford–Pettis property of $L^1(\nu) = L^1(\mathbb{T})$, [42, Ch. III, Corollary 2.14]. For details of the above claims (and more) we refer to Chapter 7. $\square$

We end this section with some statements concerning the variation of vector measures which are evaluations of spectral measures. For the definition of *band projections* in Banach lattices we refer to [2], [108], [149], [165], for example.

Proposition 3.68. *Let E be a Dedekind complete Banach lattice with o.c.-norm and let $\mathcal{M} \subseteq \mathcal{L}(E)$ be the Boolean algebra of all band projections. Let $P : \Sigma \to \mathcal{L}(E)$ be any spectral measure with range $\mathcal{R}(P) = \mathcal{M}$. For $x \in E$, let*

$$\mathcal{M}[x] := \overline{\operatorname{span}}\big\{P(A)x : A \in \Sigma\big\}$$

denote the cyclic space spanned by x relative to $\mathcal{M}$, equipped with the induced norm and lattice operations from E. If $\mathcal{M}[x]$ is an abstract L^1-space, then the vector measure $Px : \Sigma \to E$ has finite variation and $L^1(Px) = L^1(|Px|)$ with their given lattice norms being equivalent.

Proof. According to [44, Proposition 2.4(vi)] the integration map $I_{Px} : L^1(Px) \to E$ is a topological and lattice isomorphism of $L^1(Px)$ onto its range, that is, onto the cyclic space $\mathcal{M}[x]$. Hence, if $\mathcal{M}[x]$ is isomorphic to an abstract L^1-space, then so is $L^1(Px)$. According to Lemma 3.14(iii) we can conclude that Px has finite variation and $L^1(|Px|) = L^1(Px)$ with their given lattice norms being equal. $\square$

The converse of the previous result is not valid in general.

Example 3.69. Let $E := \ell^p$ for any $1 < p < \infty$. Then the band projections are precisely the multiplication operators $\{M_{\chi_A} : A \in 2^{\mathbb{N}}\}$, which we can represent via the spectral measure $P : 2^{\mathbb{N}} \to \mathcal{L}(E)$ given by $P(A) := M_{\chi_A}$ for $A \in 2^{\mathbb{N}}$. For $x \in E$, let $\operatorname{supp}(x) := \{n \in \mathbb{N} : x_n \neq 0\}$, where $x = (x_1, x_2, \dots)$ in sequence notation. Then it is routine to check that $\mathcal{M}[x]$ is lattice isomorphic to the Banach lattice $\ell^p(\operatorname{supp}(x)) := \{y \in E : \operatorname{supp}(y) \subseteq \operatorname{supp}(x)\}$. So, $\mathcal{M}[x]$ is an abstract L^1-space if and only if $\operatorname{supp}(x)$ is a finite set.

We claim that the E-valued vector measure Px has finite variation if and only if $x \in \ell^1 \subseteq E$. Indeed, if $|Px|(\mathbb{N}) < \infty$, then for every $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} |x_n| = \sum_{n=1}^{N} \big\|Px(\{n\})\big\|_E \le \big\|Px\big(\{k : k > N\}\big)\big\|_E + \sum_{n=1}^{N} \big\|Px(\{n\})\big\|_E \le |Px|(\mathbb{N})$$

and so $x \in \ell^1$. Conversely, suppose that $x \in \ell^1$. Let $\{A_j\}_{j=1}^n$ be any partition of $\mathbb{N}$. Since $\|P(A_j)x\|_{\ell^p} \le \|P(A_j)x\|_{\ell^1}$ for $1 \le j \le n$, it follows that

$$\sum_{j=1}^{n} \big\|(Px)(A_j)\big\|_E = \sum_{j=1}^{n} \Big(\sum_{k \in A_j} |x_k|^p \Big)^{1/p} \le \sum_{j=1}^{n} \Big(\sum_{k \in A_j} |x_k| \Big) = \|x\|_{\ell^1} < \infty.$$

Accordingly, $Px : 2^{\mathbb{N}} \to E$ has finite variation.

Note, for $x \in \ell^1$, that $\mu := |Px|$ is the finite measure given by $\mu(\{n\}) := |x_n|$ for $n \in \mathbb{N}$ and hence, $L^1(|Px|) \simeq \ell^1(\mu)$; see also Lemma 3.20(ii). On the other hand, $L^1(Px)$ is lattice isomorphic to $\ell^p(\operatorname{supp}(x))$. So, if $\operatorname{supp}(x)$ is an infinite set we see that $L^1(|Px|) \neq L^1(Px)$, which should be compared with Lemma 3.14(iii). $\square$

We point out that the phenomenon exhibited in Example 3.69 cannot occur for the band projections $\mathcal{M} := \{M_{\chi_A} : A \in \mathcal{B}([0,1])\}$ in $E := L^p([0,1])$ with $1 < p < \infty$. For, in this case, the cyclic space $\mathcal{M}[x]$ for $x \neq 0$ is reflexive, being isomorphic to $L^p(\mathrm{supp}(x))$, where $\mathrm{supp}(x)$ is the essential-support of x, and hence, also $L^1(Px)$ is reflexive, where $P : \mathcal{B}([0,1]) \to \mathcal{L}(E)$ is the spectral measure given by $P(A) := M_{\chi_A}$ for $A \in \mathcal{B}([0,1])$. According to Corollary 3.23(ii), the variation measure $|Px|$ is necessarily totally infinite.

3.4 Concavity of $L^1(\nu)$ and the integration operator I_ν for a vector measure ν

We provided basic facts on the concavity of Banach lattices and q-B.f.s.' and of relevant linear operators in Chapter 2. In this section we concentrate on concavity properties of the particular class of Banach lattices of the kind $L^1(\nu)$ and the associated integration operator $I_\nu : L^1(\nu) \to E$, for general vector measures ν.

To be precise, let (Ω, Σ) be a measurable space and $\nu : \Sigma \to E$ be a Banach-space-valued vector measure. For $0 < r < \infty$ we say that ν has finite r-variation if

$$|\nu|_r(\Omega) := \sup \Big(\sum_{j=1}^{n} \big\|\nu(A_j)\big\|_E^r \Big)^{1/r} < \infty,$$

where the supremum is taken over all finite Σ-partitions $\{A_j\}_{j=1}^n$ of Ω and all $n \in \mathbb{N}$. Of course, $|\nu|_1(\Omega) = |\nu|(\Omega)$ is the usual variation of ν. It is clear that

$$|\nu|_q(\Omega) \;\le\; |\nu|_r(\Omega), \qquad 0 < r < q < \infty, \tag{3.113}$$

as a consequence of the fact that $\|\psi\|_{\ell^q} \le \|\psi\|_{\ell^r}$ for $\psi \in \ell^r \subseteq \ell^q$.

There exists a vector measure with finite r-variation for *every* $0 < r < \infty$; see Example 3.72(ii) below.

We begin with the following basic fact, on which our arguments will be built.

Proposition 3.70. *Let (Ω, Σ) be a measurable space and $\nu : \Sigma \to E$ be a Banach-space-valued vector measure. Suppose that $\mu : \Sigma \to [0, \infty)$ is a control measure for ν and $X(\mu)$ is any q-B.f.s. over (Ω, Σ, μ) with $X(\mu) \subseteq L^1(\nu)$. Let $J : X(\mu) \to L^1(\nu)$ denote the natural inclusion map, necessarily positive, and $0 < q < \infty$.*

(i) *The following conditions are equivalent.*
   (a) *J is q-concave.*
   (b) *The composition $I_\nu \circ J : X(\mu) \to E$, which is exactly the restriction of I_ν to $X(\mu)$, is a q-concave operator.*

   In this case,

$$\mathbf{M}_{(q)}[J] \;=\; \mathbf{M}_{(q)}[I_\nu \circ J]. \tag{3.114}$$

(ii) *If $J : X(\mu) \to L^1(\nu)$ is q-concave, then ν has finite q-variation.*

Proving this proposition requires the following preliminary result.

Lemma 3.71. *Given are a measurable space (Ω, Σ), a Banach-space-valued vector measure $\nu : \Sigma \to E$ and a number $0 < q < \infty$. Then it follows, for every $g_1, \ldots, g_n \in L^1(\nu)$ with $n \in \mathbb{N}$, that*

$$\Big(\sum_{j=1}^{n} \|g_j\|_{L^1(\nu)}^q\Big)^{1/q} = \sup_{s_1,\ldots,s_n} \Big(\sum_{j=1}^{n} \Big\| \int_\Omega s_j g_j \, d\nu \Big\|_E^q\Big)^{1/q}, \tag{3.115}$$

where the supremum is taken over all choices of $s_j \in \operatorname{sim}\Sigma$ with $\sup_{\omega\in\Omega} \big|s_j(\omega)\big| \leq 1$ for $j = 1, \ldots, n$.

Proof. Let $s_j \in \operatorname{sim}\Sigma$ satisfy $\sup_{\omega\in\Omega} |s_j(\omega)| \leq 1$ for $j = 1, \ldots, n$. Then for each j we have $\|g_j\|_{L^1(\nu)} \geq \big\| \int_\Omega s_j g_j \, d\nu \big\|_E$, via Lemma 3.11, which implies that the left-hand side of (3.115) is greater than or equal to the right-hand side.

Let a denote the right-hand side of (3.115). To prove that

$$\Big(\sum_{j=1}^{n} \|g_j\|_{L^1(\nu)}^q\Big)^{1/q} \leq a, \tag{3.116}$$

let $\varepsilon > 0$. For each $j = 1, \ldots, n$, it follows again from Lemma 3.11 that there exists $s_j^* \in \operatorname{sim}\Sigma$ satisfying $\sup_{\omega\in\Omega} |s_j^*(\omega)| \leq 1$ and

$$\|g_j\|_{L^1(\nu)}^q \leq \Big\| \int_\Omega s_j^* g_j \, d\nu \Big\|_E^q + \frac{\varepsilon}{n}.$$

Therefore

$$\begin{aligned} \Big(\sum_{j=1}^{n} \|g_j\|_{L^1(\nu)}^q\Big)^{1/q} &\leq \Big(\sum_{j=1}^{n} \Big\| \int_\Omega s_j^* g_j \, d\nu \Big\|_E^q + \varepsilon\Big)^{1/q} \\ &\leq (a^q + \varepsilon)^{1/q}. \end{aligned} \tag{3.117}$$

Letting $\varepsilon \to 0$ in (3.117) yields (3.116) and hence, (3.115) holds. $\square$

Proof of Proposition 3.70. (i) (a) $\Rightarrow$ (b). Proposition 2.68(i) implies that $I_\nu \circ J$ is q-concave and

$$\mathbf{M}_{(q)}[I_\nu \circ J] \leq \|I_\nu\| \big(\mathbf{M}_{(q)}[J]\big) = \mathbf{M}_{(q)}[J] \tag{3.118}$$

because the integration operator I_ν is continuous and $\|I_\nu\| = 1$ via (3.99).

(b) $\Rightarrow$ (a). Fix $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in X(\mu)$. The identity (3.115) with $g_j := J(f_j) \in L^1(\nu)$ for $j = 1, \ldots, n$, condition (b) and the fact that J is the natural inclusion map together imply that

$$\begin{aligned}\Big(\sum_{j=1}^{n}\big\|J(f_j)\big\|_{L^1(\nu)}^{q}\Big)^{1/q} &= \sup_{s_1,\dots,s_n}\Big(\sum_{j=1}^{n}\Big\|\int_\Omega s_jf_j\,d\nu\Big\|_E^q\Big)^{1/q}\\ &= \sup_{s_1,\dots,s_n}\Big(\sum_{j=1}^{n}\big\|(I_\nu\circ J)(s_jf_j)\big\|_E^q\Big)^{1/q}\\ &\le \sup_{s_1,\dots,s_n}\Big(\mathbf{M}_{(q)}[I_\nu\circ J]\Big)\Big\|\Big(\sum_{j=1}^{n}|s_jf_j|^q\Big)^{1/q}\Big\|_{X(\mu)}\\ &= \Big(\mathbf{M}_{(q)}[I_\nu\circ J]\Big)\Big\|\Big(\sum_{j=1}^{n}|f_j|^q\Big)^{1/q}\Big\|_{X(\mu)},\end{aligned}$$

where the supremum is taken over all choices of $s_j\in\operatorname{sim}\Sigma$ with $\sup_{\omega\in\Omega}|s_j(\omega)|\le 1$ for $j=1,\dots,n$. So, (a) holds and $\mathbf{M}_{(q)}[J]\le\mathbf{M}_{(q)}[I_\nu\circ J]$. This, together with (3.118), establish (3.114).

(ii) Let $\big\{A(j)\big\}_{j=1}^{n}$ be any finite Σ-partition of Ω with $n\in\mathbb{N}$. Then part (i) implies that

$$\begin{aligned}&\Big(\sum_{j=1}^{n}\big\|\nu\big(A(j)\big)\big\|_E^q\Big)^{1/q} = \Big(\sum_{j=1}^{n}\big\|\big(I_\nu\circ J\big)\big(\chi_{A(j)}\big)\big\|_E^q\Big)^{1/q}\\ &\le \Big(\mathbf{M}_{(q)}[I_\nu\circ J]\Big)\Big\|\Big(\sum_{j=1}^{n}|\chi_{A(j)}|^q\Big)^{1/q}\Big\|_{X(\mu)} = \Big(\mathbf{M}_{(q)}[I_\nu\circ J]\Big)\big\|\chi_\Omega\big\|_{X(\mu)}<\infty.\end{aligned}$$

So, ν has finite q-variation. □

Before discussing implications of Proposition 3.70, let us show that its setting can occur even for $0<q<1$.

Example 3.72. Let $\Omega:=\mathbb{N}$ and $\Sigma:=2^{\mathbb{N}}$. Fix $0<q\le 1$. Let $\varphi\in\ell^q\subseteq\ell^1$ be a function satisfying $\varphi(n)>0$ for every $n\in\mathbb{N}$. Define a measure $\mu:\Sigma\to[0,\infty)$ by $\mu(\{n\}):=\varphi(n)$ for $n\in\mathbb{N}$. For the function $\psi: n\mapsto\big(\varphi(n)\big)^{q-1}$ on $\mathbb{N}$, we have

$$\ell^q(\psi d\mu)\subseteq\ell^1(\mu);$$

for the case $0<q<1$ see (2.81) with $r:=q$ whereas the case $q=1$ is obvious. Let $E:=\ell^1(\mu)$. The E-valued vector measure $\nu: A\mapsto\chi_A$ on Σ satisfies $L^1(\nu)=E=\ell^1(\mu)$ with equal norms and I_ν is the identity operator via Corollary 3.66(ii) (with $X(\mu):=\ell^1(\mu)$ there).

(i) The natural inclusion J from the q-concave q-B.f.s. $X(\mu):=\ell^q(\psi d\mu)$ (see Example 2.73(i)) into $L^1(\nu)=\ell^1(\mu)$ is q-concave by Corollary 2.69. However, J is not r-concave when $0<r<q$. Assume, on the contrary, that J is r-concave.

Then, there exists a constant $c > 0$ such that

$$\Big(\sum_{j=1}^{n} \big\|J(f_j)\big\|_E^r\Big)^{1/r} \le c\Big\|\Big(\sum_{j=1}^{n} |f_j|^r\Big)^{1/r}\Big\|_{X(\mu)} \tag{3.119}$$

for all $f_1, \dots, f_n \in X(\mu)$ with $n \in \mathbb{N}$. Let $g \in \ell^q \setminus \ell^r$. Fix $n \in \mathbb{N}$. Let $f_j := \big(g(j)/\varphi(j)\big)\chi_{\{j\}} \in X(\mu)$ for $j = 1, \dots, n$. Recalling that $J(f) = f$ for every $f \in X(\mu) = \ell^q(\psi d\mu)$, observe that $|g(j)| = \|f_j\|_{\ell^1(\mu)} = \big\|J(f_j)\big\|_E$ for $j = 1, \dots, n$. Moreover, since $f_1, \dots, f_n$ are disjointly supported, it follows that

$$\Big(\sum_{j=1}^{n} |f_j|^r\Big)^{1/r} = \sum_{j=1}^{n} |f_j|$$

and hence,

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{n} |f_j|^r\Big)^{1/r}\Big\|_{X(\mu)} &= \Big\|\sum_{j=1}^{n} |f_j|\Big\|_{\ell^q(\psi d\mu)} \\ &= \bigg(\sum_{j=1}^{n} \bigg|\frac{g(j)}{\varphi(j)}\bigg|^q \psi(j)\,\mu(\{j\})\bigg)^{1/q} = \Big(\sum_{j=1}^{n} \big|g(j)\big|^q\Big)^{1/q}.\end{aligned}$$

This and (3.119) imply that

$$\begin{aligned}\Big(\sum_{j=1}^{n} |g(j)|^r\Big)^{1/r} &= \Big(\sum_{j=1}^{n} \big\|J(f_j)\big\|_E^r\Big)^{1/r} \\ &\le c\Big\|\Big(\sum_{j=1}^{n} |f_j|^r\Big)^{1/r}\Big\|_{X(\mu)} = c\Big(\sum_{j=1}^{n} \big|g(j)\big|^q\Big)^{1/q}.\end{aligned}$$

Taking the limit as $n \to \infty$ yields that

$$\Big(\sum_{j=1}^{\infty} |g(j)|^r\Big)^{1/r} \le c\Big(\sum_{j=1}^{\infty} |g(j)|^q\Big)^{1/q} < \infty$$

because $g \in \ell^q$. This contradicts the assumption that $g \notin \ell^r$. Thus, J is not r-concave.

(ii) Letting

$$\varphi(n) := 2^{-n}, \qquad n \in \mathbb{N}, \tag{3.120}$$

we shall show that $|\nu|_r(\Omega) < \infty$ whenever $0 < r < \infty$. To this end, first consider the case when $0 < r \le 1$. We claim that

$$|\nu|_r(\Omega) \le \Big(\frac{1}{2^r - 1}\Big)^{1/r} < \infty. \tag{3.121}$$

To prove this claim, fix $n \in \mathbb{N}$ and any finite Σ-partition $\{A_j\}_{j=1}^n$ of $\Omega = \mathbb{N}$. It follows that

$$\Big(\sum_{j=1}^n \|\nu(A_j)\|_E^r\Big)^{1/r} = \Big(\sum_{j=1}^n \big(\mu(A_j)\big)^r\Big)^{1/r} = \Big(\sum_{j=1}^n \Big(\sum_{k\in A_j} 2^{-k}\Big)^r\Big)^{1/r}$$
$$\le \Big(\sum_{j=1}^n \sum_{k\in A_j} (2^{-k})^r\Big)^{1/r} = \Big(\sum_{k=1}^\infty \Big(\frac{1}{2^r}\Big)^k\Big)^{1/r} = \Big(\frac{1}{2^r-1}\Big)^{1/r} < \infty.$$

Here the inequality $\big(\sum_{k\in A_j} 2^{-k}\big)^r \le \sum_{k\in A_j}(2^{-k})^r$ has been used, which follows immediately from the general fact that, for any subset $\{a_k : k \in \mathbb{N}\} \subseteq [0,\infty)$, we have

$$\sum_{k=1}^\infty a_k \le \Big(\sum_{k=1}^\infty (a_k)^r\Big)^{1/r},$$

thanks to the assumption that $0 < r \le 1$; see (2.43). So, (3.121) holds.

Next consider the remaining case when $1 < r < \infty$. From the first case just considered, recall that $|\nu|_1(\Omega) < \infty$. Then $r > 1$ gives that

$$|\nu|_r(\Omega) \le |\nu|_1(\Omega) < \infty$$

by (3.113) with r in place of q and 1 in place of r.

(iii) Let $0 < r < q \le 1$ and let φ be as in (3.120). Then $|\nu|_r(\Omega) < \infty$ via part (ii). On the other hand, the inclusion map $J : X(\mu) = \ell^q(\psi d\mu) \to L^1(\nu) = \ell^1(\mu)$ is not r-concave via part (i). In other words, the converse of Proposition 3.70(ii), with r in place of q, does not hold in our setting because $I_\nu \circ J = J$ (as I_ν is the identity on $L^1(\nu)$). ☐

We now apply Proposition 3.70 to the spaces $X(\mu) := L^p(\nu)$ with $1 \le p \le \infty$.

Corollary 3.73. *Let (Ω, Σ) be a measurable space and $\nu : \Sigma \to E$ be a Banach-space-valued vector measure.*

(i) *Given $1 \le p \le \infty$ and $0 < q < \infty$, the natural injection $\alpha_p : L^p(\nu) \to L^1(\nu)$ is q-concave if and only if the restricted integration operator $I_\nu^{(p)} : L^p(\nu) \to E$ is q-concave.*

(ii) *Assume that $1 \le q < \infty$. Then the following conditions are equivalent.*
   (a) *The Banach lattice $L^1(\nu)$ is q-concave.*
   (b) *The associated integration operator $I_\nu : L^1(\nu) \to E$ is q-concave.*
   (c) *The Banach lattice $L^r(\nu)$ is qr-concave for some/every $1 < r < \infty$.*

Proof. (i) If $\mu : \Sigma \to [0,\infty)$ is any control measure for ν, then $L^1(\nu)$ is a B.f.s. over (Ω, Σ, μ); see Proposition 3.28(i). So, apply Proposition 3.70(i) to the case when $X(\mu) := L^p(\nu)$, in which case J corresponds to α_p and so $I_\nu \circ J = I_\nu^{(p)}$.

(ii) (a) $\Leftrightarrow$ (b). This is a special case of part (i) with $p := 1$. In this case, the inclusion map α_1 is the identity on $L^1(\nu)$.

(a) $\Rightarrow$ (c). This follows from Proposition 2.75(ii) with $X(\mu) := L^1(\nu)$ and $p := 1/r$ because $L^r(\nu) = L^1(\nu)_{[1/r]}$.

(c) $\Rightarrow$ (a). Assume that $L^r(\nu)$ is (qr)-concave for some $1 < r < \infty$. Then $L^1(\nu) = L^r(\nu)_{[r]}$ is q-concave, again by applying Proposition 2.75(ii) with $L^r(\nu)$ in place of $X(\mu)$, (qr) in place of q, and r in place of p. □

Before presenting a consequence of Corollary 3.73, let us recall that a complex Banach lattice $(Z, \|\cdot\|_Z)$ is said to be an *abstract L^p-space*, for $1 \le p < \infty$, if

$$\|x + y\|_Z^p = \|x\|_Z^p + \|y\|_Z^p \tag{3.122}$$

whenever $x, y \in Z^+$ with $x \wedge y = 0$; see [94, Ch. 5, §15, Definition 1]. A typical example of an abstract L^p-space is the L^p-space of a $[0,\infty]$-valued scalar measure. Conversely, it is known that every abstract L^p-space is lattice isometric to the L^p-space of such a scalar measure; see [94, Ch. 5, §15, Theorem 3]. In particular, every abstract L^p-space is p-convex and p-concave and its p-convexity and p-concavity constants are 1 via Example 2.73(i).

Let us collect various equivalent conditions for $L^1(\nu)$ to be an abstract L^1-space.

Proposition 3.74. *Let ν be a vector measure, defined on a measurable space (Ω, Σ), with values in a Banach space E. Then the following assertions are equivalent.*

(i) *The Banach lattice $L^1(\nu)$ is lattice isomorphic to an abstract L^1-space.*

(ii) *$L^1(\nu) = L^1(|\nu|)$ with their given norms being equivalent (in which case $|\nu|(\Omega) < \infty$).*

(iii) *$L^1(\nu)$ is 1-concave.*

(iv) *The associated integration operator $I_\nu : L^1(\nu) \to E$ is 1-concave.*

(v) *The Banach lattice $L^p(\nu)$ is lattice isomorphic to an abstract L^p-space for some/every $1 < p < \infty$.*

(vi) *$L^p(\nu) = L^p(|\nu|)$ with their given norms being equivalent for some/every value of $1 < p < \infty$ (in which case $|\nu|(\Omega) < \infty$).*

(vii) *$L^p(\nu)$ is p-concave for some/every $1 < p < \infty$.*

Proof. (i) $\Leftrightarrow$ (ii). See Lemma 3.14(iii).

(ii) $\Rightarrow$ (iii). This is clear because $L^1(|\nu|)$ is 1-concave via Example 2.73(i).

(iii) $\Rightarrow$ (i). The real part $L^1(\nu)_\mathbb{R}$ of the 1-concave Banach lattice $L^1(\nu)$ is also 1-concave; see Lemma 2.49(ii). Moreover, $L^1(\nu)_\mathbb{R}$ is equal to the space $L^1_\mathbb{R}(\nu)$ of all $\mathbb{R}$-valued, ν-integrable functions on Ω and is a real Banach lattice equipped with the induced (real) lattice norm by $L^1(\nu)$ (see Remark 3.32). Since $L^1_\mathbb{R}(\nu)$ is both 1-convex (every real Banach lattice is 1-convex, for example, via Lemma 2.49(i) and Proposition 2.77(i)) and 1-concave, it follows from [99, p. 59] that there exists a $[0,\infty]$-valued measure η for which $L^1_{\mathbb{R}}(\nu)$ and $L^1_{\mathbb{R}}(\eta)$ are lattice isomorphic as real Banach lattices. Then $L^1(\nu)$ and $L^1(\eta)$ are lattice isomorphic as complex Banach lattices via Lemma 3.8(iv) because $L^1(\nu)$ and $L^1(\eta)$ are the complexifications of $L^1_{\mathbb{R}}(\nu)$ and $L^1_{\mathbb{R}}(\eta)$ respectively. Hence, (i) holds because $L^1(\eta)$ is an abstract L^1-space.

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vii). Apply Corollary 3.73(ii) with $q := 1$ and $r := p$.

(ii) $\Leftrightarrow$ (vi). This is clear because $L^p(\nu) = L^1(\nu)_{[1/p]}$ and $L^p(|\nu|) = L^1(|\nu|)_{[1/p]}$.

(vi) $\Rightarrow$ (v). Since $L^p(|\nu|)$ is an abstract L^p-space, this follows easily.

(v) $\Rightarrow$ (i). Condition (v) means that there is a lattice isomorphism T from $(L^p(\nu), \|\cdot\|_{L^p(\nu)})$ onto an abstract L^p-space $(Z, \|\cdot\|_Z)$. So, there exist constants c_1, $c_2 > 0$ such that

$$c_1\|f\|_{L^p(\nu)} \le \big\|T(f)\big\|_Z \le c_2\|f\|_{L^p(\nu)}, \qquad f \in L^p(\nu). \tag{3.123}$$

This allows us to define an equivalent lattice norm $|\!|\!|\cdot|\!|\!|_{L^p(\nu)}$ on $L^p(\nu)$ by

$$|\!|\!|f|\!|\!|_{L^p(\nu)} := \|T(f)\|_Z, \qquad f \in L^p(\nu), \tag{3.124}$$

so that $(L^p(\nu), |\!|\!|\cdot|\!|\!|_{L^p(\nu)})$ is lattice isometric to the abstract L^p-space $(Z, \|\cdot\|_Z)$. In particular, $(L^p(\nu), |\!|\!|\cdot|\!|\!|_{L^p(\nu)})$ itself is an abstract L^p-space and hence, is a p-convex B.f.s. with p-convexity constant 1 (see the discussion prior to this proposition). From Proposition 2.23(iii) it follows that the p-th power $\big(L^p(\nu)_{[p]}, |\!|\!|\cdot|\!|\!|_{L^p(\nu)_{[p]}}\big)$ of $(L^p(\nu), |\!|\!|\cdot|\!|\!|_{L^p(\nu)})$ is a B.f.s. In other words, the lattice quasi-norm $|\!|\!|\cdot|\!|\!|_{L^1(\nu)}$, on the p-th power $L^1(\nu) = L^p(\nu)_{[p]}$ of $L^p(\nu)$ (see (3.52)), defined by

$$|\!|\!|g|\!|\!|_{L^1(\nu)} := |\!|\!|g|\!|\!|_{L^p(\nu)_{[p]}} = \big|\!\big|\!\big|\, |g|^{1/p} \big|\!\big|\!\big|^p_{L^p(\nu)}, \qquad g \in L^1(\nu), \tag{3.125}$$

is indeed a norm. This norm $|\!|\!|\cdot|\!|\!|_{L^1(\nu)}$ is equivalent to $\|\cdot\|_{L^1(\nu)}$ because of the inequalities

$$c_1^p\,\|g\|_{L^1(\nu)} \le |\!|\!|g|\!|\!|_{L^1(\nu)} \le c_2^p\,\|g\|_{L^1(\nu)}, \qquad g \in L^1(\nu). \tag{3.126}$$

To verify (3.126), fix $g \in L^1(\nu)$. Since $|g|^{1/p} \in L^p(\nu)$, it follows from (3.123) and (3.124) that

$$c_1^p\,\big\|\,|g|^{1/p}\big\|^p_{L^p(\nu)} \le \big|\!\big|\!\big|\,|g|^{1/p}\big|\!\big|\!\big|^p_{L^p(\nu)} \le c_2^p\,\big\|\,|g|^{1/p}\big\|^p_{L^p(\nu)}.$$

So, (3.126) follows from this, (3.125) and the fact that $\|g\|_{L^1(\nu)} = \big\|\,|g|^{1/p}\big\|^p_{L^p(\nu)}$.

Now, given $g, h \in L^1(\nu)^+$ with $g \wedge h = 0$, it follows that

$$\begin{aligned} |\!|\!|g+h|\!|\!|_{L^1(\nu)} &= \big|\!\big|\!\big|\,(g+h)^{1/p}\big|\!\big|\!\big|^p_{L^p(\nu)} = \big|\!\big|\!\big|g^{1/p} + h^{1/p}\big|\!\big|\!\big|^p_{L^p(\nu)} \\ &= \big|\!\big|\!\big|g^{1/p}\big|\!\big|\!\big|^p_{L^p(\nu)} + \big|\!\big|\!\big|h^{1/p}\big|\!\big|\!\big|^p_{L^p(\nu)} = |\!|\!|g|\!|\!|_{L^1(\nu)} + |\!|\!|h|\!|\!|_{L^1(\nu)} \end{aligned}$$

because $g^{1/p}$, $h^{1/p} \in L^p(\nu)^+$ and $g^{1/p} \wedge h^{1/p} = 0$ and because $|\!|\!|\cdot|\!|\!|_{L^p(\nu)}$ satisfies (3.122) (with $Z := L^p(\nu)$). Therefore, $\big(L^1(\nu), |\!|\!|\cdot|\!|\!|_{L^1(\nu)}\big)$ is an abstract L^1-space. So, (i) holds. $\square$

Example 3.75. We shall present examples of vector measures ν with values in an abstract L^1-space E such that the associated integration operator $I_\nu : L^1(\nu) \to E$ is not 1-concave. Such an example implies that, in general, we cannot remove the positivity assumption imposed on U in Corollary 2.70, with $q := 1$ (see Example 2.73(i) and the discussion prior to Proposition 3.74).

(i) Example 3.15 presents a vector measure ν, defined on $2^{\mathbb{N}}$ and with values in the 1-concave, abstract L^1-space ℓ^1, such that ν has infinite variation. So, Proposition 3.74 yields that the associated integration operator $I_\nu : L^1(\nu) \to \ell^1$ is not 1-concave.

The method adopted in Example 3.15 can easily be extended to the setting of any infinite-dimensional abstract L^1-space E in place of ℓ^1; all we need is an unconditionally summable sequence in E which is not absolutely summable. The Dvoretzky–Rogers Theorem ensures that such a sequence exists.

(ii) If a vector measure ν has totally infinite variation, then we have the proper inclusion $L^1(|\nu|) = \{0\} \subsetneqq L^1(\nu)$, so that I_ν cannot be 1-concave; see Proposition 3.74. General methods for constructing such vector measures ν (in any infinite-dimensional Banach space) can be found in [45].

(iii) In order to provide other types of examples, we first need to establish a general fact. Given are a (topological) isomorphism from a Banach space E_0 into a Banach space E (not necessarily surjective) and a vector measure $\nu_0 : \Sigma \to E_0$ defined on a measurable space (Ω, Σ). Then the composition $T \circ \nu_0 : \Sigma \to E$ is also a vector measure; see Lemma 3.27(i). Moreover, let us show that

(a) $\mathcal{N}_0(T \circ \nu_0) = \mathcal{N}_0(\nu_0)$,

(b) $L^1\big(|T \circ \nu_0|\big) = L^1(|\nu_0|)$ with their given norms being equivalent, and

(c) $L^1\big(T \circ \nu_0\big) = L^1(\nu_0)$ with their given norms being equivalent.

Part (a) has already been established in Lemma 3.27(iii) because T is injective. Furthermore, since T is an isomorphism, there exist constants C_1, $C_2 > 0$ such that

$$C_1\|x\|_{E_0} \;\le\; \|T(x)\|_E \;\le\; C_2\|x\|_{E_0}, \qquad x \in E_0.$$

This, together with the definition of the variation of a vector measure, give

$$C_1|\nu_0|(A) \;\le\; |T \circ \nu_0|(A) \;\le\; C_2|\nu_0|(A), \qquad A \in \Sigma,$$

from which (b) follows.

Finally, (c) is a consequence of the definition of integrability for a vector measure together with the facts that the range $\mathcal{R}(T)$ of T is a closed subspace of E and that E_0 and $\mathcal{R}(T)$ (equipped with the induced norm topology from E) are topologically isomorphic. Alternatively, we can apply Theorem 3.5 as follows. The continuous inclusion $L^1(\nu_0) \subseteq L^1(T\circ\nu_0)$ has been established in Lemma 3.27(iii). To verify the reverse inclusion, fix $f \in L^1(T\circ\nu_0)$. According to Theorem 3.5 applied to the vector measure $T\circ\nu_0$, we can choose a sequence $\{s_n\}_{n=1}^\infty \subseteq \operatorname{sim}\Sigma$ with $s_n \to f$ pointwise as $n\to\infty$ and such that

$$\lim_{n\to\infty} T\Big(\int_A s_n\,d\nu_0\Big) = \lim_{n\to\infty}\int_A s_n\,d(T\circ\nu_0) = \int_A f\,d(T\circ\nu_0), \qquad A\in\Sigma, \quad (3.127)$$

in the topology of E. Consequently, since T is an isomorphism, we have from (3.127) that

$$\lim_{n\to\infty}\int_A s_n\,d\nu_0 = \lim_{n\to\infty} T^{-1}\Big(\int_A s_n\,d(T\circ\nu_0)\Big) = T^{-1}\Big(\int_A f\,d(T\circ\nu_0)\Big), \qquad A\in\Sigma,$$

in the Banach space E_0. So, Theorem 3.5 (now with $\nu := \nu_0$) implies that the function $f\in L^1(\nu_0)$, from which the equality $L^1\big(T\circ\nu_0\big) = L^1(\nu_0)$ follows. The equivalence of the given norms is a consequence of the Open Mapping Theorem.

(iv) Let H be a separable Hilbert space with a complete orthonormal basis $\{e_n\}_{n=1}^\infty$. Suppose that we are given a vector measure $\nu_0:\Sigma\to H$, defined on a measurable space (Ω,Σ), whose associated integration operator $I_{\nu_0}: L^1(\nu_0)\to H$ is not 1-concave or, equivalently,

$$L^1\big(|\nu_0|\big) \neq L^1(\nu_0); \qquad (3.128)$$

see Proposition 3.74.

Let H_1 denote the closed linear span of all Rademacher functions r_n, for $n\in\mathbb{N}$ (see, e.g., [41, p. 10] for their definition), in the Hilbert space $L^2([0,1])$. Equipped with the inner product induced by $L^2([0,1])$, the subspace H_1 is a Hilbert space with orthonormal basis $\{r_n\}_{n=1}^\infty$. The linear operator $R: H\to H_1$ assigning to each e_n the function r_n, for $n\in\mathbb{N}$, is a surjective linear isometry. On the other hand, from Khinchin's inequality, [41, p. 10], it follows that the topology induced on H_1 by $L^1([0,1])$ is the same as that induced by $L^2([0,1])$. In other words, the canonical injection $S: H_1\to L^1([0,1])$ is an isomorphism onto its range. Consequently, the composition $T = S\circ R: H\to L^1([0,1])$ is also an isomorphism onto its range.

Now, the vector measure $\nu := T\circ\nu_0 : \Sigma\to L^1([0,1])$ satisfies

$$L^1(|\nu|) \neq L^1(\nu)$$

because part (iii) above (with $E_0 := H$) and (3.128) yield that

$$L^1(|\nu|) = L^1\big(|T\circ\nu_0|\big) = L^1(|\nu_0|) \neq L^1(\nu_0) = L^1(T\circ\nu_0) = L^1(\nu).$$

Therefore, it follows from Proposition 3.74 that the associated integration operator $I_\nu: L^1(\nu)\to L^1([0,1])$ is not 1-concave whereas its codomain space $L^1([0,1])$ is 1-concave (see Example 2.73(i)).

(v) In order that part (iv) is applicable, we need some examples of vector measures ν_0 taking values in a separable Hilbert space and whose associated integration operator I_{ν_0} is not 1-concave (equivalently, $L^1\big(|\nu_0|\big)\neq L^1(\nu_0)$).

(v-a) For the Volterra measure $\nu_2 : \mathcal{B}([0,1]) \to L^2([0,1])$ of order 2, we have from Example 3.26(ii) that $L^1\big(|\nu_2|\big) \neq L^1(\nu_2)$.

(v-b) Given a separable Hilbert space H, let $P : \Sigma \to \mathcal{L}(H)$ be any spectral measure defined on a measurable space (Ω, Σ) and consider the evaluation $Px : \Sigma \to H$ of P at a point $x \in H$ with the property that the range $\mathcal{R}(I_{Px})$ of the associated integration operator $I_{Px} : L^1(Px) \to H$ is infinite-dimensional. According to Proposition 3.64(i), the Banach space $L^1(Px)$ is isomorphic to $\mathcal{R}(I_{Px})$, equipped with the relative topology from H. In particular, $L^1(Px)$ is an infinite-dimensional, *reflexive* Banach space. So, $L^1(Px)$ cannot be 1-concave. In fact, if $L^1(Px)$ were 1-concave then, from Proposition 3.74, it would follow that $L^1(Px) = L^1(|Px|)$ with their given norms being equivalent. This is a contradiction because $L^1(|Px|)$, whose dimension is infinite, cannot be reflexive. Thus, we conclude that $L^1(Px)$ is not 1-concave and, by Proposition 3.74, that $I_{Px} : L^1(Px) \to H$ is not 1-concave.

(v-c) Let (Ω, Σ, μ) be a σ-decomposable, positive, finite measure space. Take any infinite-dimensional separable Hilbert space H. Let $F : \Omega \to H$ be a strongly measurable, Pettis μ-integrable function which is *not* Bochner μ-integrable. Its Pettis indefinite integral $\widetilde{\mu}_F : \Sigma \to H$ is then an H-valued vector measure, [42, Ch. II, Theorem 3.5], and has σ-finite variation, [162, Proposition 5.6(iv)]. Moreover, the variation of $\widetilde{\mu}_F$ is genuinely infinite. In fact, take a sequence of pairwise disjoint non-μ-null sets $\{A_n\}_{n=1}^\infty$ in Σ whose union equals Ω and such that $\big|\widetilde{\mu}_F\big|(A_n) < \infty$ for every $n \in \mathbb{N}$. Observe that

$$\big|\widetilde{\mu}_F\big|(A) = \int_A \|F(\omega\|_{H_0}\, d\mu(\omega), \qquad A \in \Sigma;$$

this can be verified in the same way that (3.97) was established in the proof of Proposition 3.46. So, $\big|\widetilde{\mu}_F\big|(\Omega) < \infty$ if and only if F is Bochner μ-integrable. From this observation and the assumption on F, it follows that the Pettis indefinite integral $\widetilde{\mu}_F$ must have infinite variation. Therefore, $L^1\big(|\widetilde{\mu}_F|\big) \neq L^1(\widetilde{\mu}_F)$.

Let us now show that there always exists a strongly measurable, Pettis μ-integrable function which is *not* Bochner μ-integrable. Since (Ω, Σ, μ) is σ-decomposable, choose a sequence of pairwise disjoint non-μ-null sets $\{B_n\}_{n=1}^\infty$ in Σ. Select any unconditionally summable sequence $\{x_n\}_{n=1}^\infty \subseteq H$ which is not absolutely summable; this is possible by the Dvoretzky-Rogers Theorem. Define a countably-valued function $F : \Omega \to H$ pointwise by

$$F(\omega) := \sum_{n=1}^\infty \big(1/\mu(B_n)\big)\chi_{B_n}(\omega)\, x_n, \qquad \omega \in \Omega.$$

Then

$$\int_\Omega \|F(\omega)\|_H\, d\mu(\omega) = \sum_{n=1}^\infty \|x_n\|_H = \infty$$

and hence, F is not Bochner μ-integrable. On the other hand, given $x^* \in H$, we have

$$\int_\Omega \big|\langle F(\omega),\, x^*\rangle\big|\, d\mu(\omega) = \sum_{n=1}^\infty \big|\langle x_n,\, x^*\rangle\big| < \infty$$

because $\{x_n\}_{n=1}^\infty$ is unconditionally summable in H. The reflexivity of H then implies that F is Pettis μ-integrable (see [42, p. 53]).

Since every strongly μ-measurable function from Ω to H is μ-a.e. uniformly approximated by countably-valued measurable functions, we have not lost generality in constructing such a desired function F; see [42, Ch. II, Corollary 1.3] and the proof of Theorem 3.7 in Ch. 2 of [42].

(v-d) Let us present a "concrete example" of a countably-valued Pettis integrable function which is not Bochner integrable. Consider the finite measure μ on the measurable space $(\mathbb{N}, 2^{\mathbb{N}})$ defined by $\mu(\{n\}) := 1/n^2$ for $n \in \mathbb{N}$, and let H be the separable Hilbert space $L^2([0, 2\pi])$. Given $n \in \mathbb{N}$, define $f_n \in H$ by $f_n(t) := (1/n)\exp(-int)$ for $t \in [0, 2\pi]$. Then the sequence $\{f_n\}_{n=1}^\infty$ is not absolutely summable in H because $\sum_{n=1}^\infty \|f_n\|_H = \sum_{n=1}^\infty 2\pi/n = \infty$. That $\{f_n\}_{n=1}^\infty$ is unconditionally summable in H is clear because $\big\{(2\pi)^{-1}\exp(-int)\big\}_{n=1}^\infty$ is an orthonormal sequence in H; see [46, Ch. IV, Lemma 4.9]. So the arguments in (v-c) yield that the function $F : \mathbb{N} \to H$ defined by $F(n) := \big(1/\mu(\{n\}\big)f_n = n^2 f_n$ for $n \in \mathbb{N}$ is Pettis μ-integrable but not Bochner μ-integrable. $\square$

Next we present a vector measure ν such that the associated integration operator I_ν is not 1-concave, whereas the restricted integration operator $I_\nu^{(p)}$ is 1-concave for some $1 < p < \infty$.

Example 3.76. Fix $1 < r < \infty$ and consider the Volterra measure $\nu_r : \mathcal{B}([0,1]) \to L^r([0,1])$ of order r. Then ν_r has finite variation with $d|\nu_r|(t) = (1-t)^{1/r}dt$ and

$$L^1\big((1-t)^{1/r}dt\big) \subsetneq L^1(\nu_r) \subsetneq L^1\big((1-t)\,dt\big) \tag{3.129}$$

with strict inclusions; see parts (i) and (ii) of Example 3.26. Let $r \le p < \infty$. We claim that

$$L^p(\nu_r) \subsetneq L^1(|\nu_r|) \subsetneq L^1(\nu_r) \tag{3.130}$$

with strict inclusions. Indeed, (3.129) yields that $L^p(\nu_r) \subseteq L^p\big((1-t)\,dt\big)$. Furthermore, we have $L^p\big((1-t)\,dt\big) \subseteq L^1\big((1-t)^{1/r}dt\big)$ because it follows from Hölder's inequality that

$$\int_0^1 |f(t)|(1-t)^{1/r}dt \le \Big(\int_0^1 |f(t)|^p(1-t)^{p/r}dt\Big)^{1/p} \le \Big(\int_0^1 |f(t)|^p(1-t)\,dt\Big)^{1/p} < \infty$$

whenever $f \in L^p\big((1-t)dt\big)$. Accordingly, $L^p(\nu_r) \subseteq L^1\big((1-t)^{1/r}dt\big)$. This inclusion is strict because $L^p(\nu_r)$ is p-convex (see Proposition 3.28(i)) but $L^1\big((1-t)^{1/r}dt\big)$ is not p-convex via Example 2.73(ii-a). That $L^1\big((1-t)^{1/r}dt\big) = L^1(|\nu_r|)$ and the inclusion $L^1(|\nu_r|) \subseteq L^1(\nu_r)$ is strict can be found in Example 3.26(ii-a) (alternatively see Remark 3.57(ii)).

Now let ν denote the vector measure ν_r and let $E := L^r([0,1])$. The map $I_\nu : L^1(\nu) \to E$ is not 1-concave because $L^1(\nu) \neq L^1(|\nu|)$ (see Proposition 3.74). On the other hand, the natural embedding $\alpha_p : L^p(\nu) \to L^1(\nu)$ is 1-concave. Indeed, observe from (3.130) that $\alpha_p = j_1 \circ \gamma_p$ where $\gamma_p : L^p(\nu) \to L^1(|\nu|)$ and $j_1 : L^1(|\nu|) \to L^1(\nu)$ denote the respective natural inclusion map. Since $L^1(|\nu|)$ is 1-concave (see Example 2.73(i)), the positive operator γ_p is 1-concave via Corollary 2.70. So, the continuity of j_1 yields, via Proposition 2.68(i), that α_p is 1-concave. Accordingly, $I_\nu^{(p)} : L^p(\nu) \to E$ is also 1-concave via Proposition 3.70(i) with $q := 1$, $X(\mu) := L^p(\nu)$ and $J := \alpha_p$. $\square$

The previous example provides a counterexample for 1-concave operators. We now give one for q-concave operators with $1 < q < \infty$.

Example 3.77. Fix $1 < q < \infty$. Our aim is to construct a vector measure ν such that its associated integration operator I_ν on $L^1(\nu)$ is not q-concave while the restricted integration operator $I_\nu^{(p)}$ on $L^p(\nu)$ is q-concave whenever $q \le p < \infty$.

Take any Banach-space-valued vector measure η defined on a measurable space (Ω, Σ) such that

$$L^q(\eta) \subseteq L^1(|\eta|) \subseteq L^1(\eta) \tag{3.131}$$

with strict inclusions; for example, the Volterra measure of order q suffices (as demonstrated in (3.130)). Then we have that $L^q(|\eta|) \neq L^q(\eta)$, so that $L^q(\eta)$ is not q-concave via Proposition 3.74 (with $\nu := \eta$ and with $p := q$). Moreover, (3.131) gives

$$L^{pq}(\eta) \subseteq L^p(|\eta|) \subseteq L^q(|\eta|) \subseteq L^q(\eta), \qquad q \le p < \infty. \tag{3.132}$$

We now construct the desired vector measure ν. Let $E := L^q(\eta)$ and let $\nu : \Sigma \to E$ denote the positive E-valued vector measure $A \mapsto \chi_A$, for $A \in \Sigma$. Then $L^1(\nu) = L^q(\eta)$ with equal norms and I_ν equals the identity on $L^q(\eta)$; see Corollary 3.66(ii). The integration operator I_ν, which is the identity operator on the non-q-concave space $L^q(\eta)$, is not q-concave by definition.

On the other hand, when $q \le p < \infty$, the restricted integration operator $I_\nu^{(p)} : L^p(\nu) \to E$ is q-concave. Indeed, since $L^1(\nu) = L^q(\eta)$, it follows that the space $L^p(\nu) = L^{pq}(\eta)$. This and (3.132) imply that

$$L^p(\nu) \subseteq L^q(|\eta|) \subseteq L^1(\nu).$$

Hence, the natural inclusion map $\alpha_p : L^p(\nu) \to L^1(\nu)$ can be written as $\alpha_p = j_q \circ \gamma_q$, where $\gamma_q : L^p(\nu) \to L^q(|\eta|)$ and $j_q : L^q(|\eta|) \to L^1(\nu)$ are the respective natural inclusion maps. Now the positive operator γ_q, whose codomain is q-concave (see Example 2.73(i)), is q-concave by Corollary 2.70. Apply Proposition 2.68(i) to deduce that $\alpha_p = j_q \circ \gamma_q$ is q-concave because j_q is continuous. Moreover I_ν, being the identity on $L^1(\nu)$, implies that $I_\nu^{(p)} = \alpha_p$. Thus, $I_\nu^{(p)}$ is q-concave. $\square$

Chapter 4

Optimal Domains and Integral Extensions

In the previous chapter we presented a selection of relevant aspects from the theory of vector measures and integration. Following a time honoured practice, we began with a vector measure ν and ended up with operators defined on the spaces $L^p(\nu)$ and $L^p_{\mathrm{w}}(\nu)$. In this chapter we reverse this line of development in a certain sense. Namely, we begin with an operator $T : X(\mu) \to E$, defined on some σ-order continuous q-B.f.s. $X(\mu)$ and taking values in a Banach space E, and produce from it the E-valued vector measure $m_T : A \mapsto T(\chi_A)$. This, in turn, has associated with it the B.f.s. $L^1(m_T)$ which is σ-o.c. and has the desirable property that every function from $X(\mu)$ belongs to $L^1(m_T)$ and

$$T(f) = \int_\Omega f \, dm_T, \qquad f \in X(\mu) \subseteq L^1(m_T).$$

If, in addition, the μ-null sets and m_T-null sets coincide, in which case T is called μ-determined, then the inclusion map $J_T : X(\mu) \to L^1(m_T)$ is injective and continuous and the integration operator $I_{m_T} : L^1(m_T) \to E$ is a continuous linear extension of T. Equivalently, T has a factorization according to the commutative diagram:

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, "J_T"'] & & E \,. \\ & L^1(m_T) \arrow[ur, "I_{m_T}"'] & \end{tikzcd}$$

This feature alone is already a useful property concerning T. However, far more important is the fact that this particular extension I_{m_T} of T turns out to be *optimal*, in the sense that if $Y(\mu)$ is any σ-order continuous q-B.f.s. for which $X(\mu) \subseteq Y(\mu)$ continuously and such that there exists a continuous linear operator

$T_{Y(\mu)}$ from $Y(\mu)$ into E which coincides with T on $X(\mu)$, then necessarily $Y(\mu)$ is continuously embedded in $L^1(m_T)$ and $T_{Y(\mu)}$ coincides with I_{m_T} restricted to $Y(\mu)$; see Theorem 4.14.

The optimality property of the spaces $L^1(m_T)$ mentioned above was first systematically developed for *kernel operators* T in [24] and then successfully used to investigate the Sobolev kernel operator, [25]; some of these results will be summarized below. This operator turned out to have applications to compactness results for the Sobolev embedding between certain function spaces defined on bounded domains in $\mathbb{R}^n$, [28, 29]. To further indicate the versatility of this approach to the analysis of operators defined on function spaces, we present several μ-determined operators T which arise in classical analysis and indicate, *in detail*, how to identify both their optimal domain space $L^1(m_T)$ and the extended operator I_{m_T}. To mention some explicit examples, we consider multiplication operators, Volterra operators, convolution operators in $L^p(\mathbb{T})$, composition operators, certain projections in L^p-spaces, the Hilbert transform in $L^p(\mathbb{T})$, the finite Hilbert transform in $L^p\big((-1,1)\big)$, and so on. Furthermore, Chapter 7 is entirely devoted to investigating "optimality aspects" of the Fourier transform operator and convolution operators, defined on L^p-spaces of general compact abelian groups. In addition, the main "optimality result" mentioned in the first paragraph plays a central role in Chapter 5 which deals with the factorization of operators through the B.f.s.' $L^p(m_T)$ for arbitrary $1 \le p < \infty$. Of course, some operators T are already defined on their optimal domain to begin with (i.e., $L^1(m_T) = X(\mu)$) and so no further extension is possible. However, this phenomenon is not always easy to detect a priori. One quite general result which we establish in this direction states: if $X(\mu)$ is a σ-order continuous B.f.s. and T is a μ-determined, semi-Fredholm operator, then necessarily $L^1(m_T) = X(\mu)$. In conclusion, for the interested reader we mention that some closely related results occur in the recent works [33], [34], [35], [36], [37], where the set function m_T associated to T may only be defined on a δ-*ring* of sets, rather than on a σ-algebra as in our setting. Nevertheless, it is still possible to construct an analogous space $L^1(m_T)$ with certain optimality properties and to use this feature to investigate classical operators which do not fit into the context of this monograph.

4.1 Set functions associated with linear operators on function spaces

Throughout this section let $X(\mu)$ be a q-B.f.s. based on a finite, positive measure space (Ω, Σ, μ) and E be a Banach space, unless stated otherwise.

As in the case of Banach lattices (see Chapter 3), define

$$X(\mu)_{\mathrm{a}} := \{f \in X(\mu) : |f| \ge f_n \downarrow 0 \text{ with } f_n \in X(\mu) \text{ implies } \big\|f_n\big\|_{X(\mu)} \downarrow 0\},$$

which we call the *order continuous part* of $X(\mu)$. Note that $X(\mu)_{\mathrm{a}}$ is *solid* in $X(\mu)$, that is, if $g \in X(\mu)$ and $f \in X(\mu)_{\mathrm{a}}$ satisfy $|g| \le |f|$, then also $g \in X(\mu)_{\mathrm{a}}$. Clearly, $X(\mu)$ is σ-o.c. if and only if $X(\mu)_{\mathrm{a}} = X(\mu)$. Define a positive, finitely additive set function $\eta : \Sigma \to X(\mu)$ by

$$\eta(A) := \chi_A, \qquad\qquad A \in \Sigma. \tag{4.1}$$

As usual, η is said to be *σ-additive* if, for every pairwise disjoint sequence $\{A_n\}_{n=1}^{\infty}$ in Σ, the sequence $\{\eta(A_n)\}_{n=1}^{\infty}$ is unconditionally summable in $X(\mu)$. Unless $X(\mu)$ is a B.f.s., we do *not* call η a vector measure, because we reserve the term "vector measure" for a *Banach-space*-valued, σ-additive set function, as defined in Chapter 3. If $X(\mu)$ happens to be a σ-order continuous B.f.s., then this set function η is precisely the vector measure (3.111). Similarly, if $X(\mu)$ is σ-o.c., then η is σ-additive. The following lemma determines exactly when η is σ-additive.

Lemma 4.1. *Let $X(\mu)$ be a q-B.f.s. over (Ω, Σ, μ). The set function $\eta : \Sigma \to X(\mu)$ defined by* (4.1) *is σ-additive if and only if $\chi_\Omega \in X(\mu)_{\mathrm{a}}$.*

Proof. Suppose that η is σ-additive. To prove that $\chi_\Omega \in X(\mu)_{\mathrm{a}}$, let $f_n \in X(\mu)^+$, for $n \in \mathbb{N}$, satisfy $\chi_\Omega \ge f_n \downarrow 0$ in $X(\mu)$. We need to show $\lim_{n\to\infty} \big\|f_n\big\|_{X(\mu)} = 0$. Fix $0 < \varepsilon < 1$ and let

$$A_n := \{\omega \in \Omega : f_n(\omega) > \varepsilon\}, \qquad n \in \mathbb{N}.$$

Then the sequence $\{A_n\}_{n=1}^{\infty} \subseteq \Sigma$ decreases to $A := \bigcap_{n=1}^{\infty} A_n$ and

$$\inf_{n\in\mathbb{N}} f_n(\omega) \ge \varepsilon, \qquad \omega \in A. \tag{4.2}$$

On the other hand, by $f_n \downarrow 0$ in $X(\mu)$ we mean (by definition) that $f_n(\omega) \downarrow 0$ in $\mathbb{R}$ for μ-a.e. $\omega \in \Omega$. This and (4.2) imply that $\mu(A) = 0$ or equivalently $\eta(A) = 0$. So, the σ-additive set function η satisfies $\eta(A_n) \to \eta(A) = 0$, that is,

$$\lim_{n\to\infty} \big\|\eta(A_n)\big\|_{X(\mu)} = 0. \tag{4.3}$$

Let $K > 0$ be a constant appearing in the "triangle inequality" for the quasi-norm $\|\cdot\|_{X(\mu)}$; see (Q3) in Chapter 2 with $Z := X(\mu)$. Since $0 < \varepsilon < 1$, it follows that

$$0 \le f_n \;\le\; \varepsilon\chi_{\Omega\setminus A_n} + \chi_{A_n} \;\le\; \varepsilon\chi_\Omega + \chi_{A_n}, \qquad n \in \mathbb{N}.$$

This and (4.3) yield

$$\begin{aligned}
\limsup_{n\to\infty} \|f_n\|_{X(\mu)} &\le \limsup_{n\to\infty} \|\varepsilon\chi_\Omega + \chi_{A_n}\|_{X(\mu)} \\
&\le \limsup_{n\to\infty} K\Big(\|\varepsilon\chi_\Omega\|_{X(\mu)} + \|\chi_{A_n}\|_{X(\mu)}\Big) \\
&= K\varepsilon\|\chi_\Omega\|_{X(\mu)} + \lim_{n\to\infty} K\big\|\eta(A_n)\big\|_{X(\mu)} \\
&= K\varepsilon\,\|\chi_\Omega\|_{X(\mu)}.
\end{aligned}$$

Since $0 < \varepsilon < 1$ is arbitrary, we conclude that $\lim_{n\to\infty} \|f_n\|_{X(\mu)} = 0$. So, the function $\chi_\Omega \in X(\mu)_\mathrm{a}$.

Conversely, suppose that $\chi_\Omega \in X(\mu)_\mathrm{a}$. Since $X(\mu)_\mathrm{a}$ is solid in $X(\mu)$, it is clear that

$$\mathcal{R}(\eta) := \{\chi_A : A \in \Sigma\} \subseteq X(\mu)_\mathrm{a},$$

and hence, η is σ-additive by σ-order continuity of $X(\mu)_\mathrm{a}$. □

We give an example to illustrate the previous lemma. Let $\nu : \Sigma \to E$ be any vector measure such that $L^1(\nu) \neq L^1_\mathrm{w}(\nu)$. Define $X(\mu) := L^1_\mathrm{w}(\nu)$ with μ a control measure for ν. Then the $X(\mu)$-valued set function $A \mapsto \chi_A$ is σ-additive on Σ, because $\chi_\Omega \in L^1(\nu) = X(\mu)_\mathrm{a}$ (see (3.86) with $p := 1$) but, $X(\mu)$ is not σ-o.c. via Proposition 3.38(I) with $p := 1$.

Let $T \in \mathcal{L}(X(\mu), E)$. Define a finitely additive set function $m_T : \Sigma \to E$ by

$$m_T(A) := T(\chi_A), \qquad A \in \Sigma, \tag{4.4}$$

that is, $m_T = T \circ \eta$. If η is σ-additive, then so is m_T because T is continuous. The converse is not valid in general; see (i) or (ii) of the following example.

Example 4.2. (i) Let $X(\mu) := L^\infty([0,1])$ and $E := L^\infty([0,1])$ and let

$$T : L^\infty([0,1]) \to L^\infty([0,1])$$

be the Volterra operator V_∞. Then m_T is exactly the Volterra measure ν_∞ of order ∞, and so is σ-additive. On the other hand the $X(\mu)$-valued set function $A \mapsto \chi_A$, for $A \in \mathcal{B}([0,1])$, is not σ-additive.

(ii) Fix $g \in L^1(\mathbb{T})^+$ and let $X(\mu) := L^\infty(\mathbb{T})$ and $E := L^\infty(\mathbb{T})$. Let $T : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})$ be the convolution operator $f \mapsto f * g$ for $f \in L^\infty(\mathbb{T})$. Then T is continuous and $T\big(L^\infty(\mathbb{T})\big) \subseteq C(\mathbb{T})$, [53, Proposition 3.1.4]. The set function m_T is σ-additive. In fact, let $\{A_n\}_{n=1}^\infty \subseteq \mathcal{B}(\mathbb{T})$ satisfy $A_n \downarrow \emptyset$. Then $m_T(A_n) \downarrow \emptyset$ pointwise. So, Dini's Theorem ensures that $m_T(A_n) \to 0$ uniformly on the compact space $\mathbb{T}$ and hence, m_T is σ-additive. On the other hand, the $X(\mu)$-valued set function $A \mapsto \chi_A$ on $\mathcal{B}(\mathbb{T})$ is not σ-additive.

(iii) Let $\mu : 2^\mathbb{N} \to [0, \infty)$ be a finite measure such that $\mu(\{n\}) > 0$ for all $n \in \mathbb{N}$. Then ℓ^∞ is a B.f.s. over $(\mathbb{N}, 2^\mathbb{N}, \mu)$. Let $X(\mu) := \ell^\infty$ and $E := \mathbb{C}$ and let $T : \ell^\infty \to \mathbb{C}$ be a non-zero continuous linear functional such that T restricted to c_0 is zero; see [88, pp. 424–426]. Then $m_T : 2^\mathbb{N} \to \mathbb{C}$ is not σ-additive even though T is compact. □

If it happens that $\mathcal{L}\big(X(\mu), E\big) = \{0\}$, then the *only* set function $m_T : \Sigma \to E$ available is necessarily the zero vector measure! So, to avoid trivialities we need to know when $\mathcal{L}(X(\mu), E) \neq \{0\}$.

Lemma 4.3. *Let $X(\mu)$ be a q-B.f.s. based on (Ω, Σ, μ). Then $\mathcal{L}(X(\mu), E) \neq \{0\}$ for every non-zero Banach space E if and only if $X(\mu)$ has non-trivial dual space, that is, $X(\mu)^* \neq \{0\}$.*

Proof. If $\mathcal{L}(X(\mu), E) \neq \{0\}$ for every Banach space $E \neq \{0\}$, then the choice $E := \mathbb{C}$ ensures that $X(\mu)^* \neq \{0\}$. Conversely, suppose there exists a non-zero element $\xi \in X(\mu)^*$. Given any Banach space $E \neq \{0\}$, fix a non-zero vector $x \in E$. Then $T(f) := \langle f, \xi \rangle x$ for $f \in X(\mu)$ defines a non-zero element of $\mathcal{L}(X(\mu), E)$. $\square$

Let $0 < r < 1$. Then $L^r(\mu)^* = \{0\}$ if μ is non-atomic (see Example 2.10). On the other hand, if μ is purely atomic, then this is not the case; see Example 2.11 which asserts that $\ell^r(\mu)^* = \ell^\infty$. So, we need to take extra care in dealing with *quasi*-B.f.s.', whereas every B.f.s. automatically has non-trivial dual.

4.2 Optimal domains

Throughout this section let $X(\mu)$ be a q-B.f.s., with σ-o.c. quasi-norm, based on a non-zero, positive, finite measure space (Ω, Σ, μ), and let E be a Banach space, unless otherwise stated.

Let $T : X(\mu) \to E$ be a continuous linear operator. Then the set function $m_T : \Sigma \to E$ defined by (4.4) is σ-additive because $X(\mu)$ is σ-o.c. We call m_T the *associated vector measure* of T and adopt the notation m_T as a standard one throughout the rest of this monograph. Of course, for concrete operators which are denoted by their own particular symbol, the subscript of the associated vector measure will be the symbol used for that operator.

The main aim of this section is to determine exactly when $X(\mu)$ is continuously embedded into the space $L^1(m_T)$, how T is extended to $L^1(m_T)$, and to show that $L^1(m_T)$ is *optimal* in a certain sense (see Theorem 4.14 below). Let us begin with the following general result.

Proposition 4.4. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. and $T : X(\mu) \to E$ be a continuous linear operator. The following statements hold for its associated vector measure $m_T : \Sigma \to E$.*

(i) *Every $f \in X(\mu)$ is m_T-integrable and*

$$T(f\chi_A) = \int_A f \, dm_T, \qquad A \in \Sigma. \tag{4.5}$$

(ii) *Concerning null functions we have $\mathcal{N}(\mu) \subseteq \mathcal{N}(m_T)$. Hence, the linear map*

$$J_T : X(\mu) \to L^1(m_T) \tag{4.6}$$

which assigns to each $f \in X(\mu)$ the m_T-integrable function f is well defined. Moreover, J_T is continuous and satisfies $\|J_T\| = \|T\|$.

(iii) *The linear map J_T is injective whenever $\mathcal{N}(\mu) = \mathcal{N}(m_T)$. In this case, $L^1(m_T)$ is a B.f.s. (based on (Ω, Σ, μ)) into which $X(\mu)$ is continuously embedded via the map J_T with $T = I_{m_T} \circ J_T$ and the integration opera-*

tor $I_{m_T} : L^1(m_T) \to E$ is the unique continuous linear extension of T to $L^1(m_T)$. That is, we have the commutative diagram:

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, "J_T"'] & & E . \\ & L^1(m_T) \arrow[ur, "I_{m_T}"'] & \end{tikzcd}$$

Proof. (i) Choose functions $s_n \in \operatorname{sim}\Sigma$, for $n \in N$, such that $s_n \to f$ pointwise as well as in the space $X(\mu)$. Fix $A \in \Sigma$. Because $\|\cdot\|_{X(\mu)}$ is a lattice quasi-norm, it follows that $\lim_{n\to\infty} s_n\chi_A = f\chi_A$ in the q-B.f.s. $X(\mu)$. It then follows from the continuity of T that

$$\lim_{n\to\infty} \int_A s_n \, dm_T = \lim_{n\to\infty} T(s_n\chi_A) \;=\; T(f\chi_A).$$

By Theorem 3.5 with $\nu := m_T$, the function f is m_T-integrable and (4.5) holds.

(ii) Let $g \in \mathcal{N}(\mu)$ and $A \in \Sigma$. Then $g\chi_A = 0$ (μ-a.e.), and so $0 = T(g\chi_A) = \int_A g \, dm_T$ by (4.5) with $f := g$. Hence, $g \in \mathcal{N}(m_T)$ because $A \in \Sigma$ is arbitrary. This establishes the inclusion $\mathcal{N}(\mu) \subseteq \mathcal{N}(m_T)$.

Let $f \in X(\mu)$ be an *individual* function. Any function g which is equal μ-a.e. to f is identified with f in $X(\mu)$; see Chapter 2. For such a function $g \in X(\mu)$, we have that $(f - g) \in \mathcal{N}(\mu) \subseteq \mathcal{N}(m_T)$. So, f and g determine the same element of $L^1(m_T)$ because $L^1(m_T)$ is identified with its quotient space with respect to $\mathcal{N}(m_T)$; see Chapter 3. Therefore the map $J_T : X(\mu) \to L^1(m_T)$ is well defined.

Fix $f \in X(\mu)$. Since T is continuous and $\|\cdot\|_{X(\mu)}$ is a lattice quasi-norm, it follows from part (i) that

$$\sup_s \left\| \int_\Omega f s \, dm_T \right\|_E \;=\; \sup_s \Big\| T(fs) \Big\|_E \;\le\; \sup_s \|T\| \cdot \|fs\|_{X(\mu)} \;=\; \|T\| \cdot \|f\|_{X(\mu)}, \tag{4.7}$$

where the supremum is taken over all $s \in \operatorname{sim}\Sigma$ with $|s(\omega)| \le 1$ for every $\omega \in \Omega$. The left-hand side of (4.7) equals $\|f\|_{L^1(m_T)}$ by Lemma 3.11 with $\nu := m_T$. Consequently, we have

$$\big\|J_T(f)\big\|_{L^1(m_T)} \;=\; \|f\|_{L^1(m_T)} \;\le\; \|T\| \cdot \|f\|_{X(\mu)}, \qquad f \in X(\mu),$$

which implies that $J_T : X(\mu) \to L^1(m_T)$ is continuous and satisfies $\|J_T\| \le \|T\|$. On the other hand, the equality $\|I_{m_T}\| = 1$ (see the beginning of Section 3.3) yields that

$$\|T\| \;=\; \|I_{m_T} \circ J_T\| \;\le\; \|I_{m_T}\| \cdot \|J_T\| \;=\; \|J_T\|.$$

So, we conclude that $\|J_T\| = \|T\|$.

(iii) Assume that $\mathcal{N}(\mu) = \mathcal{N}(m_T)$. Let $f \in X(\mu)$ be any function satisfying $J_T(f) = 0$. By definition, $J_T(f) = 0$ means that the function $f = J_T(f)$ is m_T-null and hence, $f \in \mathcal{N}(\mu)$. So, J_T is injective.

The fact that $T = I_{m_T} \circ J_T$, obtained in part (i), can then be rephrased as follows: $I_{m_T} : L^1(m_T) \to E$ is a continuous linear extension of $T : X(\mu) \to E$ to the larger domain space $L^1(m_T)$ into which $X(\mu)$ is continuously embedded via the injection J_T. Moreover, I_{m_T} is the unique extension of T to $L^1(m_T)$ because $J_T(\operatorname{sim}\Sigma) = \operatorname{sim}\Sigma$ is dense in $L^1(m_T)$. $\square$

We say that a continuous linear operator $T : X(\mu) \to E$ is *μ-determined* if $\mathcal{N}(\mu) = \mathcal{N}(m_T)$. Such an operator T is necessarily non-zero whenever $X(\mu) \neq \{0\}$. Indeed, if $T = 0$, then it follows that $m_T \equiv 0$ and so $\mathcal{N}(m_T) = \mathcal{L}^0(\Sigma)$. Then $\mathcal{N}(\mu) = \mathcal{N}(m_T)$ yields $X(\mu) = \{0\}$, which is a contradiction!

The following criteria for T to be μ-determined will be useful in the sequel.

Lemma 4.5. *Let $X(\mu)$ be σ-order continuous and $T : X(\mu) \to E$ be a continuous linear operator.*

(i) *The operator T is μ-determined if and only if $\mathcal{N}_0(\mu) = \mathcal{N}_0(m_T)$ (that is, the μ-null and m_T-null sets are the same) if and only if μ is a control measure for m_T.*

(ii) *The linear operator T is μ-determined if and only if the linear operator $J_T : X(\mu) \to L^1(m_T)$ defined in (4.6) is μ-determined.*

(iii) *The operator T is μ-determined if it is injective on the subset $\{\chi_A : A \in \Sigma\}$ of its domain $X(\mu)$. In particular, if T is injective on the whole domain $X(\mu)$, then T is μ-determined.*

(iv) *The operator T is μ-determined if it is injective on the positive cone $X(\mu)^+$.*

(v) *Assume further that E is a Banach lattice and that $T : X(\mu) \to E$ is a μ-determined positive operator. Then T has the property that $f = 0$ whenever $f \in X(\mu)^+$ satisfies $T(f) = 0$.*

Proof. (i) That T is μ-determined is equivalent to the identity $\mathcal{N}_0(\mu) = \mathcal{N}_0(m_T)$; see Remark 3.4(ii). Moreover, it follows from Pettis' Theorem, [42, Ch. I, Theorem 2.1], that μ is a control measure for m_T if and only if $\mathcal{N}_0(\mu) = \mathcal{N}_0(m_T)$.

(ii) Consider the vector measure $m_{J_T} : \Sigma \to L^1(m_T)$ associated with J_T, that is, $m_{J_T}(A) = J_T(\chi_A) = \chi_A$ for $A \in \Sigma$. Then, given $A \in \Sigma$, it follows that

$$A \in \mathcal{N}_0(m_{J_T}) \iff \chi_A \in \mathcal{N}(m_T) \iff A \in \mathcal{N}_0(m_T).$$

This, and part (i), establish (ii).

(iii) Suppose that $A \in \mathcal{N}_0(m_T)$. Then $\chi_A \in X(\mu)$ and $T(\chi_A) = m_T(A) = 0$. By the hypotheses on T we conclude that $\chi_A = 0$ in $X(\mu)$, that is, $A \in \mathcal{N}_0(\mu)$. So, $\mathcal{N}_0(m_T) \subseteq \mathcal{N}_0(\mu)$. By Proposition 4.4(ii) we can establish the fact that $\mathcal{N}_0(m_T) = \mathcal{N}_0(\mu)$ and so T is μ-determined by part (i).

(iv) Since $\{\chi_A : A \in \Sigma\} \subseteq X(\mu)^+$, part (iv) follows from (iii).

(v) Let $f \in X(\mu)^+$ satisfy $T(f) = 0$. We need to show that $f = 0$ (μ-a.e.), i.e., $f \in \mathcal{N}(\mu)$. To this end define $A(n) := \{\omega \in \Omega : f(\omega) > 1/n\}$ for each $n \in \mathbb{N}$. Since $\chi_{A(n)} \leq nf$ and T is positive, it follows that

$$0 \leq m_T\big(A(n)\big) = T\big(\chi_{A(n)}\big) \leq n \cdot T(f) = 0.$$

So, $A(n) \in \mathcal{N}_0(m_T)$. Therefore $f^{-1}(\mathbb{C} \setminus \{0\}) = \bigcup_{n=1}^{\infty} A(n) \in \mathcal{N}_0(m_T)$. Remark 3.4(ii) yields $f \in \mathcal{N}(m_T)$. Since T is μ-determined, we have $f \in \mathcal{N}(\mu)$. □

It may be worthwhile to point out that, for a positive μ-determined operator $T : X(\mu) \to E$ (with E a Banach lattice), Lemma 4.5(v) is *not* equivalent to T being injective on $X(\mu)^+$. Indeed, for $X(\mu) := L^1([0,1])$ and $E := \mathbb{C}$, the operator $T(f) := \int_{[0,1]} f\,d\mu$, for $f \in L^1([0,1])$ and μ being Lebesgue measure on $[0,1]$, is positive and μ-determined (as $m_T = \mu$) but, $T\big(\chi_{[0,\,1/2]}\big) = T\big(\chi_{(1/2,\,1]}\big)$, for example. This also shows that the converses of parts (iii) and (iv) of Lemma 4.5 fail to be valid.

Let us give several varied examples of μ-determined operators. We begin with linear functionals.

Example 4.6. Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ). Fix $g \in X(\mu)'$. With $E := \mathbb{C}$, define a continuous linear functional $T : X(\mu) \to E$ by

$$T(f) := \int_\Omega fg\,d\mu, \qquad f \in X(\mu).$$

Then, the associated (vector) measure m_T equals the indefinite integral $\mu_g : A \mapsto \int_A g\,d\mu$ on Σ. We claim that T is μ-determined if and only if $g(\omega) \neq 0$ for μ-a.e. $\omega \in \Omega$. In fact, it follows from Lemma 4.5(i) that T is μ-determined if and only if $\mathcal{N}_0(m_T) = \mathcal{N}_0(\mu)$ or equivalently $\mathcal{N}_0(\mu_g) = \mathcal{N}_0(\mu)$. On the other hand, it is routine to verify that $\mathcal{N}_0(\mu_g) = \mathcal{N}_0(\mu)$ if and only if $g(\omega) \neq 0$ for μ-a.e. $\omega \in \Omega$. This establishes our claim. □

It is straightforward to determine whether or not multiplication operators are μ-determined.

Example 4.7. Let (Ω, Σ, μ) be a positive, finite measure space.

(i) Fix $g \in L^\infty(\mu)$ and $1 \leq r < \infty$. For the multiplication operator given by $M_g : f \mapsto fg$ from $X(\mu) := L^r(\mu)$ into $E := L^r(\mu)$, it is immediate from the definition of $m_{M_g}(A) = \chi_A g$ for $A \in \Sigma$ that M_g is μ-determined if and only if $g(\omega) \neq 0$, for μ-a.e. $\omega \in \Omega$, if and only if M_g is injective. Now, suppose that M_g *is* μ-determined. Set $\nu^{(g)} := m_{M_g}$; this vector measure is denoted by Pg in Corollary 3.66(iii) which gives that $L^1(\nu^{(g)}) = \{f \in L^0(\mu) : gf \in L^r(\mu)\}$ and that the integration operator $I_{\nu^{(g)}} : L^1(\nu^{(g)}) \to L^r(\mu)$ is a surjective linear isometry onto its range.

If g is bounded away from 0, that is, $1/g \in L^\infty(\nu)$, then $L^1(\nu^{(g)}) = L^r(\mu)$. On the other hand, if $1/g \notin L^\infty(\nu)$, then

$$L^1(\nu^{(g)}) \;=\; (1/g)\cdot L^r(\mu) := \big\{f/g : f \in L^r(\mu)\big\}. \tag{4.8}$$

In this case, $M_g : L^r(\mu) \to L^r(\mu)$ has the proper extension $I_{\nu^{(g)}} : L^1(\nu^{(g)}) \to L^r(\mu)$ and we have the following commutative diagram:

$$\begin{tikzcd}
X(\mu) = L^r(\mu) \arrow[rr, "M_g"] \arrow[dr, "J_{M_g}"'] & & E = L^r(\mu) \, . \\
& L^1(\nu^{(g)}) \arrow[ur, "I_{\nu^{(g)}}"'] &
\end{tikzcd}$$

If $g^{-1}(\{0\}) \notin \mathcal{N}_0(\mu)$, then M_g is an example of an operator that is *not* μ-determined. Indeed, every $f \in L^r(\mu)$ whose μ-essential support is contained in $g^{-1}(\{0\})$ and is a non-μ-null set belongs to $\mathcal{N}_0(m_{M_g})$ but *not* to $\mathcal{N}(\mu)$.

(ii) Part (i) is a special case of a more general result. Let $X(\mu)$ be a σ-order continuous q-B.f.s. and $Y(\mu)$ be a B.f.s. with σ-o.c. norm. Take any non-zero function $g \in \mathcal{M}\big(X(\mu), Y(\mu)\big)$, if it exists. Again it is clear that M_g is μ-determined if and only if $g(\omega) \neq 0$, for μ-a.e. $\omega \in \Omega$, if and only if M_g is injective. In this case, let $\nu^{(g)} : \Sigma \to Y(\mu)$ again denote the vector measure associated with M_g. From Corollary 3.66(iii), with $Y(\mu)$ in place of $X(\mu)$ and $\nu^{(g)}$ in place of Pg, it follows that

$$L^1(\nu^{(g)}) \;=\; (1/g)\cdot Y(\mu) := \big\{f/g : f \in Y(\mu)\big\}$$

and that the integration operator $I_{\nu^{(g)}} : L^1(m_g) \to Y(\mu)$ is a surjective linear isometry onto its range.

(iii) What happens if we do *not* assume that the B.f.s. $Y(\mu)$, in (ii) above, is σ-o.c.? This is reduced to the σ-o.c. case. Indeed, fix $g \in L^0(\mu)$. Still under the assumption that $X(\mu)$ is σ-o.c. we show that

$$g\cdot X(\mu) \subseteq Y(\mu) \qquad \text{if and only if} \qquad g\cdot X(\mu) \subseteq Y(\mu)_{\mathrm{a}}. \tag{4.9}$$

Clearly $g\cdot X(\mu) \subseteq Y(\mu)_{\mathrm{a}}$ implies that $g\cdot X(\mu) \subseteq Y(\mu)$. Conversely, assume that $g\cdot X(\mu) \subseteq Y(\mu)$. Let $f \in X(\mu)$ be arbitrary and assume that $h_n \in Y(\mu)$ for $n \in \mathbb{N}$ satisfy $|fg| \geq h_n \downarrow 0$. For each $n \in \mathbb{N}$ define a Σ-measurable function h_n' by $h_n'(\omega) := h_n(\omega)/|g(\omega)|$ for every $\omega \in \Omega$ with $g(\omega) \neq 0$ and by $h_n'(\omega) := 0$ otherwise. Then $|f| \geq h_n'$ and hence, $\{h_n'\}_{n=1}^\infty \subseteq X(\mu)$. Moreover, $h_n' \downarrow 0$. So, $X(\mu)$ being σ-o.c. yields that $\lim_{n\to\infty} \|h_n'\|_{X(\mu)} = 0$. It follows that $\lim_{n\to\infty} \|h_n\|_{Y(\mu)} = \lim_{n\to\infty} \big\|M_g(h_n')\big\|_{Y(\mu)} = 0$ because $M_g : X(\mu) \to Y(\mu)$ is continuous. Thus, $fg \in Y(\mu)_{\mathrm{a}}$, which verifies that $g\cdot X(\mu) \subseteq Y(\mu)_{\mathrm{a}}$ and hence, (4.9) holds. We conclude that, if $Y(\mu)$ is not σ-o.c., then we can replace $Y(\mu)$ with $Y(\mu)_{\mathrm{a}}$, which enables us to apply (ii). $\square$

We now exhibit a μ-determined operator on a "*genuine*" q-B.f.s.

Example 4.8. Let $0 < r < 1$ and select $\varphi \in \ell^r \subseteq \ell^1$ satisfying $\varphi(n) > 0$ for each $n \in \mathbb{N}$. Then, the scalar measure μ defined by $\mu(A) := \sum_{n\in\mathbb{N}} \varphi(n)$ for $A \in 2^{\mathbb{N}}$ is surely finite. With $\psi(n) := \big(\varphi(n)\big)^{r-1}$ for $n \in \mathbb{N}$, it follows that $X(\mu) := \ell^r(\psi d\mu) \subseteq E := \ell^1(\mu)$ continuously; see (2.81). The natural inclusion $T : \ell^r(\psi\, d\mu) \to \ell^1(\mu)$ is a μ-determined operator on the q-B.f.s. $\ell^r(\psi d\mu)$. We recall that $\ell^r(\psi d\mu)$ is not normable because it is isomorphic to the non-normable q.B.f.s. $\ell^r(\mu)$; see Example 2.11.

This setting is also a special case of part (i) of Example 4.7 with $E := \ell^1(\mu)$ and $g := \chi_{\mathbb{N}}$. Hence, $L^1(m_T) = \ell^1(\mu)$ with equal norms and $I_{m_T} = \mathrm{id}_{\ell^1(\mu)}$.

The fact that μ is purely atomic in this example is crucial. Indeed, if μ were non-atomic then, given $\psi \in L^0(\mu) \setminus \{0\}$, we would have $\mathcal{M}\big(L^r(\psi d\mu), L^1(\mu)\big) = \{0\}$ because the identity $\big(L^r(\psi\, d\mu)\big)^* = \{0\}$ yields $\mathcal{L}\big(L^r(\psi\, d\mu), L^1(\mu)\big) = \{0\}$; see Lemma 4.3. $\square$

Let us return to one of our favourites, the *Volterra operator*.

Example 4.9. Let $1 \le r < \infty$. With μ denoting Lebesgue measure on $[0,1]$ and $X(\mu) := L^r([0,1])$ and $E := L^r([0,1])$, the Volterra operator $V_r : L^r([0,1]) \to L^r([0,1])$ (see (3.27)) is μ-determined. In fact, this follows from Lemma 4.5(iii) and the fact that V_r is injective on $L^r([0,1])$ because the derivative of $V_r(f)$ equals f (μ-a.e.) for every $f \in L^r([0,1])$. $\square$

Recall that the Fourier transform operator was introduced in Example 3.67.

Example 4.10. Let μ be normalized Haar measure on the circle group $\mathbb{T}$. Given $1 \le r < \infty$, let $X(\mu) := L^r(\mathbb{T})$ and $E := c_0(\mathbb{Z})$. Then $F_{r,0} : L^r(\mathbb{T}) \to c_0(\mathbb{Z})$ denotes the restriction of the Fourier transform operator $F_{1,0} : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$; see Example 3.67. Since $F_{1,0}$ is injective, so is $F_{r,0}$ and hence, $F_{r,0}$ is μ-determined via Lemma 4.5(iii). The associated vector measure of $F_{r,0}$ is the same as that of $F_{1,0}$, namely the $c_0(\mathbb{Z})$-valued measure $A \mapsto F_{1,0}(\chi_A)$ on $\mathcal{B}(\mathbb{T})$ as given in Example 3.67. It will be shown in Chapter 7 that $L^1(m_{F_{r,0}}) = L^1(\mathbb{T})$ for all $1 \le r < \infty$. $\square$

Convolution operators, which also form an important class of operators in harmonic analysis, are always μ-determined.

Example 4.11. Let μ and r, as well as $X(\mu)$, be as in Example 4.10. Let $g \in L^1(\mathbb{T}) \setminus \{0\}$ and define the convolution operator $C_g^{(r)} : L^r(\mathbb{T}) \to L^r(\mathbb{T})$ by

$$C_g^{(r)}(f) := f * g, \qquad f \in L^r(\mathbb{T}). \tag{4.10}$$

Then, with $E := X(\mu)$, the operator $C_g^{(r)} : X(\mu) \to E$ is μ-determined, [123, Lemma 2.2(ii)]. However, $C_g^{(r)}$ may not be injective. In fact, $C_g^{(r)}$ is injective if and only if the Fourier transform $\widehat{g} : \mathbb{Z} \to \mathbb{C}$ of g satisfies $\widehat{g}(n) = 0$ for all $n \in \mathbb{Z}$. This is so because the injectivity of $F_{r,0} : L^r(\mathbb{T}) \to c_0(\mathbb{Z})$ yields that

$$C_g^{(r)}(f) = 0 \iff F_{r,0}(f * g) = \widehat{f}\,\widehat{g} = 0, \qquad f \in L^r(\mathbb{T}).$$

Let $m_g^{(r)} : \mathcal{B}(\mathbb{T}) \to E$ denote the vector measure associated with $C_g^{(r)}$. Then it follows from [123, Theorem 1.1 and Proposition 4.1(i)] that

$$L^r(\mathbb{T}) \subseteq L^1\big(m_g^{(r)}\big) \subseteq L^1(\mathbb{T})$$

with the first inclusion being strict when $1 < r < \infty$; see also Chapter 7. So, if $1 < r < \infty$, then the associated integration operator $I_{m_g^{(r)}} : L^1\big(m_g^{(r)}\big) \to E$ is a genuine extension of $C_g^{(r)}$ and we have the commutative diagram:

$$\begin{tikzcd} L^r(\mathbb{T}) \arrow[rr, "C_g^{(r)}"] \arrow[dr, "J_{C_g^{(r)}}"'] & & L^r(\mathbb{T}) \arrow[r, "F_{r,0}"] & c_0(\mathbb{Z}). \\ & L^1(m_g^{(r)}) \arrow[ur, "I_{m_g^{(r)}}"'] & & \end{tikzcd}$$

□

As to be expected, *composition operators* are good candidates to be μ-determined.

Example 4.12. Let μ denote Lebesgue measure on $\Omega := [0,1]$. As customary, we write $dt = d\mu(t)$. Given any function $g \in L^0(\mu)$ with $g(t) > 0$, for μ-a.e. $t \in [0,1]$, let $L^2(g(t)\,dt)$ denote the weighted L^2-space, over $[0,1]$, with weight g. In other words, $L^2(g(t)\,dt)$ is the B.f.s., based on $(\Omega, \mathcal{B}(\Omega), \mu)$, equipped with the norm

$$f \longmapsto \left(\int_0^1 |f(t)|^2 g(t)\,dt\right)^{1/2}, \quad f \in L^2(g(t)dt).$$

We shall consider two different weighted L^2-spaces. Let $\alpha \geq 1$ and set $X(\mu) := L^2([0,1]) = L^2(\mu)$ and $E := L^2(t^\alpha\,dt)$. For every $f \in X(\mu)$, we have

$$\int_0^1 \big|f(t^2)\big|^2 t^\alpha\,dt = \frac{1}{2}\int_0^1 \big|f(u)\big|^2 u^{(\alpha-1)/2}\,du \leq \frac{1}{2}\int_0^1 \big|f(u)\big|^2 du < \infty. \tag{4.11}$$

So, we can define $T \in \mathcal{L}\big(L^2([0,1]), E\big)$ by

$$(Tf)(t) := f(t^2), \qquad t \in [0,1], \quad f \in X(\mu),$$

in which case $\|T\| \leq 1/\sqrt{2}$. Clearly, T is injective and hence, μ-determined via Lemma 4.5(iii). Since $t \mapsto 2^{-1}t^{(\alpha-1)/2}$ belongs to $L^\infty([0,1])$, it is clear that the natural inclusion $X(\mu) \subseteq L^2(2^{-1}t^{(\alpha-1)/2}dt)$ is continuous.

We claim, with equal norms, that

$$L^1(m_T) = L^2\big(2^{-1}t^{(\alpha-1)/2}\,dt\big). \tag{4.12}$$

Indeed, observe first that T admits a natural isometric linear extension

$$\widetilde{T} : L^2\big(2^{-1}t^{(\alpha-1)/2}\,dt\big) \to E$$

which assigns to each $f \in L^2\big(2^{-1}t^{(\alpha-1)/2}\,dt\big)$ the function $\widetilde{T}(f) : t \mapsto f(t^2)$ on $[0,1]$; this follows from the equality in (4.11). To verify the inclusion

$$L^2\big(2^{-1}t^{(\alpha-1)/2}\,dt\big) \subseteq L^1(m_T),$$

fix $f \in L^2\big(2^{-1}t^{(\alpha-1)/2}\,dt\big)$. With $f_n := f\chi_{[1/(n+1),1]}$ for $n \in \mathbb{N}$, it follows that $\lim_{n\to\infty} f_n = f$ pointwise and hence, by the Dominated Convergence Theorem, also in the norm of $L^2\big(2^{-1}t^{(\alpha-1)/2}\,dt\big)$. In particular, given $A \in \Sigma$, we have that $\lim_{n\to\infty} f_n\chi_A = f\chi_A$ in $L^2\big(2^{-1}t^{(\alpha-1)/2}\,dt\big)$ and hence, by continuity of $\widetilde{T}$, that $\lim_{n\to\infty} \big\|\widetilde{T}(f_n\chi_A) - \widetilde{T}(f\chi_A)\big\|_E = 0$. Since $t \mapsto 1/t^{(\alpha-1)/2}$ is bounded away from 0 on $[1/(n+1),1]$, it follows that $f_n \in X(\mu) \subseteq L^1(m_T)$ for each $n \in \mathbb{N}$. Accordingly, $\widetilde{T}(f_n\chi_A) = T(f_n\chi_A) = \int_A f_n\,dm_T$ via Proposition 4.4(i). Now apply Theorem 3.5, with $\nu := m_T$, to deduce that $f \in L^1(m_T)$ and $I_{m_T}(f) = \widetilde{T}f$. Thus, $L^2\big(2^{-1}t^{(\alpha-1)/2}\,dt\big) \subseteq L^1(m_T)$ and the integration operator $I_{m_T} : L^1(m_T) \to E$ is an extension of $\widetilde{T}$.

To verify the reverse inclusion $L^1(m_T) \subseteq L^2\big(2^{-1}t^{(\alpha-1)/2}\,dt\big)$, let $f \in L^1(m_T)$. Choose $s_n \in \operatorname{sim}\Sigma$ for $n \in \mathbb{N}$ such that $|s_n| \le |f|$ and $s_n \to f$ pointwise as $n \to \infty$. The Monotone Convergence Theorem for the positive measure $2^{-1}t^{(\alpha-1)/2}dt$ yields that

$$\frac{1}{2}\int_0^1 |f(t)|^2\, t^{(\alpha-1)/2}dt \;=\; \lim_{n\to\infty}\frac{1}{2}\int_0^1 |s_n(t)|^2\, t^{(\alpha-1)/2}dt.$$

From (4.11), with s_n in place of f, and the Lebesgue Dominated Convergence Theorem for the vector measure m_T (see Theorem 3.7(i)), it then follows that

$$\begin{aligned}\frac{1}{2}\int_0^1 \big|f(t)\big|^2\, t^{(\alpha-1)/2}dt \;&=\; \lim_{n\to\infty}\frac{1}{2}\int_0^1 \big|s_n(t)\big|^2\, t^{(\alpha-1)/2}dt \\ =\; \lim_{n\to\infty}\big\|T(s_n)\big\|_E^2 \;&=\; \lim_{n\to\infty}\big\|I_{m_T}(s_n)\big\|_E^2 = \;\big\|I_{m_T}(f)\big\|_E^2 < \infty. \end{aligned} \tag{4.13}$$

Accordingly, $f \in L^2\big(2^{-1}t^{(\alpha-1)/2}\,dt\big)$ which justifies the inclusion

$$L^1(m_T) \subseteq L^2\big(2^{-1}t^{(\alpha-1)/2}\,dt\big)$$

and hence, completes the proof of the identity (4.12) as vector spaces.

Now, (4.13) and Lemma 3.13 applied to the positive vector measure m_T imply that

$$\big\|f\big\|_{L^1(m_T)} = \;\big\|I_{m_T}(|f|)\big\|_E \;=\; \left(\frac{1}{2}\int_0^1 \big|f(t)\big|^2\, t^{(\alpha-1)/2}dt\right)^{1/2}, \qquad f \in L^1(m_T);$$

in other words, $L^1(m_T)$ and $L^2\big(2^{-1}t^{(\alpha-1)/2}\,dt\big)$ have equal norms. Observe, also from (4.13), that $I_{m_T} : L^1(m_T) \to E$ is a linear isometry. Moreover, I_{m_T} is surjective. Indeed, if $h \in E = L^2(t^\alpha\,dt)$, then the function $f : t \mapsto h(\sqrt{t})$ on $[0,1]$

belongs to $L^2\big(2^{-1}t^{(\alpha-1)/2}\,dt\big)$ and satisfies $I_{m_T}(f) = h$. Hence, $I_{m_T} : L^1(m_T) \to E$ is an isometric isomorphism.

Finally, since $L^1(m_T)$ is reflexive and m_T is non-atomic, it follows from Corollary 3.23, with $\nu := m_T$, that $|m_T|$ is totally infinite. $\square$

Our final example is the *reflection operator*.

Example 4.13. Let μ be Lebesgue measure on $\Omega := [-1,1]$. Given $1 \le r < \infty$, set $X(\mu) := L^r([-1,1])$ and $E := L^r([-1,1])$. Define the (linear) reflection operator $R : L^r([-1,1]) \to L^r([-1,1])$ by

$$(Rf)(t) := \ f(-t), \qquad t \in [-1,1], \quad f \in L^r([-1,1]),$$

and then define $Q \in \mathcal{L}(X(\mu), E)$ by

$$Q := \ \frac{1}{2}\Big(\mathrm{id} + R\Big)$$

with id denoting the identity operator from $X(\mu)$ to E. Then Q is a norm 1 *projection* of $L^r([-1,1])$ onto its closed subspace consisting of all the *even functions* and hence, is surely not injective. But, Q is μ-determined. In fact, let $A \in \mathcal{N}_0(m_Q)$. Then also $A \cap [0,1] \in \mathcal{N}_0(m_Q)$ and so

$$0 \ = \ m_Q\big(A \cap [0,1]\big) \ = \ \frac{1}{2}\Big(\chi_{A\cap[0,1]} + \chi_{(-A)\cap[-1,0]}\Big)$$

in $X(\mu)$, which implies that $\chi_{A\cap[0,1]} = 0$ (μ-a.e), that is, $\mu\big(A \cap [0,1]\big) = 0$. By a similar argument $\mu\big(A\cap[-1,0]\big) = 0$, so that $\mu(A) = 0$. Hence, Q is μ-determined.

Next, let us prove that $L^1(m_Q) = L^r([-1,1])$. By Proposition 4.4 applied to $T := Q$ we know that $L^r([-1,1]) = X(\mu) \subseteq L^1(m_Q)$. Let $J_Q : L^r([-1,1]) \to L^1(m_Q)$ denote the natural injection. To prove the reverse inclusion $L^1(m_Q) \subseteq L^r([-1,1])$, define $S : L^1(m_Q) \to L^1(m_Q)$ by

$$S(f) := \ \frac{1}{2} f\chi_{[0,1]} \ - \ J_Q\Big(\chi_{[0,1]} \cdot I_{m_Q}\big(f\chi_{[0,1]}\big)\Big), \qquad f \in L^1(m_Q).$$

Note that S is well defined, because the codomain space $L^r([-1,1])$ of I_{m_Q} is equal to the domain space of J_Q, and that S satisfies $\|S\| \le (1/2) + \|J_Q\| \cdot \|I_{m_Q}\|$. Direct calculation shows that $S(\chi_A) = 0$ for each $A \in \mathcal{B}(\Omega)$ and so S vanishes on the dense subspace $\operatorname{sim}\mathcal{B}(\Omega)$ of $L^1(m_Q)$, that is, $S = 0$. Accordingly,

$$\frac{1}{2} f\chi_{[0,1]} \ = \ J_Q\Big(\chi_{[0,1]} \cdot I_{m_Q}\big(f\chi_{[0,1]}\big)\Big) \ \in \ J_Q\big(L^r([-1,1])\big), \qquad f \in L^1(m_Q).$$

This means that $\chi_{[0,1]} \cdot L^1(m_Q) \subseteq J_Q\big(L^r([-1,1])\big) = L^r([-1,1])$. A similar argument gives $\chi_{[-1,0)} \cdot L^1(m_Q) \subseteq L^r([-1,1])$. So, the identity $L^1(m_Q) = L^r([-1,1])$ holds. In particular, Q is already defined on its optimal domain. $\square$

Before we show that every μ-determined operator $T \in \mathcal{L}(X(\mu), E)$ can be uniquely extended to $L^1(m_T)$, still with values in E, let us give an illustrative example. Let μ be Lebesgue measure on $[0,1]$ and $T : L^2([0,1]) \to \mathbb{C}$ be a continuous linear operator/functional. Can we extend T to a larger B.f.s. with σ-o.c. norm? Yes, and the procedure is indeed well known. By standard Hilbert space theory, there exists $g \in L^2([0,1])$ such that $T(f) = \int_0^1 fg\,d\mu$ for $f \in L^2([0,1])$. By Hölder's inequality we can extend T to the (typically) larger domain $L^1(g(t)\,dt)$ which is σ-o.c. Here observe that the $\mathbb{C}$-valued "vector measure" m_T associated with T is the weighted Lebesgue measure with weight g, i.e., $dm_T(t) = g(t)\,dt$, so that $L^1(g(t)\,dt) = L^1(m_T)$. It turns out that such an extension is available to *all* μ-determined operators $T \in \mathcal{L}(X(\mu), E)$ with $X(\mu)$ a σ-order continuous q-B.f.s. and E a Banach space. The associated vector measure m_T plays a crucial role. As observed in Proposition 4.4, the q-B.f.s. $X(\mu)$ is embedded into $L^1(m_T)$ via the continuous injection J_T and the integration operator $I_{m_T} : L^1(m_T) \to E$ is a continuous linear extension of T. The crucial point about this particular extension I_{m_T} to the particular space $L^1(m_T)$ is that it is *optimal*, as we now verify.

Theorem 4.14. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ). Let $T : X(\mu) \to E$ be a Banach-space-valued, continuous linear operator.*

Suppose that T is μ-determined. Then the σ-order continuous B.f.s. $L^1(m_T)$ is the largest space amongst all q-B.f.s.' with σ-o.c. quasi-norm (based on (Ω, Σ, μ)) into which $X(\mu)$ is continuously embedded and to which T admits an E-valued, continuous linear extension. Moreover, such an extension is unique and is precisely the integration operator $I_{m_T} : L^1(m_T) \to E$.

Proof. Let $Y(\mu)$ be any σ-order continuous q-B.f.s. such that $X(\mu) \subseteq Y(\mu)$, in which case the natural embedding is continuous by Lemma 2.7, and such that T admits a continuous linear extension $\widetilde{T} : Y(\mu) \to E$. Then

$$m_{\widetilde{T}}(A) \;=\; \widetilde{T}(\chi_A) \;=\; T(\chi_A) \;=\; m_T(A), \qquad A \in \Sigma,$$

that is, $m_{\widetilde{T}} = m_T$ and hence, $\widetilde{T}$ is also μ-determined. By Proposition 4.4 with $\widetilde{T}$ in place of T and $Y(\mu)$ in place of $X(\mu)$, we conclude that $Y(\mu) \subseteq L^1(m_{\widetilde{T}}) = L^1(m_T)$ and that $I_{m_{\widetilde{T}}} = I_{m_T}$ is a continuous linear extension of $\widetilde{T}$. So, $L^1(m_T)$ is the largest q-B.f.s. which contains $X(\mu)$ and to which T admits an E-valued, continuous linear extension. That I_{m_T} is the unique continuous linear extension of T to $L^1(m_T)$ is immediate from Proposition 4.4(iii). $\square$

In view of Theorem 4.14, the space $L^1(m_T)$ is called the *order continuous optimal domain* of T or simply, the optimal domain of T. Recall that order continuity and σ-order continuity are the same for the B.f.s. $L^1(m_T)$ over (Ω, Σ, μ); see Remark 2.5.

Remark 4.15. The main motivation of Theorem 4.14 comes from Theorem 3.1 in [24] which implies, in our terminology, that a Banach-spaced-valued continuous linear operator T on a σ-order continuous *Banach* function space $X(\mu)$ is uniquely extended to $L^1(m_T)$, and that $L^1(m_T)$ is the largest σ-order continuous B.f.s. containing $X(\mu)$ and to which T is extended. The condition that T is μ-determined, although not explicitly stated, is surely implicit in Theorem 3.1 of [24] as can be easily seen from its proof.

In our theorem, we are admitting a σ-order continuous *quasi*-B.f.s. $X(\mu)$ to begin with. A Banach-space-valued continuous linear operator T on $X(\mu)$ is then uniquely extended to its largest σ-order continuous q-B.f.s. $L^1(m_T)$ which turns out *necessarily* to be a B.f.s. $\square$

A simple but useful consequence of the previous theorem is the following one, in which the assumptions are the same as that of Theorem 4.14.

Corollary 4.16. *Suppose that $Y(\mu)$ is a σ-order continuous q-B.f.s. over (Ω, Σ, μ) such that $X(\mu) \subseteq Y(\mu)$. Then the following conditions for a μ-determined operator $T : X(\mu) \to E$ are equivalent.*

(i) *There exists a constant $C > 0$ such that*

$$\|Tf\|_E \le C\|f\|_{Y(\mu)}, \qquad f \in X(\mu) \subseteq Y(\mu), \tag{4.14}$$

(ii) *$X(\mu) \subseteq Y(\mu) \subseteq L^1(m_T)$ with continuous inclusions.*

Proof. (i) $\Rightarrow$ (ii). It follows from (i) that the operator $T : X(\mu) \to E$ is continuous when $X(\mu)$ is equipped with the relative topology induced by $Y(\mu)$. Since $X(\mu)$ contains the subspace $\operatorname{sim}\Sigma$ of $Y(\mu)$, which is dense in $Y(\mu)$, we can uniquely extend T from $X(\mu)$ to $Y(\mu)$. According to Theorem 4.14, $L^1(m_T)$ is the *largest* σ-order continuous q-B.f.s. to which T has an E-valued, continuous extension and so, $Y(\mu) \subseteq L^1(m_T)$. Observe that $X(\mu)$, $Y(\mu)$ and $L^1(m_T)$ are all q-B.f.s.' over (Ω, Σ, μ) and so, the natural inclusions in part (ii) are both continuous via Lemma 2.7. This establishes part (ii).

(ii) $\Rightarrow$ (i). The restriction of I_{m_T} to $Y(\mu)$, which is the composition of I_{m_T} with the natural inclusion map from $Y(\mu)$ into $L^1(m_T)$, is continuous. So, there is $C > 0$ such that $\|Tf\|_E = \|I_{m_T}(f)\|_E \le C\|f\|_{Y(\mu)}$ for $f \in X(\mu)$. $\square$

We have exhibited several classical μ-determined operators whose order continuous optimal domain is genuinely larger than their original domain (see Examples 4.7, 4.8, 4.10, 4.11 and 4.12 as well as Example 4.9 together with Example 3.26). It can also happen that the original domain $X(\mu)$ of T is already its optimal domain. For instance, see Example 4.13 or, the Fourier transform operator $F_{1,0} : L^1(\mathbb{T}) \to c_0(\mathbb{Z})$ in Example 3.67. For both of these examples, the corresponding operator T is not an isomorphism. On the other hand, if $T : X(\mu) \to E$ is an isomorphism, then it is to be expected that $X(\mu)$ is already optimal, i.e., $X(\mu) = L^1(m_T)$. Proposition 4.18 below isolates a class of operators T, including but not exclusive to isomorphisms, with the property that $L^1(m_T)$ equals the original domain of T. Let us clarify some technical terms. Let $S : Z_1 \to Z_2$ be a continuous linear operator between Banach spaces. The subspace $S^{-1}(\{0\})$ of Z_1 is called the *null space* (or kernel) of S. We say that S is a *semi-Fredholm operator* if its null space $S^{-1}(\{0\})$ is finite-dimensional and if its range

$$\mathcal{R}(S) := \{S(z) : z \in Z_1\}$$

is closed in Z_2. If, in addition, the quotient space $Z_2/\mathcal{R}(S)$ has finite dimension, then S is called a *Fredholm operator*. In particular, every isomorphism from one Banach space onto another Banach space is a Fredholm operator. Let us provide some examples.

Example 4.17. Let μ be normalized Haar measure on the circle group $\mathbb{T}$. Fix $1 \le r < \infty$.

(i) Given $a \in \mathbb{T}$, recall from Chapter 1 that $\tau_a : f \mapsto f(\cdot - a)$ denotes the translation operator by a, acting in $L^r(\mathbb{T})$. Then τ_a is an isometric isomorphism of $L^r(\mathbb{T})$ onto itself and is surely μ-determined. Moreover, $L^r(\mathbb{T})$ is already the o.c. optimal domain $L^1(m_{\tau_a})$ of τ_a, that is,

$$L^1(\mathbb{T}) \;=\; L^1(m_{\tau_a}). \tag{4.15}$$

In fact, let $f \in L^1(m_{\tau_a})$. Then, given any $g \in L^{p'}(\mathbb{T}) = L^p(\mathbb{T})^*$, we have

$$\langle m_{\tau_a}, g\rangle(A) = \int_A g(t+a)\,dt \quad \text{for every} \quad A \in \mathcal{B}(\mathbb{T}),$$

where dt stands for normalized Haar measure on $\mathbb{T}$. Therefore, from (I-1) with $\nu := m_{\tau_a}$ (see § 3.1) and $E := L^p(\mathbb{T})$, it follows that f belongs to the Köthe dual $L^p(\mathbb{T})$ of $L^{p'}(\mathbb{T})$. This yields the inclusion $L^1(m_{\tau_a}) \subseteq L^p(\mathbb{T})$ and hence, (4.15) holds because the reverse inclusion $L^p(\mathbb{T}) \subseteq L^1(m_{\tau_a})$ is guaranteed by Proposition 4.4(iii) with $X(\mu) := L^p(\mathbb{T})$, $T := \tau_a$ and $E := L^p(\mathbb{T})$. The identity (4.15) can also be obtained by Proposition 4.18 below.

(ii) Fix $g \in L^1(\mathbb{T}) \setminus \{0\}$ and $a \in \mathbb{T}$, and let the notation be as in Example 4.11. The convolution operator $C_g^{(r)} : L^r(\mathbb{T}) \to L^r(\mathbb{T})$ is compact, [48, Corollary 6]. So, since τ_a is an isomorphism of $L^r(\mathbb{T})$ onto itself, the operator $T := \tau_a + C_g^{(r)}$ is Fredholm, [150, Theorem 5.10]. It then follows that $L^1(m_T) = L^r(\mathbb{T})$, because of Proposition 4.18 below. □

Proposition 4.18. *Let $X(\mu)$ be a σ-order continuous B.f.s. over a positive, finite measure space (Ω, Σ, μ) and E be a Banach space. If $T : X(\mu) \to E$ is a μ-determined, semi-Fredholm operator, then $L^1(m_T) = X(\mu)$.*

Proof. By Proposition 4.4, we have that $T = I_{m_T} \circ J_T$, which is a semi-Fredholm operator by assumption. Then the natural *injection* $J_T : X(\mu) \to L^1(m_T)$ (see Proposition 4.4(iii)) is necessarily semi-Fredholm and so, in particular, has closed range; see, for instance, [150, Theorem 5.32]. But, $J_T\big(X(\mu)\big)$ contains the dense subspace $\operatorname{sim}\Sigma$ of $L^1(m_T)$ and hence, must be equal to $L^1(m_T)$. In other words, $L^1(m_T) = X(\mu)$. □

Remark 4.19. (i) Assume that (Ω, Σ, μ) is a *non-atomic*, positive, finite measure space and that $X(\mu)$ is a B.f.s. with σ-o.c. norm. Then every Banach-space-valued, *semi-Fredholm* operator on $X(\mu)$ is necessarily μ-determined. We actually claim a stronger result namely: *if μ is non-atomic and $T : X(\mu) \to E$ is any Banach-space-valued continuous linear operator satisfying* $\dim T^{-1}(\{0\}) < \infty$, *then T is necessarily μ-determined.* Indeed, assume on the contrary that there exists a set $A \in \mathcal{N}_0(m_T) \setminus \mathcal{N}_0(\mu)$. Then the subspace $\chi_A \cdot X(\mu) := \{\chi_A f : f \in X(\mu)\}$ of $X(\mu)$ is infinite-dimensional (as μ is non-atomic) and contained in $T^{-1}(\{0\})$ because $T(\chi_A f) = \int_A f\, dm_T = 0$ for all $f \in X(\mu)$; see Proposition 4.4(i). This contradicts the assumption that $\dim T^{-1}(\{0\}) < \infty$. So, T is necessarily μ-determined.

(ii) The criterion in part (i) may fail for the case of a purely atomic measure μ. For example, let the notation be as in Example 2.11 with $1 \le r < \infty$. Then the continuous projection $f \mapsto (0, f(2), f(3), \dots)$ on the B.f.s. $X(\mu) := \ell^r(\mu)$ into $E := \ell^r(\mu)$ is a Fredholm operator but, it is not μ-determined. Therefore, the assumption of μ-determinedness of T in Proposition 4.18 is necessary. $\square$

Let us present some further examples of classical Fredholm operators on function spaces.

Example 4.20. Let μ be normalized Haar measure on the circle group $\mathbb{T}$. Fix $1 < r < \infty$. Define the function $\operatorname{sgn} : \mathbb{Z} \to \mathbb{R}$ by $\operatorname{sgn}(k) := k/|k|$ for $k \in \mathbb{Z} \setminus \{0\}$ and $\operatorname{sgn}(0) := 0$.

Let $X(\mu) := L^r(\mathbb{T})$ and $E := L^r(\mathbb{T})$. Then the continuous linear operator $H_r : L^r(\mathbb{T}) \to L^r(\mathbb{T})$ given by

$$(H_r f)\widehat{\ } = (i \operatorname{sgn}) \cdot \widehat{f}, \qquad f \in L^r(\mathbb{T}),$$

is called the *Hilbert transform* on $L^r(\mathbb{T})$; see [17, Proposition 9.1.8]. In other words, H_r is a Fourier multiplier operator with multiplier $i \operatorname{sgn}$ (see [95] or Section 7.3 of Chapter 7 for the definition of Fourier multiplier operators). An important fact is that

$$H_r^2 f = -f + \widehat{f}(0)\chi_{\mathbb{T}}, \qquad f \in L^r(\mathbb{T}),$$

[17, Proposition 9.1.11]. This enables us to determine the range

$$\mathcal{R}(H_r) = \{f \in L^r(\mathbb{T}) : \widehat{f}(0) = 0\},$$

[17, pp. 339–340], so that $\mathcal{R}(H_r)$ is a closed subspace of $L^r(\mathbb{T})$ with codimension 1. This is so because we can write $\mathcal{R}(H_r) = (\pi_0 \circ F_{r,0})^{-1}(\{0\})$ by using the Fourier transform map $F_{r,0} : L^r(\mathbb{T}) \to c_0(\mathbb{Z})$ (see Example 4.10) and the coordinate functional $\pi_0 : c_0(\mathbb{Z}) \to \mathbb{C}$ of evaluation at $0 \in \mathbb{Z}$ with both π_0 and $F_{r,0}$ continuous.

Next we claim that the null space $H_r^{-1}(\{0\})$ is the 1-dimensional space spanned by the constant function $\chi_{\mathbb{T}}$, that is,

$$H_r^{-1}(\{0\}) = \mathbb{C}\chi_{\mathbb{T}}. \tag{4.16}$$

In fact, since $F_{r,0} : L^r(\mathbb{T}) \to c_0(\mathbb{Z})$ is injective, we have that a function $f \in L^r(\mathbb{T})$ belongs to $H_r^{-1}(\{0\})$ if and only if $0 = (F_{r,0} \circ H_r)(f) = (-i\,\mathrm{sgn}\,) \cdot \widehat{f}$ if and only if $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$ if and only if $f \in \mathbb{C}\chi_{\mathbb{T}}$. That is, (4.16) holds. Therefore H_r is a Fredholm operator. Since μ is non-atomic, the operator H_r is μ-determined (see Remark 4.19(i)). So, Proposition 4.18 yields that $L^r(\mathbb{T})$ is already the order continuous optimal domain of H_r. □

The usual definition of the Hilbert transform on the circle group $\mathbb{T}$ is via the Cauchy principal value; see [17, Definition 9.0.1]. In Example 4.20 above we have instead adopted the definition via the Fourier transform. This is no longer possible for the finite Hilbert transform, which we now discuss.

Example 4.21. Let Ω denote the open interval $(-1, 1)$ and μ be Lebesgue measure on Ω. Fix $1 < r < \infty$. Given $f \in L^r(\mu)$, the Cauchy principal value

$$(T_r f)(t) := \frac{1}{\pi}\,\mathrm{PV}\!\int_{-1}^{1} \frac{f(u)}{t-u}\,du = \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\left(\int_{-1}^{t-\varepsilon} + \int_{t+\varepsilon}^{1}\right)\frac{f(u)}{t-u}\,du$$

exists for μ-a.e. $t \in \Omega$. The resulting function $T_r f$ is called the *finite Hilbert transform* of f and belongs to $L^r(\mu)$, and the so-defined operator $T_r : f \mapsto T_r f \in L^r(\mu)$ is continuous from $L^r(\mu)$ into itself as a consequence of M. Riesz's Theorem, [17, Proposition 8.1.9]. Let $\mathbf{w}$ denote the function $t \mapsto \sqrt{1-t^2}$ on Ω. Since μ is non-atomic and $\dim T_r^{-1}(\{0\})$ is either 0 or 1, as seen from parts (i)–(iii) below, the operator T_r is μ-determined by Remark 4.19.

(i) Assume that $1 < r < 2$. Then $T_r \in \mathcal{L}\big(L^r(\mu)\big)$ is a Fredholm operator because T_r is surjective and $T_r^{-1}(\{0\}) = \{c/\mathbf{w} : c \in \mathbb{C}\}$. This is well known, [80, §13]. For an alternative proof see [118, Proposition 2.4]. It follows from Proposition 4.18 that $L^r(\mu)$ is already the order continuous optimal domain of T_r, that is, $L^1(m_{T_r}) = L^r(\mu)$.

(ii) Assume that $2 < r < \infty$. Again $T_r \in \mathcal{L}\big(L^r(\mu)\big)$ is a Fredholm operator because T_r is injective and $\mathcal{R}(T_r) = \{f \in L^r(\mu) : \int_{-1}^{1} f(t)/\mathbf{w}(t)\,dt = 0\}$. This is also well known, [80, §13]. An alternative proof has been given in [118, Proposition 2.6]. Again from Proposition 4.18 we have that $L^r(\mu)$ is already the order continuous optimal domain of T_r, that is, $L^1(m_{T_r}) = L^r(\mu)$.

(iii) The case $r = 2$ is different. The operator $T_2 \in \mathcal{L}\big(L^2(\mu)\big)$ is not Fredholm, although it is injective. Indeed, its range $\mathcal{R}(T_2)$ is a proper dense subspace of $L^2(\mu)$; see [118, § 3] for this fact and a description of $\mathcal{R}(T_2)$. The injection T_2 is μ-determined via Lemma 4.5(iii). The precise identification of the order continuous optimal domain $L^1(m_{T_2})$ of T_2 seems to be an open question. Of course, Proposition 4.4 gives that $L^2(\mu) \subseteq L^1(m_{T_2})$. Furthermore, we have that

$$L^1(m_{T_2}) \subseteq \bigcap_{1<r<2} L^r(\mu). \tag{4.17}$$

Indeed, let $1 < r < 2$ and $\alpha^{(r)} : L^2(\mu) \to L^r(\mu)$ denote the natural embedding. Then $m_{T_r} = \alpha^{(r)} \circ m_{T_2}$, which implies that $L^1(m_{T_2}) \subseteq L^1(m_{T_r})$ and $\int_A f\,dm_{T_r} =$

$\int_A f\, dm_{T_2}$ for every $f \in L^1(m_{T_2})$ and every Borel set $A \subseteq \Omega$; see Lemma 3.27. Hence, (4.17) holds because $L^1(m_{T_r}) = L^r(\mu)$ via part (i). Thus, we have that

$$L^2(\mu) \subseteq L^1(m_{T_2}) \subseteq \bigcap_{1<r<2} L^r(\mu). \qquad \square$$

The assumptions of Proposition 4.18 are that $T \in \mathcal{L}(X(\mu), E)$ is both μ-determined and semi-Fredholm. We cannot replace T being semi-Fredholm with the weaker requirement that T has closed range.

Example 4.22. (i) Let μ be Lebesgue measure on $[0,1]$ and $X(\mu) := L^2(\mu)$. Define $T : X(\mu) \to \mathbb{C}$ to be the continuous linear functional $f \mapsto \int_0^1 f(t)\, dt$. Then the range of T equals $\mathbb{C}$ and hence, is surely closed. Since $m_T = \mu$, the operator T is clearly μ-determined. However, its optimal domain $L^1(m_T) = L^1(\mu)$, which is *strictly larger* than $X(\mu) = L^2(\mu)$.

(ii) Let $\Omega := [-1,1]$ and μ be Lebesgue measure in Ω. Let $X(\mu)$ be the sublattice of $L^0(\mu)$ consisting of all $f \in L^0(\mu)$ such that

$$\|f\|_{X(\mu)} := \Big(\int_{-1}^0 |f(t)|^3\, dt\Big)^{1/3} + \Big(\int_0^1 |f(t)|^2\, dt\Big)^{1/2} < \infty.$$

Then $f \mapsto \|f\|_{X(\mu)}$, for $f \in X(\mu)$, is a σ-o.c. lattice norm for which $X(\mu)$ is a B.f.s. over $(\Omega, \mathcal{B}(\Omega), \mu)$. Observe that $X(\mu) \subsetneq L^2([-1,1])$ with a continuous inclusion. Let $T : X(\mu) \to L^2([-1,1])$ denote the restriction to $X(\mu)$ of the continuous projection $Q : L^2([-1,1]) \to L^2([-1,1])$, as defined in Example 4.13 for $r := 2$. Then $\mathcal{R}(T) = \mathcal{R}(Q)$ because, given any (even) function $g \in \mathcal{R}(Q)$, we observe that $T(2g\chi_{[0,1]}) = g$ with $2g\chi_{[0,1]} \in X(\mu)$. So, T has closed range.

On the other hand, $L^1(m_T)$ is *strictly larger* than $X(\mu)$ because $m_T = m_Q$ and hence, $L^1(m_T) = L^1(m_Q) = L^2([-1,1])$ as verified in Example 4.13. $\square$

4.3 Kernel operators

Kernel operators on $\Omega := [0,1]$, with $\Sigma := \mathcal{B}(\Omega)$ and μ being Lebesgue measure on Σ provide a rich and interesting supply of μ-determined operators whose optimal domain, in many cases, can be precisely described.

Let $K : \Omega \times \Omega \to [0, \infty)$ be a Borel measurable function such that,

$$\forall t \in \Omega, \text{ the function } K_t : u \mapsto K(t,u), \text{ for } u \in \Omega, \text{ is } \mu\text{-integrable.} \tag{4.18}$$

Let T_K be the associated operator defined via the formula

$$(T_K f)(t) := \int_\Omega f(u)\, K_t(u)\, du, \qquad t \in \Omega, \tag{4.19}$$

for any Σ-measurable function $f : \Omega \to \mathbb{R}$ for which the right-hand side of (4.19) is defined as a real number for a.e. $t \in \Omega$. Clearly $T_K f \geq 0$ whenever $f \geq 0$ and $T_K f$ is defined. Observe that $T_K f$ is defined pointwise on Ω for every $f \in L^\infty_{\mathbb{R}}(\mu)$. Under the assumption on the kernel K that

$$\lim_{n\to\infty} K_{t_n} = K_{t_0} \text{ weakly in the B.f.s. } L^1_{\mathbb{R}}(\mu) \text{ whenever } t_0 \in \Omega \text{ and } \lim_{n\to\infty} t_n = t_0, \tag{4.20}$$

it turns out that $T_K : L^\infty_{\mathbb{R}}(\mu) \to C_{\mathbb{R}}(\Omega)$ continuously; see [24, Proposition 4.1(a)] and the proof of [24, Proposition 5.1(a)]. Here $C_{\mathbb{R}}(\Omega)$ is the Banach space of all continuous, $\mathbb{R}$-valued functions on Ω equipped with the sup-norm.

The following result characterizes μ-determinedness of $T_K : L^\infty_{\mathbb{R}}(\mu) \to C_{\mathbb{R}}(\Omega)$ in terms of the kernel K itself. Note that the domain space of T_K is *not* σ-o.c. but, the associated vector measure $m_{T_K} : \mathcal{B}(\Omega) \to C_{\mathbb{R}}(\Omega)$ is still σ-additive (because of (4.20) rather than σ-order continuity of the domain space) and so μ-determinedness of T_K is still meaningful.

Proposition 4.23. *Let $K : \Omega \times \Omega \to [0, \infty)$ satisfy the assumptions* (4.18) *and* (4.20) *and let $T_K : L^\infty_{\mathbb{R}}(\mu) \to C_{\mathbb{R}}(\Omega)$ be given by* (4.19). *Whenever $u \in \Omega$, let $K^{(u)}$ denote the function $t \mapsto K(t, u)$, for $t \in \Omega$. Then T_K is μ-determined if and only if $0 < \int_\Omega K^{(u)}(t)\, dt < \infty$ for a.e. $u \in \Omega$.*

Proof. For every $A \in \Sigma$, we have from (4.19) that

$$m_{T_K}(A) = T_K(\chi_A) : t \longmapsto \int_A K(t, u)\, du, \qquad t \in \Omega. \tag{4.21}$$

Moreover, Fubini's Theorem yields

$$\int_\Omega m_{T_K}(A)(t)\, dt = \int_\Omega \left(\int_A K(t, u)\, du \right) dt = \int_\Omega \chi_A(u) \left(\int_\Omega K^{(u)}(t)\, dt \right) du. \tag{4.22}$$

Assume now that $0 < \int_\Omega K^{(u)}(t)\, dt < \infty$ for a.e. $u \in \Omega$. Let $A \in \mathcal{N}_0(m_{T_K})$. In particular, $m_{T_K}(A) = 0$ in $C_{\mathbb{R}}(\Omega)$ and so (4.22) implies that

$$\int_\Omega \chi_A(u) \left(\int_\Omega K^{(u)}(t)\, dt \right) du = 0$$

and hence, $\mu(A) = 0$. Thus, $\mathcal{N}_0(m_{T_K}) \subseteq \mathcal{N}_0(\mu)$. As the reverse inclusion always holds, we have $\mathcal{N}_0(m_{T_K}) = \mathcal{N}_0(\mu)$. So, T_K is μ-determined via Remark 3.4(ii).

Conversely assume that T_K is μ-determined, that is, $\mathcal{N}_0(m_{T_K}) = \mathcal{N}_0(\mu)$. Let

$$A := \left\{ u \in \Omega : \int_\Omega K^{(u)}(t)\, dt = 0 \right\}.$$

Since $m_{T_K}(A) \in C_{\mathbb{R}}(\Omega)$ is a non-negative function, it follows from (4.22) that $m_{T_K}(A)(t) = 0$ for a.e. $t \in \Omega$ and hence, by continuity of $m_{T_K}(A)$, for all $t \in \Omega$. So,

m_{T_K} being a positive vector measure, it follows that $A \in \mathcal{N}_0(m_{T_K})$. By hypothesis, also $A \in \mathcal{N}_0(\mu)$. But,

$$\int_\Omega K^{(u)}(t)\,dt < \infty \qquad \text{for a.e. } u \in \Omega. \tag{4.23}$$

In fact, since $T_K(\chi_\Omega) \in C_\mathbb{R}(\Omega) \subseteq L^1_\mathbb{R}(\mu)$, Fubini's Theorem applied to the function $K \geq 0$ gives that

$$\int_\Omega \Big(\int_\Omega K^{(u)}(t)\,dt \Big) du = \int_\Omega \Big(\int_\Omega K(t,u)\,du \Big)\,dt = \|T_K(\chi_\Omega)\|_{L^1(\mu)} < \infty.$$

So (4.23) holds. Then (4.23), together with the definition of the set A and the fact that $A \in \mathcal{N}_0(\mu)$, imply that $0 < \int_\Omega K^{(u)}(t)\,dt < \infty$ for a.e. $u \in \Omega$. $\square$

Remark 4.24. (i) The condition (4.20) in Proposition 4.23 is only needed as a necessary condition to ensure that $m_{T_K} : \mathcal{B}(\Omega) \to C_\mathbb{R}(\Omega)$ is σ-additive. Accordingly, it can be replaced with the *assumption* that m_{T_K} is σ-additive and Proposition 4.23 is then still valid.

(ii) Under the same assumptions on the kernel K as given in Proposition 4.23, the operator $T_K : L^\infty_\mathbb{R}(\mu) \to C_\mathbb{R}(\Omega)$ admits a natural extension $\tilde{T}_K : L^\infty(\mu) \to C(\Omega)$. The proof of Proposition 4.23 can be adapted with little change to draw the corresponding conclusion: $\tilde{T}_K$ is μ-determined if and only if $0 < \int_\Omega K^{(u)}(t)\,dt < \infty$ for a.e. $u \in \Omega$.

Example 4.25. Classical examples of kernels on $[0,1] \times [0,1]$ which satisfy the assumptions (4.18) and (4.20) are the following ones.

(a) $K(t,u) := \chi_\Delta(t,u)$, the Volterra kernel, where

$$\Delta := \Big\{(t,u) \in [0,1] \times [0,1] : 0 \leq u \leq t\Big\}.$$

We have already considered the Volterra operators which have this kernel and their associated Volterra measures in several examples earlier (see Examples 3.10, 3.26, 3.45, 3.49(iv), 4.2(i) and 4.9).

(b) $K(t,u) := \exp\big(-\lambda(u-t)\big) \cdot \chi_{[t,1]}(u)$ with $\lambda \in \mathbb{R}$, which arises in the nilpotent, left translation semigroup, [24, Example 4.4], [77, §19.4].

(c) $K(t,u) := |t-u|^{\alpha-1}$ for $0 < \alpha < 1$, arising in the Riemann–Liouville fractional semigroup, [24, pp. 51–52, Example 4.6].

(d) $K(t,u) := \arctan(u/t)$ for $t \neq 0$ and $K(0,u) := \pi/2$, the Poisson semigroup kernel, [24, Example 4.5], [77, p. 579].

(e) $K(t,u) := \chi_{[t,1]}(u) \cdot u^{(1/n)-1}$ for $n \in \mathbb{N}$ fixed, the Sobolev kernel, [25, p. 132], [50]. $\square$

The μ-determinedness of the kernel operators T_K given by (4.19) has been assumed only implicitly in [24] and [25]. Also, to be "completely correct", the condition that T is μ-determined (for μ being Lebesgue measure on $[0,1]$) should be added to Theorem 3.1 in [24].

A real B.f.s. $X_{\mathbb{R}}(\mu)$ over (Ω, Σ, μ), with $\Omega := [0,1]$, $\Sigma := \mathcal{B}(\Omega)$ and μ denoting Lebesgue measure, is called *rearrangement invariant* (briefly, r.i.) if it has the Fatou property and the property that, whenever $f \in X_{\mathbb{R}}(\mu)$ and g is μ-equimeasurable with f, then $g \in X_{\mathbb{R}}(\mu)$ and $\|g\|_{X_{\mathbb{R}}(\mu)} = \|f\|_{X_{\mathbb{R}}(\mu)}$, [13, Ch. 1, Definition 1.1 and Ch. 2, Definition 4.1]. It follows that $L^\infty_{\mathbb{R}}(\mu) \subseteq X_{\mathbb{R}}(\mu) \subseteq L^1_{\mathbb{R}}(\mu)$ continuously. Still assuming (4.18) and (4.20), we saw above that the kernel operator $T_K : L^\infty_{\mathbb{R}}(\mu) \to C_{\mathbb{R}}(\Omega)$ as given by (4.19) is continuous and hence, is also continuous if considered as taking its values in $L^\infty_{\mathbb{R}}(\mu)$ (because $C_{\mathbb{R}}(\Omega) \subseteq L^\infty_{\mathbb{R}}(\mu)$ continuously). Henceforth, we consider $T_K : L^\infty_{\mathbb{R}}(\mu) \to L^\infty_{\mathbb{R}}(\mu)$. Since $K \geq 0$, we have

$$\|T_K f\|_{L^1_{\mathbb{R}}(\mu)} \leq \int_\Omega \bigg(\int_\Omega |f(u)| K(t,u)\, du \bigg) dt = \int_\Omega |f(u)| \mathbf{w}(u)\, du,$$

for each $f \in L^1_{\mathbb{R}}(\mu)$, where $\mathbf{w}$ is the weight function

$$\mathbf{w}(u) := \int_0^1 K(t,u)\, dt = \big\|K^{(u)}\big\|_{L^1_{\mathbb{R}}(\mu)}, \qquad u \in [0,1]. \tag{4.24}$$

So, if $\mathbf{w} \in L^\infty_{\mathbb{R}}(\mu)$, that is, $\big\{K^{(u)} : u \in [0,1]\big\} \subseteq L^1_{\mathbb{R}}(\mu)$ is a bounded set, then T_K maps $L^1_{\mathbb{R}}(\mu)$ continuously into $L^1_{\mathbb{R}}(\mu)$. By the Interpolation Theorem, [13, Ch. 3, Proposition 1.10 and Theorem 2.2], it follows that T_K maps $X_{\mathbb{R}}(\mu)$ continuously into $X_{\mathbb{R}}(\mu)$ for *every* r.i. Banach function space $X_{\mathbb{R}}(\mu)$; in this case we denote the corresponding operator by $T_{K,X_{\mathbb{R}}(\mu)}$. Since the range $\mathcal{R}(m_{T_{K,X_{\mathbb{R}}(\mu)}}) \subseteq C_{\mathbb{R}}(\Omega) \subseteq X_{\mathbb{R}}(\mu)$ for all r.i. spaces $X_{\mathbb{R}}(\mu)$, it follows from Proposition 4.23 that the vector measure $m_{T_{K,X_{\mathbb{R}}(\mu)}} : \mathcal{B}(\Omega) \to X_{\mathbb{R}}(\mu)$ is μ-*determined* for each r.i. space $X_{\mathbb{R}}(\mu)$, provided that $\mathbf{w} \in L^\infty_{\mathbb{R}}(\mu)$. Under some mild conditions, the optimal domain spaces $L^1_{\mathbb{R}}(m_{T_{K,X_{\mathbb{R}}(\mu)}})$ have some interesting “concrete descriptions” which we now present.

We recall the **K**-method of J. Peetre. If (X_0, X_1) is a compatible pair of Banach spaces, then the **K**-functional of $f \in X_0 + X_1$ is, for $t > 0$,

$$\mathbf{K}\Big(t, f; X_0, X_1\Big) := \inf \Big\{ \|f_0\|_{X_0} + t\|f_1\|_{X_1} \ : \ f = f_0 + f_1;\ f_0 \in X_0,\ f_1 \in X_1 \Big\}. \tag{4.25}$$

From a r.i. norm ρ on $[0,1]$ (see [13, Ch. 1, Definition 1.1 and Ch. 2, Definition 4.1]) we can generate interpolation spaces $(X_0, X_1)_\rho$; see [13, Ch. 5, Definitions 1.13 and 1.18]. These spaces have a monotonicity property. Indeed, let (Y_0, Y_1) be another pair of compatible Banach spaces. If $f \in (X_0, X_1)_\rho$ and for every $t > 0$ we have $\mathbf{K}(t, g; Y_0, Y_1) \leq \mathbf{K}(t, f; X_0, X_1)$, then $g \in (Y_0, Y_1)_\rho$ and $\|g\|_{(Y_0,Y_1)_\rho} \leq \|f\|_{(X_0,X_1)_\rho}$; see [13, Ch. 5, Theorem 1.19 and Ch. 5, inequality (1.47)]. Every r.i. space $X_{\mathbb{R}}(\mu)$ on Ω arises as $X_{\mathbb{R}}(\mu) = \big(L_{\mathbb{R}}(\mu), L^\infty_{\mathbb{R}}(\mu)\big)_\rho$ for a suitable norm ρ; see [13, Ch. 5, Theorem 1.17].

The kernel $K \geq 0$ on $\Omega \times \Omega$ is called *non-decreasing* if $\{K_t : t \in [0,1]\}$, as given in (4.19), is non-decreasing in the sense that $K_{t_1} \leq K_{t_2}$ pointwise a.e. on $[0,1]$ whenever $0 \leq t_1 \leq t_2 \leq 1$. Similarly, we define *non-increasing* kernels. The Volterra kernel of Example 4.25(a) is non-decreasing whereas the kernels in (b), (d), (e) of Example 4.25 are non-increasing. For non-decreasing kernels K with the property that there exists a constant $\beta > 0$ such that, for every $r > 0$ and every $u \in [0,1]$,

$$\int_{\max\{0,\ 1-r\}}^{1} K(t,u)\,dt \ \geq \ \beta \cdot \min\left\{ rK(1,u),\ \int_0^1 K(t,u)\,dt \right\}, \tag{4.26}$$

and for non-increasing kernels K with the property that there exists a constant $\beta > 0$ such that, for every $r > 0$ and $u \in [0,1]$,

$$\int_0^{\min\{1,r\}} K(t,u)\,dt \ \geq \ \beta \cdot \min\left\{ rK(0,u),\ \int_0^1 K(t,u)\,dt \right\}, \tag{4.27}$$

rather precise information concerning $L^1_{\mathbb{R}}\big(m_{T_K,X_{\mathbb{R}}(\mu)}\big)$ is available. We point out that the Volterra kernel of Example 4.25(a) satisfies (4.26) and that the kernels in (b), (e) of Example 4.25 both satisfy (4.27). Define two further weight functions by

$$\xi(u) := K(1,u) \quad \text{and} \quad \eta(u) := K(0,u), \qquad u \in [0,1], \tag{4.28}$$

and let $L^1_{\mathbf{w}}(\mu)$, $L^1_{\xi}(\mu)$ and $L^1_{\eta}(\mu)$ denote the spaces $L^1_{\mathbb{R}}\big(\mathbf{w}(u)\,du\big)$, $L^1_{\mathbb{R}}\big(\xi(u)\,du\big)$ and $L^1_{\mathbb{R}}\big(\eta(u)\,du\big)$ respectively, where the weights $\mathbf{w}$, ξ and η are defined by (4.24) and (4.28). The following result, [24, Theorem 5.11 and 5.12], provides an alternate description of the optimal domain spaces of the μ-determined kernel operators $T_{K,X_{\mathbb{R}}(\mu)} \in \mathcal{L}(X_{\mathbb{R}}(\mu))$. Recall that $\Omega := [0,1]$.

Proposition 4.26. *Let μ be Lebesgue measure on Ω and $K : \Omega \times \Omega \to [0,\infty)$ be a kernel which satisfies* (4.18) *and for which $\mathbf{w} \in L^\infty_{\mathbb{R}}(\mu)$. Suppose that the set function m_{T_K} defined on $\mathcal{B}(\Omega)$ by* (4.21) *is a $C_{\mathbb{R}}(\Omega)$-valued vector measure. Given a r.i. norm ρ on Ω, let $X_{\mathbb{R}}(\mu)$ denote the r.i. space $\big(L^1_{\mathbb{R}}(\mu),\ L^\infty_{\mathbb{R}}(\mu)\big)_\rho$ on Ω, assumed to have σ-o.c. norm, and consider the kernel operator $T_{K,X_{\mathbb{R}}(\mu)} \in \mathcal{L}\big(X_{\mathbb{R}}(\mu)\big)$.*

(i) *Suppose that K is non-decreasing and satisfies* (4.26)*. Then*

$$L^1_{\mathbb{R}}\big(m_{T_K,X_{\mathbb{R}}(\mu)}\big) = \Big(L^1_{\mathbf{w}}(\mu),\ L^1_{\xi}(\mu)\Big)_\rho.$$

(ii) *Suppose that K is non-increasing and satisfies* (4.27)*. Then*

$$L^1_{\mathbb{R}}\big(m_{T_K,X_{\mathbb{R}}(\mu)}\big) = \Big(L^1_{\mathbf{w}}(\mu),\ L^1_{\eta}(\mu)\Big)_\rho.$$

The procedure for applying Proposition 4.26 starts by identifying the spaces $L^1_{\mathbf{w}}(\mu)$, $L^1_\xi(\mu)$ and $L^1_\eta(\mu)$, and then checking that either the condition (4.26) or (4.27) holds, depending on the monotonicity properties of K. Next, the **K**-functional with respect to the pair $\big(L^1_{\mathbf{w}}(\mu), L^1_\xi(\mu)\big)$ or $\big(L^1_{\mathbf{w}}(\mu), L^1_\eta(\mu)\big)$ has to be computed, via the formula (4.25), say. Finally the corresponding r.i. norm ρ determines the space $L^1_{\mathbb{R}}(m_{T_{K,X_{\mathbb{R}}(\mu)}})$ as given by Proposition 4.26.

For example, the kernel K of Example 4.25(b) satisfies the hypotheses required in Proposition 4.26(ii) whenever $\lambda < 0$. For this example we have $\mathbf{w}(u) = (1 - e^{-\lambda u})/\lambda$ and $\eta(u) = e^{-\lambda u}$, on $[0,1]$, and so $L^1_{\mathbf{w}}(\mu) = L^1_{\mathbb{R}}(u\,du)$ and $L^1_\eta(\mu) = L^1_{\mathbb{R}}([0,1])$, with equivalent norms. The **K**-functional is then

$$\mathbf{K}\Big(t, f; L^1_{\mathbf{w}}(\mu), L^1_\eta(\mu)\Big) = \int_0^1 f(u)\,\min\{u,t\}\,du, \qquad t \in [0,1].$$

Consider the case when $X_{\mathbb{R}}(\mu)$ is a Lorentz space $L^{p,q}_{\mathbb{R}}([0,1])$, with $1 < p < \infty$ and $1 \le q < \infty$, which is the real Lorentz space $L^{p,q}_{\mathbb{R}}(\mu)$ defined in Example 2.76(ii); see also [13, Ch. 4, §4] and [69, Ch. 1, §1.4]). Then the corresponding space $L^1_{\mathbb{R}}\big(m_{T_{K,X_{\mathbb{R}}(\mu)}}\big)$ is precisely the space of functions f satisfying

$$\int_0^\infty \left(t^{(1/p)-1} \int_0^1 |f(u)|\,\min\{u,t\}\,du \right)^q \frac{dt}{t} < \infty;$$

see [13, Ch. 5, Definition 1.7]. This implies that the norm of $L^1_{\mathbb{R}}\big(m_{T_{K,X_{\mathbb{R}}(\mu)}}\big)$ is equivalent to $\int_0^1 |f(u)|\,u^{1/p}\,du$, [24, Remark 5.14].

It should be noted that $X_{\mathbb{R}}(\mu)$ being r.i. does *not* necessarily imply that $L^1_{\mathbb{R}}\big(m_{T_{K,X_{\mathbb{R}}(\mu)}}\big)$ is r.i. For instance, this phenomenon occurs for the kernel K of Example 4.25(c) with $X_{\mathbb{R}}(\mu) = L^\infty_{\mathbb{R}}([0,1])$, [24, Example 5.15(b)].

For more details and further results along these lines we refer to [24]. The particular *Sobolev kernel* K, as given in Example 4.25(e), is investigated in detail in [25]. In addition to the description given by Proposition 4.26(ii), it is also shown in [25, Proposition 3.4], still for the Sobolev kernel, that the space $L^1_{\mathbb{R}}\big(m_{T_{K,X_{\mathbb{R}}(\mu)}}\big)$ consists of all functions $f \in L^0_{\mathbb{R}}(\mu)$ satisfying

$$\int_0^1 |f(u)|\,u^{(1/u)-1} \left(\int_0^u g(t)\,dt \right) du \;<\; \infty$$

for every decreasing function $g \in \big(X_{\mathbb{R}}(\mu)'\big)^+$.

For particular r.i. spaces $X_{\mathbb{R}}(\mu)$ this becomes more explicit. For instance, if $X_{\mathbb{R}}(\mu) = L^\infty_{\mathbb{R}}([0,1])$, then

$$L^1_{\mathbb{R}}\big(m_{T_{K,X_{\mathbb{R}}(\mu)}}\big) \;=\; L^1_{\mathbb{R}}\big(|m_{T_{K,X_{\mathbb{R}}(\mu)}}|\big) \;=\; L^1_{\mathbb{R}}\big(u^{(1/n)-1}\,du\big),$$

[25, Proposition 2.1], whereas for $X_{\mathbb{R}}(\mu) = L^1_{\mathbb{R}}([0,1])$ we have

$$L^1_{\mathbb{R}}\big(m_{T_{K,X_{\mathbb{R}}(\mu)}}\big) \;=\; L^1_{\mathbb{R}}\big(|m_{T_{K,X_{\mathbb{R}}(\mu)}}|\big) \;=\; L^1_{\mathbb{R}}(u^{1/n}\,du),$$

[25, Corollary 4.3]. Actually, the latter example is a special case of a more general fact. Let φ be any increasing, concave function on $[0,1]$ with $\varphi(0)=0$. Then the r.i. space

$$\Lambda_\varphi := \Big\{ f \in L^0_{\mathbb{R}}(\mu) : \|f\|_{\Lambda_\varphi} = \int_0^1 f^*(t)\, d\varphi(t) < \infty \Big\},$$

where f^* is the *decreasing rearrangement* of f, is called the *Lorentz space* associated with φ, [91]. Recall that the *fundamental function* $\varphi_{X_{\mathbb{R}}(\mu)}$ of a r.i. space $X_{\mathbb{R}}(\mu)$ is defined by $u \mapsto \|\chi_{[0,u]}\|_{X_{\mathbb{R}}(\mu)}$ for $u \in [0,1]$. It is increasing and quasiconcave with $\lim_{u\to 0^+} \varphi_{X_{\mathbb{R}}(\mu)}(u) \geq 0$. Actually, we may assume that $\varphi_{X_{\mathbb{R}}(\mu)}$ is concave, [13, Ch. 2, Proposition 5.11]. The space $\Lambda_{X_{\mathbb{R}}(\mu)} := \Lambda_\varphi$ for $\varphi := \varphi_{X_{\mathbb{R}}(\mu)}$ is the *smallest* r.i. space having the same fundamental function as $X_{\mathbb{R}}(\mu)$ and satisfies $\Lambda_{X_{\mathbb{R}}(\mu)} \subseteq X_{\mathbb{R}}(\mu)$ continuously, [91, pp. 118–119]. It turns out that the vector measure $m_{T_{K,\Lambda_{X_{\mathbb{R}}(\mu)}}}$ has finite variation, given by

$$A \mapsto \int_A u^{(1/n)-1}\, \varphi_{X_{\mathbb{R}}(\mu)}(u)\, du, \qquad A \in \mathcal{B}(\Omega),$$

[25, Proposition 3.1], and that

$$L^1_{\mathbb{R}}\big(m_{T_{K,\Lambda_{X_{\mathbb{R}}(\mu)}}}\big) \;=\; L^1_{\mathbb{R}}\big(|m_{T_{K,\Lambda_{X_{\mathbb{R}}(\mu)}}}|\big) \;=\; L^1_{\mathbb{R}}\big(u^{(1/n)-1}\varphi_{X_{\mathbb{R}}(\mu)}(u)\, du\big),$$

[25, Corollary 4.3].

Let us now give an example of a non-μ-determined kernel operator.

Example 4.27. Let μ be Lebesgue measure on $[0,1]$ and $\Sigma := \mathcal{B}([0,1])$. Let

$$\Delta := \Big\{ (t,u) \in [0,1]^2 : t \leq u \leq 2^{-1} \Big\} \bigcup \Big(\big[2^{-1},1\big] \times \big[0,2^{-1}\big] \Big)$$

and $K(t,u) := \chi_\Delta(t,u)$ for $(t,u) \in [0,1]^2$. Given $1 \leq r < \infty$, define the associated kernel operator $T_K : L^r([0,1]) \to L^r([0,1])$ by

$$(T_K f)(t) := \int_0^1 K(t,u) f(u)\, du, \qquad t \in [0,1], \tag{4.29}$$

for every $f \in L^r([0,1])$. Then the associated vector measure $m_T : \Sigma \to L^r([0,1]))$ satisfies $(2^{-1},1] \in \mathcal{N}_0(m_T) \setminus \mathcal{N}_0(\mu)$. Hence, T_K is not μ-determined. This can also be seen from Proposition 4.23 because $\int_\Omega K^{(u)}(t)\, dt = 0$ for every $u \in (2^{-1},1]$. $\square$

In the above example, T_K is not μ-determined because the measure μ specifying its domain space $L^r(\mu)$, which is a B.f.s. over the measure space $([0,1],\Sigma,\mu)$, is defined on a set which is "too large" in relation to the operator T_K. However, if we restrict T_K to the L^r-space over $[0,2^{-1}]$, then the resulting operator is $\mu_{[0,2^{-1}]}$-determined, where $\mu_{[0,2^{-1}]}$ is the measure μ restricted to $\Sigma \cap [0,2^{-1}]$. This feature occurs for all non-μ-determined operators (and general finite measures μ).

Proposition 4.28. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ) and let $T : X(\mu) \to E$ be a Banach-space-valued, continuous linear operator which is not μ-determined.*

(i) *There exists a set $\Omega_1 \in \Sigma$ such that both $\Omega \setminus \Omega_1 \in \mathcal{N}_0(m_T) \setminus \mathcal{N}_0(\mu)$ and also $\mathcal{N}_0(m_T) \cap \Omega_1 = \mathcal{N}_0(\mu) \cap \Omega_1$. Consequently, $T(f\chi_{\Omega\setminus\Omega_1}) = 0$ or equivalently $T(f) = T(f\chi_{\Omega_1})$ for every $f \in X(\mu)$.*

(ii) *Let $\mu_1 := \mu|_{\Omega_1 \cap \Sigma}$. Regard the complemented subspace of $X(\mu)$ given by $X(\mu_1) := \{f\chi_{\Omega_1} : f \in X(\mu)\}$, in a natural way, as a q-B.f.s. over the finite measure space $(\Omega_1, \Sigma \cap \Omega_1, \mu_1)$ and let $i_1 : X(\mu_1) \to X(\mu)$ denote the natural embedding (i.e., each $h \in X(\mu_1)$ is extended to Ω by defining it to be 0 on $\Omega \setminus \Omega_1$). Then the operator $T \circ i_1 : X(\mu_1) \to E$ is μ_1-determined.*

(iii) *The set Ω_1 is maximal in the sense that if $\Omega_2 \in \Sigma$ is another set such that $\Omega \setminus \Omega_2 \in \mathcal{N}_0(m_T) \setminus \mathcal{N}_0(\mu)$ and $\mathcal{N}_0(m_T) \cap \Omega_2 = \mathcal{N}_0(\mu) \cap \Omega_2$, then the symmetric difference $\Omega_1 \triangle \Omega_2$ of Ω_1 and Ω_2 is μ-null.*

Proof. (i) Let $\lambda : \Sigma \to [0, \infty)$ be a control measure for $m_T : \Sigma \to E$, that is, $\mathcal{N}_0(\lambda) = \mathcal{N}_0(m_T)$. Since $\mathcal{N}_0(\mu) \subseteq \mathcal{N}(m_T)$, we have $\lambda \ll \mu$, that is, λ is absolutely continuous with respect to μ. So, we may consider the Radon–Nikodým derivative $g := d\lambda/d\mu$, in which case $\lambda(A) = \int_A g \, d\mu$, for each $A \in \Sigma$, with $g \in L^1(\mu)^+$. Let $\Omega_1 := \Omega \setminus g^{-1}(\{0\})$. Given $A \in \Sigma$, we claim that

$$A \in \mathcal{N}_0(m_T) \iff \mu(A \cap \Omega_1) = 0. \tag{4.30}$$

In fact, $A \in \mathcal{N}_0(m_T)$ if and only if $\lambda(A) = 0$ if and only if $g(\omega) = 0$ for λ-a.e $\omega \in A$, which is equivalent to $\mu(A \cap \Omega_1) = 0$. So, (4.30) holds. It is clear from (4.30) that $\mathcal{N}_0(m_T) \cap \Omega_1 = \mathcal{N}_0(\mu) \cap \Omega_1$ and that $\Omega \setminus \Omega_1 \in \mathcal{N}_0(m_T)$. Moreover, $\mu(\Omega \setminus \Omega_1) > 0$. For, if $\mu(\Omega \setminus \Omega_1) = 0$, then (4.30) would imply that $\mathcal{N}_0(m_T) = \mathcal{N}_0(\mu)$, which is not the case as T is not μ-determined; see Lemma 4.5(i). Thus $\Omega \setminus \Omega_1 \in \mathcal{N}_0(m_T) \setminus \mathcal{N}_0(\mu)$.

Next, given $s \in \operatorname{sim} \Sigma$, we have

$$T(s\chi_{\Omega\setminus\Omega_1}) = 0. \tag{4.31}$$

Indeed, write $s = \sum_{j=1}^{k} a_j \chi_{A_j}$ for some $a_j \in \mathbb{C}$ and $A_j \in \Sigma$ with $j = 1, \ldots, k$ and $k \in \mathbb{N}$. Then

$$T(s\chi_{\Omega\setminus\Omega_1}) = \sum_{j=1}^{k} a_j m_T\big(A_j \cap (\Omega \setminus \Omega_1)\big) = 0$$

because $\Omega \setminus \Omega_1 \in \mathcal{N}_0(m_T)$.

Now take a general function $f \in X(\mu)$ and find a sequence $\{s_n\}_{n=1}^{\infty} \subseteq \operatorname{sim} \Sigma$ such that $|s_n| \le |f|$ for $n \in \mathbb{N}$ and $s_n \to f$ pointwise as $n \to \infty$. Then Proposition 4.4(i) and the Lebesgue Dominated Convergence Theorem for m_T (see Theorem

3.7(i)) yield that

$$T(f\chi_{\Omega\setminus\Omega_1}) = \int_\Omega f\chi_{\Omega\setminus\Omega_1}\,dm_T = \lim_{n\to\infty}\int_\Omega s_n\chi_{\Omega\setminus\Omega_1}\,dm_T = \lim_{n\to\infty} T(s_n\chi_{\Omega\setminus\Omega_1}) = 0$$

because of (4.31) with $s := s_n$ for $n \in \mathbb{N}$.

(ii) The operator $T \circ i_1$ is μ_1-determined because part (i) yields that $\mathcal{N}_0(m_{(T\circ i_1)}) = \mathcal{N}_0(\mu_1)$.

(iii) By assumption $\Omega_1 \setminus \Omega_2 \subseteq \Omega \setminus \Omega_2 \in \mathcal{N}_0(m_T)$ and hence, $\Omega_1 \setminus \Omega_2 \in \mathcal{N}_0(m_T) \cap \Omega_1 = \mathcal{N}_0(\mu) \cap \Omega_1$ via part (i). Similarly $\Omega_2 \setminus \Omega_1 \in \mathcal{N}_0(m_T) \cap \Omega_2 = \mathcal{N}_0(\mu) \cap \Omega_2$ by exchanging the role of Ω_1 with that of Ω_2. Consequently, we have that $\Omega_1 \triangle \Omega_2 = \big(\Omega_1 \setminus \Omega_2\big) \cup \big(\Omega_2 \setminus \Omega_1\big) \in \mathcal{N}_0(\mu)$. $\square$

Given a σ-order continuous q-B.f.s. $X(\mu)$ over a positive, finite measure space (Ω, Σ, μ) and a Banach-space-valued, continuous linear operator $T : X(\mu) \to E$, a set $\Omega_1 \in \Sigma$ is said to be an *essential carrier* of T if $\Omega \setminus \Omega_1 \in \mathcal{N}_0(m_T)$ and $\mathcal{N}_0(m_T) \cap \Omega_1 = \mathcal{N}_0(\mu) \cap \Omega_1$. If T happens to be μ-determined, then the whole set Ω is an essential carrier of T.

Chapter 5

p-th Power Factorable Operators

Let $X(\mu)$ be a q-B.f.s. with σ-o.c. norm and E be a Banach space. A continuous linear operator $T : X(\mu) \to E$ is *p-th power factorable* (cf. Definition 5.1 below), for $1 \le p < \infty$, if there exists $T_{[p]} \in \mathcal{L}\big(X(\mu)_{[p]}, E\big)$ which coincides with T on $X(\mu) \subseteq X(\mu)_{[p]}$. There is no a priori reason to suspect any connection between the p-th power factorability of T and its associated E-valued vector measure $m_T : A \mapsto T(\chi_A)$. The aim of this chapter is to convince the reader that such a connection does indeed exist and has some far-reaching consequences. Henceforth, assume that T is also μ-determined.

As noted in earlier chapters, the B.f.s.' $L^p(m_T)$ satisfy $L^p(m_T)_{[p]} = L^1(m_T)$. It turns out that the restriction map $I_{m_T}^{(p)} : L^p(m_T) \to E$ of the integration operator $I_{m_T} : L^1(m_T) \to E$ to $L^p(m_T) \subseteq L^1(m_T)$ is *always* p-th power factorable. However, it does *not* follow in general that $I_{m_T}^{(p)} : L^p(m_T) \to E$ is an extension of the original operator $T : X(\mu) \to E$. The difficulty is that the containments $X(\mu) \subseteq L^1(m_T)$ and $L^p(m_T) \subseteq L^1(m_T)$, which always hold, need *not* imply that $X(\mu) \subseteq L^p(m_T)$; see Example 5.9(i) and Proposition 5.22, for example. However, as we shall see, many operators of interest coming from various branches of analysis are p-th power factorable and we will focus our attention on these. The crucial fact in this regard is that T is p-th power factorable if and only if $X(\mu) \subseteq L^p(m_T)$ or, equivalently, if and only if $I_{m_T}^{(p)} : L^p(m_T) \to E$ is an extension of T. This observation, together with several other equivalences, is established in Theorem 5.7.

The spaces $L^p(m_T)$ also possess an important *optimality property* although, for $p > 1$, this is more involved than for $p = 1$. Namely, it is shown in Theorem 5.11 that, if T is p-th power factorable, then $L^p(m_T)$ is maximal amongst all σ-order continuous q-B.f.s.' $Y(\mu)$ which continuously contain $X(\mu)$ and such that T has an E-valued extension $T_{Y(\mu)} : Y(\mu) \to E$ which is itself p-th power factorable.

Whenever m_T is σ-decomposable, the containment $L^p(m_T) \subseteq L^1(m_T)$ is *proper*; see Chapter 3. Accordingly, the factorization of T through $L^p(m_T)$, when available, rather than through $L^1(m_T)$ (always possible), provides *extra* information about T. For instance, if $p > 1$, then T is necessarily weakly compact. Or, if T is p-th power factorable and m_T has σ-finite variation with $I_{m_T} : L^1(m_T) \to E$ weakly compact, then T is actually compact; see Corollary 5.18. In the setting of E a Banach lattice and $T : X(\mu) \to E$ a positive operator, it turns out that T is p-convex whenever it is p-th power factorable (cf. Proposition 5.24). Suppose that m_T has finite variation. Then $T : X(\mu) \to E$ has an extension to the *classical* B.f.s. $L^p(|m_T|)$ of a scalar measure, rather than $L^p(m_T)$, if and only if $X(\mu) \subseteq L^p(|m_T|)$ which, in turn, is equivalent to $d|m_T|/d\mu \in (X(\mu)_{[p]})'$; see Proposition 5.13. Such factorizations of T via a classical space $L^p(|m_T|)$ are reminiscent of the Maurey–Rosenthal factorization theory. Indeed, this link is actively pursued in Chapter 6, where the results of this chapter play a vital role. A second motivation is the application of the results of this chapter to the factorization theory of convolution (and related) operators acting in function spaces over a compact abelian group. This is one of the main themes of Chapter 7. So, let us turn to establishing the highlights of this chapter.

5.1 p-th power factorable operators

Throughout, let $(X(\mu), \|\cdot\|_{X(\mu)})$ be a *σ-order continuous* q-B.f.s. based on a positive, finite measure space (Ω, Σ, μ) and E be a Banach space, unless stated otherwise.

Let $1 \leq p < \infty$. The p-th power $X(\mu)_{[p]}$ of $X(\mu)$, equipped with the quasi-norm $\|\cdot\|_{X(\mu)_{[p]}}$, is a σ-order continuous q-B.f.s. over (Ω, Σ, μ), as established in Chapter 2; see (2.46), (2.47), Lemma 2.21(iii) and Proposition 2.22. Moreover, $X(\mu)$ is p-convex if and only if its p-th power $X(\mu)_{[p]}$ admits a lattice norm equivalent to $\|\cdot\|_{X(\mu)_{[p]}}$. If $X(\mu)$ is p-convex and its p-convexity constant is 1, then $\|\cdot\|_{X(\mu)_{[p]}}$ itself is a lattice norm. These facts are in Proposition 2.23(ii).

In view of the continuous inclusion $X(\mu) \subseteq X(\mu)_{[p]}$ (see Lemma 2.21(iv)), let

$$i_{[p]} : X(\mu) \to X(\mu)_{[p]} \tag{5.1}$$

denote the continuous inclusion map. Since $\operatorname{sim}\Sigma$ is dense in both $X(\mu)$ and $X(\mu)_{[p]}$, we have that $X(\mu) = i_{[p]}\big(X(\mu)\big)$ is dense in $X(\mu)_{[p]}$.

Definition 5.1. Let $X(\mu)$ be a σ-order continuous q-B.f.s. and E be a Banach space. Given $1 \leq p < \infty$, we say that a continuous linear operator $T : X(\mu) \to E$ is *p-th power factorable* if there exists a continuous linear operator $T_{[p]} : X(\mu)_{[p]} \to E$ such that

$$T \;=\; T_{[p]} \circ i_{[p]}. \tag{5.2}$$

In other words, $T_{[p]}$ is a continuous linear extension of T to the q-B.f.s. $X(\mu)_{[p]}$ and the following diagram commutes:

$$\begin{tikzcd}
X(\mu) \arrow[rr, "T"] \arrow[dr, "i_{[p]}"'] & & E\,. \\
& X(\mu)_{[p]} \arrow[ur, "T_{[p]}"'] &
\end{tikzcd}$$

In the above definition, the extension $T_{[p]}$ of a p-th power factorable operator $T : X(\mu) \to E$ to $X(\mu)_{[p]}$ is unique because $i_{[p]}$ is continuous and $i_{[p]}\big(X(\mu)\big)$ is dense in $X(\mu)_{[p]}$. It is clear that the collection of all E-valued, p-th power factorable operators on $X(\mu)$ equals the set

$$\mathcal{L}\big(X(\mu)_{[p]}, E\big) \circ i_{[p]} := \big\{S \circ i_{[p]} \,:\, S \in \mathcal{L}\big(X(\mu)_{[p]},\, E\big)\big\}. \tag{5.3}$$

In particular, for $p = 1$, the collection of all E-valued, 1-th power factorable operators on $X(\mu)$ is exactly $\mathcal{L}(X(\mu), E)$.

Given $1 \le p < \infty$, the collection of all p-th power factorable operators from a σ-order continuous q-B.f.s. $X(\mu)$ into a Banach space E is a linear subspace of $\mathcal{L}(X(\mu), E)$. In fact, if $T, S \in \mathcal{L}(X(\mu), E)$ are p-th power factorable and $a, b \in \mathbb{C}$, then $(aT + bS)$ is p-th power factorable and $\big(aT + bS\big)_{[p]} = aT_{[p]} + bS_{[p]}$.

Example 5.2. Let $\nu : \Sigma \to E$ be a Banach-space-valued measure and μ be any control measure for ν. Let $1 \le p < \infty$ and, in the notation of Proposition 3.31(iii), let $T := I_\nu \circ \alpha_p$ denote the restriction of the integration operator $I_\nu : L^1(\nu) \to E$ to $X(\mu) := L^p(\nu)$. Then the operator $T : X(\mu) \to E$ is p-th power factorable because $L^p(\nu)_{[p]} = L^1(\nu)$, via (3.52), and $T_{[p]} = I_\nu$. Of course, in this case $i_{[p]} = \alpha_p$. $\square$

We present two preliminary results.

Lemma 5.3. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. and E be a Banach space. Suppose that $1 \le p < \infty$. Then a continuous linear operator $T : X(\mu) \to E$ is p-th power factorable if and only if there is a constant $C > 0$ such that*

$$\big\|T(f)\big\|_E \le C\|f\|_{X(\mu)_{[p]}} = C\big\|\,|f|^{1/p}\big\|^p_{X(\mu)}, \qquad f \in X(\mu) \subseteq X(\mu)_{[p]}. \tag{5.4}$$

Proof. Let T be p-th power factorable. Let $f \in X(\mu)$, in which case $i_{[p]}(f) = f$. Since $T_{[p]} : X(\mu)_{[p]} \to E$ is continuous, it follows from (5.2) that

$$\begin{aligned}\big\|T(f)\big\|_E = \Big\|\big(T_{[p]} \circ i_{[p]}\big)(f)\Big\|_E &\le \big\|T_{[p]}\big\| \cdot \big\|i_{[p]}(f)\big\|_{X(\mu)_{[p]}} \\ &= \big\|T_{[p]}\big\| \cdot \|f\|_{X(\mu)_{[p]}} = \big\|T_{[p]}\big\| \cdot \big\|\,|f|^{1/p}\big\|^p_{X(\mu)}.\end{aligned}$$

In other words, (5.4) holds with $C := \big\|T_{[p]}\big\|$.

Conversely, assume that there is a constant $C > 0$ satisfying (5.4). This implies that T admits a unique E-valued continuous linear extension to the closure of $X(\mu)$ in $X(\mu)_{[p]}$. But, $X(\mu)$ is dense in $X(\mu)_{[p]}$, as observed just prior to Definition 5.1, which implies that T is p-th power factorable (by definition). $\square$

Lemma 5.4. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. and E be a Banach space. Suppose that $1 \leq p < \infty$ and $T : X(\mu) \to E$ is a p-th power factorable operator. Given any Banach space Z and $S \in \mathcal{L}(E, Z)$, the composition $S \circ T : X(\mu) \to Z$ is also a p-th power factorable operator and $(S \circ T)_{[p]} = S \circ T_{[p]}$.*

Proof. Since T admits the unique continuous linear extension $T_{[p]} : X(\mu)_{[p]} \to E$, the operator $S \circ T_{[p]} : X(\mu)_{[p]} \to Z$ is also a continuous linear extension of $S \circ T$ to $X(\mu)_{[p]}$. So, $S \circ T$ is p-th power factorable and $(S \circ T)_{[p]} = S \circ T_{[p]}$. □

There are p-th power factorable operators which are not μ-determined. The operator T in Example 4.27 provides an example. To be precise, with μ denoting Lebesgue measure and for $1 \leq r < \infty$, the operator $T_K : L^r([0,1]) \to L^r([0,1])$ given by (4.29) is not μ-determined. But, T is r-th power factorable. Indeed, in this case $X(\mu) := L^r([0,1])$ and so the extension $(T_K)_{[r]} : X(\mu)_{[r]} = L^1([0,1]) \to L^r([0,1])$ of $T : X(\mu) \to L^r([0,1])$ is also given by the right-hand side of (4.29), now with $f \in L^1([0,1])$.

Our primary concern is the class of μ-determined operators. In view of Proposition 4.28, this is no real restriction. So, given $1 \leq p < \infty$, let $\mathcal{F}_{[p]}\big(X(\mu), E\big)$ denote the set of all *μ-determined*, p-th power factorable operators from $X(\mu)$ into E. In particular, $\mathcal{F}_{[1]}\big(X(\mu), E\big)$ is exactly the set of all μ-determined operators from $X(\mu)$ into E. Since the zero operator is not μ-determined, we see that $\mathcal{F}_{[p]}(X(\mu), E)$ is not a vector space.

Example 5.5. Let $1 \leq p < \infty$. Suppose that $\nu : \Sigma \to E$ is a Banach-space-valued vector measure and that μ is a control measure for ν, that is, $\mathcal{N}_0(\mu) = \mathcal{N}_0(\nu)$. Let $I_\nu^{(p)} : L^p(\nu) \to E$ denote the restriction of the integration operator $I_\nu : L^1(\nu) \to E$ to $L^p(\nu)$; see Example 5.2. Of course, $I_\nu^{(1)} = I_\nu$.

(i) For $T := I_\nu^{(p)}$ and $X(\mu) := L^p(\nu)$, we have that $T \in \mathcal{F}_{[p]}(X(\mu), E)$. Indeed, recall the associated vector measure $m_T : \Sigma \to E$ of T is defined by $m_T(A) := T(\chi_A)$ for $A \in \Sigma$; see (4.4). Then $m_T = \nu$ and so, $\mathcal{N}_0(m_T) = \mathcal{N}_0(\nu) = \mathcal{N}_0(\mu)$, that is, T is μ-determined. Moreover, Example 5.2 shows that the operator $I_\nu^{(p)} = T \in \mathcal{F}_{[p]}(L^p(\nu), E)$.

(ii) Consider the natural injection $\alpha_p : L^p(\nu) \to L^1(\nu)$ as given by (3.58). For $T := \alpha_p$, $X(\mu) := L^p(\nu)$ and $E := L^1(\nu)$, we have that $T \in \mathcal{F}_{[p]}(X(\mu), E)$. In fact, observe first that $m_T(A) = \alpha_p(\chi_A) = \chi_A \in L^1(\nu)$ for every $A \in \Sigma$. So, given $A \in \Sigma$, we have

$$A \in \mathcal{N}_0(m_T) \quad \Longleftrightarrow \quad \chi_A = 0 \in L^1(\nu) \quad \Longleftrightarrow \quad A \in \mathcal{N}_0(\nu) = \mathcal{N}_0(\mu),$$

which implies that $\mathcal{N}_0(m_T) = \mathcal{N}_0(\mu)$, that is, T is μ-determined. Next, the identity operator id on $L^1(\nu)$ is a continuous linear extension of $T = \alpha_p$ to $L^1(\nu) = L^p(\nu)_{[p]} = X(\mu)_{[p]}$, by (3.52). So, $T = \alpha_p$ is p-th power factorable and $T_{[p]} = \mathrm{id}$.

(iii) Let $T := \alpha_p$, $X(\mu) := L^p(\nu)$ and $E := L^1(\nu)$ be as in part (ii), in which case T is μ-determined. Let $1 \leq q \leq p$ be fixed. Then $T \in \mathcal{F}_{[q]}(X(\mu), E)$. Indeed, it is routine to check that $X(\mu)_{[q]} = L^{p/q}(\nu)$. According to Lemma 2.21(iv) we have $X(\mu) \subseteq X(\mu)_{[q]}$ continuously, that is, $L^p(\nu) \subseteq L^{p/q}(\nu)$ continuously. Since $1 \le p/q$, the operator $\alpha_{p/q} : L^{p/q}(\nu) \to E$ is a continuous linear extension of $T = \alpha_p$ to $L^{p/q}(\nu) = X(\mu)_{[q]}$. So, $T \in \mathcal{F}_{[q]}(X(\mu), E)$ with $T_{[q]} = \alpha_{p/q}$. $\square$

If $E = \{0\}$ then, of course, $\mathcal{L}(X(\mu), E) = \{0\}$. It can also happen that $\mathcal{F}_{[p]}(X(\mu), E) = \{0\}$ with E a non-trivial Banach space.

Example 5.6. Assume that (Ω, Σ, μ) is a non-atomic, positive, finite measure space and that $1 \le r < \infty$. Let $X(\mu) := L^r(\mu)$ and E be a non-trivial Banach space.

(i) Given $1 \le p < \infty$, it follows from Example 2.10 (with (r/p) in place of p) that

$$\mathcal{L}\big(X(\mu)_{[p]}, E\big) = \{0\} \quad \Longleftrightarrow \quad p > r \tag{5.5}$$

because $X(\mu)_{[p]} = L^{r/p}(\mu)$.

(ii) Given $1 \le p < \infty$, the collection of all p-th power factorable operators from $X(\mu)$ into E equals the set $\mathcal{L}(X(\mu)_{[p]}, E) \circ i_{[p]}$ (see (5.3)). But, (5.5) yields that

$$\mathcal{L}(X(\mu)_{[p]}, E) \circ i_{[p]} = \{0\} \quad \Longleftrightarrow \quad p > r.$$

In other words, every E-valued p-th power factorable operator on $X(\mu)$ is necessarily the zero operator if and only if $p > r$.

(iii) We shall show that

$$\mathcal{F}_{[p]}\big(X(\mu), E\big) = \{0\} \quad \Longleftrightarrow \quad p > r. \tag{5.6}$$

To this end, assume first that $p \le r$. Choose a function $g \in L^{r/p}(\mu)'$ such that $g(\omega) \ne 0$ for every $\omega \in \Omega$. Fix $x_0 \in E \setminus \{0\}$. Define

$$S(f) := \langle f, g\rangle x_0 = \Big(\int_\Omega f g \, d\mu\Big) x_0, \qquad f \in X(\mu)_{[p]}.$$

Since $m_S(A) = \big(\int_A g\, d\mu\big) x_0$ with $x_0 \ne 0$ and $|g| > 0$, it is clear that S is μ-determined; see also Example 4.6. The operator $T := S \circ i_{[p]}$ is also μ-determined because $m_T = m_S$. Since S is the continuous linear extension of T to $X(\mu)_{[p]}$, we have $T \in \mathcal{F}_{[p]}(X(\mu), E) \setminus \{0\}$. This proves the implication ($\Rightarrow$) in (5.6) by a contrapositive argument.

Conversely, assume that $p > r$. Then

$$\mathcal{F}_{[p]}\big(X(\mu), E\big) \subseteq \mathcal{L}(X(\mu)_{[p]}, E) \circ i_{[p]} = \{0\}$$

via the definition of $\mathcal{F}_{[p]}\big(X(\mu), E\big)$ and part (ii). So, we have established (5.6). $\square$

5.2 Connections with $L^p(m_T)$

Now let $T : X(\mu) \to E$ be a Banach-space-valued, μ-determined operator on a σ-order continuous q-B.f.s. $X(\mu)$. Recall the associated vector measure $m_T : \Sigma \to E$ is defined by $m_T(A) := T(\chi_A)$ for $A \in \Sigma$. Since T is μ-determined, the space $L^1(m_T)$ is a σ-order continuous B.f.s. (based on (Ω, Σ, μ)) into which $X(\mu)$ is continuously embedded, via the natural inclusion $J_T : X(\mu) \to L^1(m_T)$, and the integration operator $I_{m_T} : L^1(m_T) \to E$ is a unique continuous linear extension of T satisfying $T = I_{m_T} \circ J_T$ (see Proposition 4.4). Given $1 \le p < \infty$, recall that $L^p(m_T)$ is also a σ-order continuous B.f.s. based on (Ω, Σ, μ) (see Proposition 3.28(i)) with

$$L^p(m_T) \;=\; L^1(m_T)_{[1/p]} \qquad \text{and} \qquad L^p(m_T)_{[p]} \;=\; L^1(m_T); \tag{5.7}$$

see (3.49) and (3.52), respectively. By Example 5.5(i) above, with $\nu := m_T$, the restriction

$$I^{(p)}_{m_T} : L^p(m_T) \to E \tag{5.8}$$

of the integration operator I_{m_T} to $L^p(m_T)$ is μ-determined and p-th power factorable, namely

$$I^{(p)}_{m_T} \;\in\; \mathcal{F}_{[p]}\big(L^p(m_T), E\big) \tag{5.9}$$

with $(I^{(p)}_{m_T})_{[p]} = I_{m_T}$. The following result shows that T is p-th power factorable if and only if $X(\mu) \subseteq L^p(m_T)$. This will enable us to factorize both T and J_T via the *p-convex space* B.f.s $L^p(m_T)$ (see Proposition 3.28); for some immediate consequences, see Remark 5.8 and Propositions 5.24 and 5.25 below. Such factorizations turn out to be useful in the investigation of T; see Chapter 6. Note that Theorem 5.7 also involves the B.f.s. $L^p_{\mathrm{w}}(m_T)$ which was studied in Chapter 3.

Theorem 5.7. *Given are $1 \le p < \infty$, a σ-order continuous q-B.f.s. $X(\mu)$ based on a positive, finite measure space (Ω, Σ, μ), and a Banach space E. The following assertions for a μ-determined operator $T : X(\mu) \to E$ are equivalent.*

(i) *T is p-th power factorable, that is, $T \in \mathcal{F}_{[p]}(X(\mu), E)$.*

(ii) *There is a constant $C > 0$ such that*

$$\big\|T(f)\big\|_E \;\le\; C\,\|f\|_{X(\mu)_{[p]}} \;=\; C\big\|\,|f|^{1/p}\big\|^p_{X(\mu)}, \qquad f \in X(\mu) \subseteq X(\mu)_{[p]}.$$

(iii) *$X(\mu) \subseteq L^p(m_T)$ with a continuous inclusion.*

(iv) *$X(\mu)_{[p]} \subseteq L^1(m_T)$ with a continuous inclusion.*

(v) *$X(\mu) \subseteq L^p_{\mathrm{w}}(m_T)$ with a continuous inclusion.*

(vi) *$X(\mu)_{[p]} \subseteq L^1_{\mathrm{w}}(m_T)$ with a continuous inclusion.*

(vii) *For every $x^* \in E^*$, the Radon–Nikodým derivative $d\langle m_T, x^*\rangle / d\mu$ belongs to the Köthe dual $\big(X(\mu)_{[p]}\big)'$ of the q-B.f.s. $X(\mu)_{[p]}$.*

(viii) *The natural injection $J_T : X(\mu) \to L^1(m_T)$ is p-th power factorable, that is, $J_T \in \mathcal{F}_{[p]}\big(X(\mu), L^1(m_T)\big)$.*

Proof. (i) $\Leftrightarrow$ (ii). This equivalence has already been established in Lemma 5.3.

(ii) $\Leftrightarrow$ (iv). This equivalence is a consequence of Corollary 4.16, with $X(\mu)_{[p]}$ in place of $Y(\mu)$, because we already know that $X(\mu) \subseteq X(\mu)_{[p]}$ via Lemma 2.21(iv).

(iii) $\Leftrightarrow$ (iv). From Lemma 2.20(ii) and the identities (5.7), it follows that

$$X(\mu) \subseteq L^p(m_T) \quad \Longleftrightarrow \quad X(\mu)_{[p]} \subseteq L^1(m_T). \tag{5.10}$$

Moreover, each respective inclusion map in (5.10) is continuous via Lemma 2.7.

(iii) $\Leftrightarrow$ (v). It suffices to prove that $X(\mu) \subseteq L^p(m_T)$ if and only if $X(\mu) \subseteq L^p_{\mathrm{w}}(m_T)$, because continuity of the respective inclusion is again guaranteed by Lemma 2.7. Moreover, since $L^p(m_T) \subseteq L^p_{\mathrm{w}}(m_T)$ by definition, we only need to establish that $X(\mu) \subseteq L^p(m_T)$ whenever $X(\mu) \subseteq L^p_{\mathrm{w}}(m_T)$. To prove this, first observe the fact, for the order continuous parts of general q-B.f.s' $Y_1(\mu)$ and $Y_2(\mu)$ over (Ω, Σ, μ), that

$$Y_1(\mu) \subseteq Y_2(\mu) \quad \Longrightarrow \quad Y_1(\mu)_{\mathrm{a}} \subseteq Y_2(\mu)_{\mathrm{a}}. \tag{5.11}$$

This is a consequence of the continuity of the natural inclusion from $Y_1(\mu)$ into $Y_2(\mu)$ (see Lemma 2.7), the definition of the order continuous part of a q-B.f.s. (see Chapter 4), and the fact that $Y_1(\mu) \subseteq Y_2(\mu)$ implies that $Y_1(\mu)$ is solid in $Y_2(\mu)$. Now, the σ-order continuity of $X(\mu)$ yields $X(\mu) = X(\mu)_{\mathrm{a}}$. Moreover, $L^p(m_T) = L^p_{\mathrm{w}}(m_T)_{\mathrm{a}}$ by (3.86) with $\nu := m_T$. By applying (5.11), with $Y_1(\mu) := X(\mu)$ and $Y_2(\mu) := L^p_{\mathrm{w}}(m_T)$, we have the desired implication

$$X(\mu) \subseteq L^p_{\mathrm{w}}(m_T) \quad \Longrightarrow \quad X(\mu) = X(\mu)_{\mathrm{a}} \subseteq L^p_{\mathrm{w}}(m_T)_{\mathrm{a}} = L^p(m_T).$$

(v) $\Leftrightarrow$ (vi). This can be proved as for the equivalence (iii) $\Leftrightarrow$ (iv) because

$$L^p_{\mathrm{w}}(m_T) = L^1_{\mathrm{w}}(m_T)_{[1/p]} \qquad \text{and} \qquad L^p_{\mathrm{w}}(m_T)_{[p]} = L^1_{\mathrm{w}}(m_T);$$

see (3.81).

(vi) $\Rightarrow$ (vii). Let $x^* \in E^*$. Since T is μ-determined, we have $\big|\langle m_T, x^*\rangle\big| \ll \mu$. Moreover,

$$\int_\Omega |f| \cdot \frac{d|\langle m_T, x^*\rangle|}{d\mu}\, d\mu = \int_\Omega |f|\, d|\langle m_T, x^*\rangle| < \infty, \qquad f \in L^1_{\mathrm{w}}(m_T).$$

Since $X(\mu)_{[p]} \subseteq L^1_{\mathrm{w}}(m_T)$, we can conclude that $\frac{d|\langle m_T, x^*\rangle|}{d\mu} \in \big(X(\mu)_{[p]}\big)'$.

(vii) $\Rightarrow$ (vi). Let $f \in X(\mu)_{[p]}$. Then (vii) implies that

$$\int_\Omega |f|\, d|\langle m_T, x^*\rangle| = \int_\Omega |f| \cdot \frac{d|\langle m_T, x^*\rangle|}{d\mu}\, d\mu < \infty, \qquad x^* \in E^*.$$

Therefore, $f \in L^1_{\mathrm{w}}(m_T)$ by definition.

(iii) $\Rightarrow$ (viii). The given assumption implies that $X(\mu)_{[p]} \subseteq L^1(m_T)$ continuously; see Lemmas 2.7 and 2.20(ii). So, let

$$\beta_{[p]} : X(\mu)_{[p]} \to L^1(m_T) \tag{5.12}$$

denote the natural embedding. Then $\beta_{[p]}$ is a continuous linear extension of $J_T : X(\mu) \to L^1(m_T)$ to the larger domain space $X(\mu)_{[p]}$; see Lemma 2.21(iv). By Definition 5.1, with J_T in place of T and $L^1(m_T)$ in place of E, we can conclude that J_T is p-th power factorable and $\beta_{[p]} = \big(J_T\big)_{[p]}$. Since T is μ-determined, Proposition 4.4(iii) shows that J_T is injective. Then J_T is μ-determined by either part (ii) or part (iii) of Lemma 4.5 (with J_T in place of T). So, $J_T \in \mathcal{F}_{[p]}\big(X(\mu), L^1(m_T)\big)$.

(viii) $\Rightarrow$ (i). Since $T = I_{m_T} \circ J_T$, with $J_T \in \mathcal{F}_{[p]}\big(X(\mu), L^1(m_T)\big)$ and $I_{m_T} \in \mathcal{L}(L^1(m_T), E)$, the operator T is p-th power factorable via Lemma 5.4. $\square$

Remark 5.8. Let the assumptions be as in Theorem 5.7.

(I) As already noted in the proof of Theorem 5.7, the following conditions are also equivalent to any one of (i)–(viii).

(iii)′ $X(\mu) \subseteq L^p(m_T)$ as vector sublattices of $L^0(\mu)$.

(iv)′ $X(\mu)_{[p]} \subseteq L^1(m_T)$ as vector sublattices of $L^0(\mu)$.

(v)′ $X(\mu) \subseteq L^p_{\mathrm{w}}(m_T)$ as vector sublattices of $L^0(\mu)$.

(vi)′ $X(\mu)_{[p]} \subseteq L^1_{\mathrm{w}}(m_T)$ as vector sublattices of $L^0(\mu)$.

(II) Assume that $T \in \mathcal{F}_{[p]}(X(\mu), E)$ for some $1 \le p < \infty$. We shall present some factorizations of T and of the natural injection $J_T : X(\mu) \to L^1(m_T)$. To this end, let

$$J_T^{(p)} : X(\mu) \to L^p(m_T) \qquad \text{and} \qquad \alpha_p : L^p(m_T) \to L^1(m_T) \tag{5.13}$$

denote the canonical inclusion maps. Here, the inclusion $X(\mu) \subseteq L^p(m_T)$ is guaranteed by Theorem 5.7(iii). We also need the inclusion $i_{[p]} : X(\mu) \to X(\mu)_{[p]}$ (as given in (5.1)), the extension $T_{[p]} : X(\mu)_{[p]} \to E$ of T (see (5.2)), the restricted integration operator $I_{m_T}^{(p)} : L^p(m_T) \to E$ (see (5.8) and (5.9)) and the inclusion map $\beta_{[p]} : X(\mu)_{[p]} \to L^1(m_T)$ of (5.12), guaranteed by Theorem 5.7(iv).

(i) *Every* μ-determined operator T admits the factorization $T = I_{m_T} \circ J_T$ via Proposition 4.4(iii). If, *in addition*, $T \in \mathcal{F}_{[p]}(X(\mu), E)$, then we have the further factorizations

$$T \;=\; T_{[p]} \circ i_{[p]} \qquad \text{and} \qquad T \;=\; I_{m_T}^{(p)} \circ J_T^{(p)}.$$

The first factorization is direct from Definition 5.1 (see (5.2)) and the second is from (5.13) via Theorem 5.7(iii).

So, the following diagram commutes:

$$\begin{tikzcd}
X(\mu) \arrow[r, "J_T^{(p)}"] \arrow[d, "i_{[p]}"'] \arrow[dr, "T"] & L^p(m_T) \arrow[d, "I_{m_T}^{(p)}"] \\
X(\mu)_{[p]} \arrow[r, "T_{[p]}"] & E.
\end{tikzcd}$$

(ii) Still assuming that $T \in \mathcal{F}_{[p]}(X(\mu), E)$, we have that the natural injection $J_T : X(\mu) \to L^1(m_T)$ admits the factorizations

$$J_T \;=\; \beta_{[p]} \circ i_{[p]} \qquad \text{and} \qquad J_T \;=\; \alpha_p \circ J_T^{(p)},$$

because $X(\mu) \subseteq L^p(m_T)$ by Theorem 5.7(iii) and $X(\mu)_{[p]} \subseteq L^1(m_T)$ by Theorem 5.7(iv).

$$\begin{tikzcd}
X(\mu) \arrow[r, "J_T^{(p)}"] \arrow[d, "i_{[p]}"'] \arrow[dr, "J_T"] & L^p(m_T) \arrow[d, "\alpha_p"] \\
X(\mu)_{[p]} \arrow[r, "\beta_{[p]}"] & L^p(m_T).
\end{tikzcd}$$

(iii) Given $T \in \mathcal{F}_{[p]}(X(\mu), E)$, the canonical map $J_T^{(p)} : X(\mu) \to L^p(m_T)$ is p-convex. Indeed, the p-convexity of $L^p(m_T)$ (see Proposition 3.28(i)) implies that every continuous linear map from $X(\mu)$ into $L^p(m_T)$ is necessarily p-convex; see Corollary 2.64. $\square$

We can determine exactly when the Volterra operators are p-th power factorable.

Example 5.9. Let μ be Lebesgue measure on $\Omega := [0, 1]$ and let $\Sigma := \mathcal{B}([0, 1])$. Given $1 \leq r < \infty$, write $L^r([0, 1]) = L^r(\mu)$ as usual. We consider the Volterra operator $V_r : L^r([0, 1]) \to L^r([0, 1])$ defined by

$$V_r(f)(t) := \int_0^t f(u)\, du, \qquad t \in [0, 1],\ f \in L^r([0, 1]).$$

With $X(\mu) := L^r([0, 1])$ and $E := L^r([0, 1])$, we shall determine when V_r is p-th power factorable. We already know that V_r is μ-determined; see Example 4.9. Observe that the associated vector measure $m_{V_r} : A \mapsto V_r(\chi_A)$ on Σ is equal to the Volterra measure ν_r of order r, that is,

$$m_{V_r}(A) \;=\; V_r(\chi_A) \;=\; \nu_r(A), \qquad A \in \Sigma; \tag{5.14}$$

see Example 3.10. Recall from Example 3.26 that the variation measure $|\nu_r|$ of ν_r is finite and is given by $d|\nu_r|(t) = (1-t)^{1/r}dt$. So,

$$L^r([0,1]) \subseteq L^1([0,1]) \subseteq L^1\big((1-t)^{1/r}dt\big) \subseteq L^1(\nu_r). \tag{5.15}$$

Observe that $L^q([0,1]) \subseteq L^1([0,1]) \subseteq L^1\big((1-t)dt\big)$ for $1 \le q \le \infty$ and that

$$L^q([0,1]) \nsubseteq L^1\big((1-t)\,dt\big) \qquad \text{whenever} \quad 0 < q < 1, \tag{5.16}$$

which follows from the fact that the function $t \mapsto t^{-1} \cdot \chi_{(0,1]}(t)$ belongs to $L^q([0,1])$ but not to $L^1([0,1])$.

(i) Let $r := 1$. With $X(\mu) := L^1([0,1])$ we have, for $1 < p < \infty$, that $L^1(|\nu_1|) = L^1(\nu_1)$ (see Example 3.26) and so

$$X(\mu)_{[p]} \;=\; L^{1/p}([0,1]) \nsubseteq L^1\big((1-t)\,dt\big) \;=\; L^1(\nu_1) = L^1(m_{V_1})$$

by (5.16) with $q := 1/p$. Accordingly, Theorem 5.7 with $T := V_1$ implies that V_1 is *not* p-th power factorable.

(ii) Fix $1 < r < \infty$. Given $1 < p < \infty$, we claim that V_r is p-th power factorable if and only if $p \le r$. In fact, assume first that V_r is p-th power factorable. Since $V_r \ne 0$, it follows from Example 5.6(ii) that $p \le r$.

Conversely, assume that $p \le r$. Then again with $X(\mu) := L^r([0,1])$ we have (via (5.15)) that

$$X(\mu)_{[p]} = L^{r/p}([0,1]) \subseteq L^1([0,1]) \subseteq L^1(\nu_r) = L^1(m_{V_r}).$$

Accordingly, V_r is p-th power factorable by Theorem 5.7 (with $T := V_r$ and the space $E := L^r([0,1])$ there).

(iii) Again let $1 < r < \infty$. Given $1 < p \le r$, the operator V_r is p-th power factorable via part (ii). It then follows from Theorem 5.7 (with $T := V_r$) that $X(\mu) = L^r([0,1]) \subseteq L^p(m_{V_r}) = L^p(\nu_r)$ and, of course, $L^p(\nu_r) \subseteq L^1(\nu_r)$ holds (even for an arbitrary vector measure in place of ν_r). The B.f.s. $L^p(\nu_r)$ is not the only p-convex B.f.s. over (Ω, Σ, μ) having this property. In other words, there exist other *p-convex space* B.f.s.' $Y(\mu)$ such that

$$L^r([0,1]) \subseteq Y(\mu) \subseteq L^1(m_{V_r}). \tag{5.17}$$

To see this, fix any positive number $\gamma < (1-p^{-1})$ and let $Y_\gamma(\mu) := L^p(t^{p\gamma}dt)$, which is a p-convex B.f.s. (see Example 2.73(i)). For $f \in Y_\gamma(\mu)$, Hölder's inequality yields

$$\int_0^1 |f(t)|\,dt \le \Big(\int_0^1 |f(t)|^p\,t^{p\gamma}\,dt\Big)^{1/p}\Big(\int_0^1 t^{-p'\gamma}\,dt\Big)^{1/p'} < \infty$$

and hence, by (5.15) and $p \le r$, we have

$$L^r([0,1]) \subseteq L^p([0,1]) \subseteq Y_\gamma(\mu) \subseteq L^1([0,1]) \subseteq L^1(\nu_r).$$

So, (5.17) does indeed hold. On the other hand,

$$Y_\gamma(\mu)_{[p]} \;=\; L^1(t^{p\gamma}dt) \;\nsubseteq\; L^1\big((1-t)^{1/r}dt\big) \;\subseteq\; L^1(\nu_r),$$

because $p\gamma > 0$. So, we have

$$Y_\gamma(\mu) = \big(Y_\gamma(\mu)_{[p]}\big)_{[1/p]} \nsubseteq L^1(\nu_r)_{[1/p]} = L^p(\nu_r) = L^p(m_{V_r})$$

via Lemma 2.20 and (5.7) with $m_{V_r} = \nu_r$ in place of m_T there. Thus, $Y_\gamma(\mu) = L^p(t^{p\gamma}\,dt)$ is a p-convex B.f.s. satisfying (5.17) but, it is *not* contained in $L^p(\nu_r)$. Moreover, it is also the case that

$$L^p(m_{V_r}) = L^p(\nu_r) \nsubseteq Y_\gamma(\mu),$$

because $L^p\big((1-t)^{1/r}dt\big) = L^p(|\nu_r|) \subseteq L^p(\nu_r)$ but, $L^p((1-t)^{1/r}dt) \nsubseteq L^p(t^{p\gamma}dt) = Y(\mu)_\gamma$. For the last claim, just consider the function $t \mapsto (1-t)^{-1/rp}\cdot\chi_{[0,1)}(t)$ on $[0,1]$.

Now let us show that $Y_\gamma(\mu) \neq L^1(\nu_r)$. To this end, it suffices to prove that $L^1(\nu_r)$ is not p-convex because $Y_\gamma(\mu) = L^p(t^{p\gamma}\,dt)$ is p-convex. Assume that $L^1(\nu_r)$ is p-convex. Then its closed subspace

$$\chi_{[0,\,1/2]}\cdot L^1(\nu_r) = \Big\{f \in L^1(\nu_r) : f \text{ vanishes on } [1/2,1]\Big\},$$

which is also a Banach lattice when equipped with the induced lattice norm, must be p-convex. However, (3.45) together with the fact that $L^1(\nu_1) = L^1\big((1-t)\,dt\big)$ (see Example 3.26(i)) yield that

$$\chi_{[0,\,1/2]}\cdot L^1(\nu_r) = L^1\big([0,\,1/2]\big)$$

which is not p-convex via Example 2.73(ii-a). Therefore, $L^1(\nu_r)$ is not p-convex either. Thus $Y_\gamma(\mu) \neq L^1(\nu_r)$.

(iv) Let the notation be as in part (iii). Recall from part (iii) that the space $Y_\gamma(\mu) = L^p(t^{p\gamma}dt)$ is a σ-o.c. p-convex B.f.s. satisfying

$$Y_\gamma(\mu) \nsubseteq L^p(\nu_r) \qquad\text{and}\qquad L^p(\nu_r) \nsubseteq Y_\gamma(\mu) \subsetneqq L^1(\nu_r).$$

Here, we shall exhibit a σ-o.c. p-convex B.f.s. $Z(\mu)$, by using $Y_\gamma(\mu)$, such that

$$L^p(\nu_r) \;\subsetneqq\; Z(\mu) \;\subsetneqq\; L^1(\nu_r). \tag{5.18}$$

Some of the details will be left to the reader. Define

$$Z(\mu) :=\; L^p(\nu_r) + Y_\gamma(\mu),$$

that is, $Z(\mu)$ is the linear span of $L^p(\nu_r)$ and $Y_\gamma(\mu)$, formed in $L^1(\nu_r)$. Given $f \in Z(\mu)$, define

$$\|f\|_{Z(\mu)} := \inf\Big\{\big(\|g\|^p_{L^p(\nu_r)} + \|h\|^p_{Y_\gamma(\mu)}\big)^{1/p} : f = g+h,\ g \in L^p(\nu_r),\ h \in Y_\gamma(\mu)\Big\}.$$

Then, $\|\cdot\|_{Z(\mu)}$ is a lattice norm for which $Z(\mu)$ is a Banach lattice. Of course, the order in $Z(\mu)^+$ is the pointwise μ-a.e. one. If $p=1$ and if $L^p(\nu_r)$ and $Y_\gamma(\mu)$ were over $\mathbb{R}$, then such a lattice norm would be equal to the one given in [99, Definition 2.g.2]. By using the fact that both $L^p(\nu_r)$ and $Y_\gamma(\mu)$ are B.f.s.' over (Ω,Σ,μ), it can be shown that $(Z(\mu),\|\cdot\|_{Z(\mu)})$ is also a B.f.s. over (Ω,Σ,μ). Since both $L^p(\nu_r)$ and $Y_\gamma(\mu)$ are p-convex, the definition of the lattice norm $\|\cdot\|_{Z(\mu)}$ yields that $Z(\mu)$ is also p-convex.

To verify that $Z(\mu)$ is σ-o.c., let $Z(\mu)^+\ni f_n\downarrow 0$. We can find $g_1\in L^p(\nu_r)^+$ and $h_1\in Y_\gamma(\mu)^+$ such that $f_1=g_1+h_1$. Indeed, noting that f_1 is $\mathbb{R}^+$-valued, first take $\mathbb{R}$-valued functions $g_0\in L^p(\nu_r)$ and $h_0\in Y_\gamma(\mu)$ such that $f_1=g_0+h_0$. Let $A:=g_0^{-1}([0,\infty))$ and $B:=h_0^{-1}([0,\infty))$. Since $\Omega\setminus(A\cup B)$ is μ-null, we can write

$$f_1=\Big(g_0\chi_{A\cap B}+(g_0+h_0)\chi_{A\setminus B}\Big)+\Big(h_0\chi_{A\cap B}+(g_0+h_0)\chi_{B\setminus A}\Big).$$

Let

$$g_1:=g_0\chi_{A\cap B}+(g_0+h_0)\chi_{A\setminus B}\quad\text{and}\quad h_1:=h_0\chi_{A\cap B}+(g_0+h_0)\chi_{B\setminus A}.$$

Since $h_0\chi_{A\setminus B}\le 0$, we have

$$g_0\chi_{A\setminus B}\ge(g_0+h_0)\chi_{A\setminus B}=f_1\chi_{A\setminus B}\ge 0,$$

from which it follows that $(g_0+h_0)\chi_{A\setminus B}\in L^p(\nu_r)^+$ and hence, $g_1\in L^p(\nu_r)^+$. Similarly, $h_1\in Y_\gamma(\mu)^+$. So, we can write $f_1=g_1+h_1$ with the desired properties. Now, with the understanding that $0/0=0$, it follows that

$$L^p(\nu_r)^+\ni\big((f_n/f_1)\cdot g_1\big)\downarrow 0\qquad\text{and}\qquad Y_\gamma(\mu)^+\ni\big((f_n/f_1)\cdot h_1\big)\downarrow 0.$$

Moreover, $f_n=(f_n/f_1)\cdot g_1+(f_n/f_1)\cdot h_1$ with $(f_n/f_1)\cdot g_1\in L^p(\nu_r)^+$ (since $0\le f_n/f_1\le 1$ and $g_1\in L^p(\nu_r)^+$) and $(f_n/f_1)\cdot h_1\in Y_\gamma(\mu)^+$ (since $0\le f_n/f_1\le 1$ and $h_1\in Y_\gamma(\mu)^+$). Therefore the σ-order continuity of $L^p(\nu_r)$ and $Y_\gamma(\mu)$ yields that

$$\|f_n\|_{Z(\mu)}\le\Big(\big\|(f_n/f_1)\cdot g_1\big\|^p_{L^p(\nu_r)}\;+\;\big\|(f_n/f_1)\cdot h_1\big\|^p_{Y_\gamma(\mu)}\Big)^{1/p}\to 0$$

as $n\to\infty$. Thus, $Z(\mu)$ is σ-o.c.

By the definition of $Z(\mu)$ we have that $L^p(\nu_r)\subseteq Z(\mu)\subseteq L^1(\nu_r)$. Moreover, $L^p(\nu_r)\ne Z(\mu)$ because $Y_\gamma(\mu)\nsubseteq L^p(\nu_r)$. Finally $Z(\mu)\ne L^1(\nu_r)$, which is a consequence of the fact that $Z(\mu)$ is p-convex whereas $L^1(\nu_r)$ is not (see part (iii) above). Thus, we have established (5.18).

(v) When $p=r$, we can find a Rybakov functional $x_0^*\in E^*=L^{p'}([0,1])$ such that

$$L^p(m_{V_p})\;=\;L^p(\nu_p)\;\subseteq\;L^p\big(|\langle\nu_p,x_0^*\rangle|\big)\;\subseteq\;L^1(\nu_p). \tag{5.19}$$

This is a consequence of Theorem 6.41 in Chapter 6 and is a case in which Question (B) below has an affirmative answer (see Remark 6.42(ii)).

Let us verify that both inclusions in (5.19) are strict. First note that $L^p\big(|\langle\nu_p,\ x_0^*\rangle|\big)$ is both p-convex and p-concave; see Example 2.73(i). Hence, $L^p\big(|\langle\nu_p, x_0^*\rangle|\big)$ cannot be equal to $L^p(\nu_p)$ which is not p-concave by the fact that $L^p(\nu_p) \neq L^p(|\nu_p|)$ (see Proposition 3.74). Moreover, $L^1(\nu_p)$ is not p-convex (see part (iii)) and hence, is not equal to $L^p\big(|\langle\nu_p, x_0^*\rangle|\big)$. □

Given $1 < p \le r < \infty$, Example 5.9(iv) provides a p-convex B.f.s. $Z(\mu)$ strictly larger than $L^p(m_{V_r}) = L^p(\nu_r)$. So, does there exist a largest p-convex B.f.s. within $L^1(\nu_r)$? Since $L^1(\nu_r)$ is not p-convex, the answer is no by part (ii) of the following fact.

Proposition 5.10. *Let $\nu : \Sigma \to E$ be a Banach-space-valued vector measure defined on a measurable space (Ω, Σ) and let $\mu : \Sigma \to [0,\infty)$ be a control measure for ν. Assume that $1 < p < \infty$.*

(i) *The space $L^1(\nu)$ is a union of p-convex B.f.s.' Namely,*

$$L^1(\nu) = \bigcup \big\{ g \cdot L^p(\nu) : g \in L^{p'}(\nu),\ g \ge 1 \big\}. \tag{5.20}$$

In particular, $L^1(\nu)$ is the union of all p-convex B.f.s.' (over (Ω,Σ,μ)) contained in $L^1(\nu)$.

(ii) *Either $L^1(\nu)$ is p-convex or there exists no largest p-convex B.f.s. (over (Ω,Σ,μ)) inside $L^1(\nu)$.*

Proof. (i) Let $g \in L^{p'}(\nu)$ satisfy $g \ge 1$ on Ω. By Proposition 3.43(i) we have that $g \in \mathcal{M}\big(L^p(\nu), L^1(\nu)\big)$ and hence,

$$g \cdot L^p(\nu) := \ \big\{ gf : f \in L^p(\nu) \big\}$$

is an ideal of $L^1(\nu)$. Equipped with the lattice norm

$$f \mapsto \|f/g\|_{L^p(\nu)}, \qquad f \in\ g \cdot L^p(\nu),$$

the ideal $g \cdot L^p(\nu)$ is lattice isometric to the p-convex B.f.s. $L^p(\nu)$; see Proposition 3.28(i) for the latter claim. Consequently, $g \cdot L^p(\nu)$ is also p-convex. Moreover, $\operatorname{sim}\Sigma \subseteq g \cdot L^p(\nu)$ because

$$s \ =\ g \cdot \frac{s}{g}\ \in\ g \cdot L^\infty(\nu)\ \subseteq\ g \cdot L^p(\nu), \qquad s \in \operatorname{sim}\Sigma.$$

Therefore, $g \cdot L^p(\nu)$ is a B.f.s. over (Ω,Σ,μ).

To establish (5.20), take an arbitrary function $f \in L^1(\nu)$. Define

$$\operatorname{sgn} f := f/|f| \qquad \text{pointwise on} \qquad \Omega,$$

with the understanding that $0/0 = 0$. Since $|f|^{1/p} \in L^p(\nu)$ and

$$f = (\operatorname{sgn} f)\,|f| = (\operatorname{sgn} f)\,|f|^{1/p}\,|f|^{1/p'} = \Big(|f|^{1/p'} + 1\Big) \cdot \frac{(\operatorname{sgn} f)|f|^{1/p'}}{|f|^{1/p'} + 1} \cdot |f|^{1/p},$$

it follows, with $1 \leq g := \big(|f|^{1/p'} + 1\big) \in L^{p'}(\nu)$, that $f \in g \cdot L^p(\nu)$. Thus, $L^1(\nu)$ is contained in the right-hand side of (5.20). The reverse containment is a consequence of the fact that

$$L^{p'}(\nu) \cdot L^p(\nu) = L^1(\nu)_{[1/p']} \cdot L^1(\nu)_{[1/p]} \subseteq L^1(\nu)$$

via Lemma 2.21(i). So, (5.20) holds. This establishes part (i).

(ii) If there is a largest p-convex B.f.s. $Z(\mu)$ inside $L^1(\nu)$, then $Z(\mu)$ necessarily contains the right-hand side of (5.20) and consequently, $Z(\mu) = L^1(\nu)$. Thus, (ii) holds. $\square$

5.3 Optimality

Let $1 \leq p < \infty$ and $T \in \mathcal{F}_{[p]}(X(\mu), E)$ for a σ-order continuous q-B.f.s. $X(\mu)$ and a Banach space E. Theorem 5.7 guarantees that

$$X(\mu) \subseteq L^p(m_T) \subseteq L^1(m_T), \tag{5.21}$$

and hence, using the notation of Remark 5.8(II)(ii), we can factorize J_T as

$$J_T = \alpha_p \circ J_T^{(p)}. \tag{5.22}$$

This observation suggests the following questions.

(A) Proposition 5.10(ii), with $\nu := m_T$, says that we do not have the largest p-convex B.f.s. within $L^1(m_T)$ unless $L^1(m_T)$ itself is p-convex. On the other hand, there are many p-convex B.f.s.' within $L^1(m_T)$, other than $L^p(\nu)$. Is there some additional property which ensures that $L^p(m_T)$ is the largest of all such p-convex B.f.s.'?

(B) For every Rybakov functional $x^* \in \mathbf{R}_{m_T}[E^*]$, we clearly have $L^p(m_T) \subseteq L^p\big(|\langle m_T,\, x^*\rangle|\big)$. Can we factorize the inclusion map $\alpha_p : L^p(m_T) \to L^1(m_T)$ through $L^p\big(|\langle m_T,\, x_0^*\rangle|\big)$ for some suitable $x_0^* \in \mathbf{R}_{m_T}[E^*]$ via inclusions? That is, does there always exist $x_0^* \in \mathbf{R}_{m_T}[E^*]$ such that

$$L^p(m_T) \subseteq L^p\big(|\langle m_T,\, x_0^*\rangle|\big) \subseteq L^1(m_T)? \tag{5.23}$$

Of course, only the second inclusion is the relevant one.

(C) Assume that m_T has finite variation. Then $L^p(|m_T|) \subseteq L^p(m_T)$ is a non-trivial B.f.s. over (Ω, Σ, μ). Can we factorize $J_T^{(p)}$ through $L^p(|m_T|)$ via inclusions? In other words, do we have

$$X(\mu) \subseteq L^p(|m_T|) \subseteq L^p(m_T)? \tag{5.24}$$

Here only the first inclusion is relevant.

To answer (A), we shall present a property which ensures that $L^p(m_T)$ is *optimal* in a certain sense. An operator $T \in \mathcal{F}_{[p]}\big(X(\mu), E\big)$ is said to admit an $\mathcal{F}_{[p]}$-*extension* to a B.f.s. $Y(\mu)$ $\big($ based on (Ω, Σ, μ) $\big)$ satisfying $X(\mu) \subseteq Y(\mu)$ if T admits a continuous linear extension $\widetilde{T} \in \mathcal{F}_{[p]}(Y(\mu), E)$. Such an extension is unique because $X(\mu)$, which contains $\operatorname{sim} \Sigma$, is necessarily dense in $Y(\mu)$. Since $L^p(m_T)$ is p-convex, the following result provides an answer to Question (A).

Theorem 5.11. *Let $1 \le p < \infty$ and $T \in \mathcal{F}_{[p]}\big(X(\mu), E\big)$. Then the B.f.s. $L^p(m_T)$ is the largest one within the class of all σ-order continuous q-B.f.s.' $\big($based on $(\Omega, \Sigma, \mu)\big)$ to which T admits an $\mathcal{F}_{[p]}$-extension.*

Proof. Suppose that $Y(\mu)$ is a σ-order continuous q-B.f.s. to which T admits an $\mathcal{F}_{[p]}$-extension. Denote by $\widetilde{T} \in \mathcal{F}_{[p]}\big(Y(\mu), E\big)$ such an extension. Then $m_{\widetilde{T}} = m_T$. In particular, $\widetilde{T}$ is also μ-determined. Since $\widetilde{T} : Y(\mu) \to E$ is a continuous linear extension of T to the σ-order continuous q-B.f.s. $Y(\mu)$, it follows from Theorem 4.14, with $\widetilde{T}$ in place of T and $Y(\mu)$ in place of $X(\mu)$, that $Y(\mu) \subseteq L^1(m_{\widetilde{T}}) = L^1(m_T)$. Now apply Theorem 5.7, with $\widetilde{T}$ in place of T and $Y(\mu)$ in place of $X(\mu)$, to deduce that $Y(\mu) \subseteq L^p(m_{\widetilde{T}}) = L^p(m_T)$. This proves the theorem. ☐

Now let us attend to Question (B). An affirmative answer would enable us to factorize both $T \in \mathcal{F}_{[p]}(X(\mu), E)$ and $J_T : X(\mu) \to E$ via $L^p\big(|\langle m_T, x_0^* \rangle|\big)$. This would have a further advantage because $L^p\big(|\langle m_T, x_0^* \rangle|\big)$, being the L^p-space of the *scalar* measure $|\langle m_T, x_0^* \rangle|$, has "nicer" properties than $L^p(m_T)$. For example, the B.f.s. $L^p\big(|\langle m_T, x_0^* \rangle|\big)$ is always p-convex and p-concave; see Example 2.73(i). Unfortunately, part (ii) of Example 5.12 below serves as a counterexample, that is, it is not always possible to find a Rybakov functional $x_0^* \in \mathbf{R}_{m_T}[E^*]$ satisfying (5.23). A sufficient condition guaranteeing an affirmative answer to Question (B) will be provided in Chapter 6; see Theorem 6.41 and Remark 6.42(ii). We now present the stated counterexample where we first exhibit, in part (i), a Banach lattice E and an E-valued vector measure ν such that $L^p\big(|\langle \nu, x^* \rangle|\big) \not\subseteq L^1(\nu)$ for all $x^* \in \mathbf{R}_\nu[E^*]$. We shall then apply this in part (ii) to an appropriate σ-order continuous q-B.f.s. $X(\mu)$ and operator $T \in \mathcal{F}_{[p]}(X(\mu), E)$.

Example 5.12. (i) Let $\lambda : \mathcal{B}(\mathbb{R}^2) \to [0, \infty]$ denote Lebesgue measure. With

$$B_n := \big[-n, n\big]^2 \setminus \big[-n+1,\ n-1\big]^2 \subseteq \mathbb{R}^2$$

and $a_n := n^{-3}$ for $n \in \mathbb{N}$, define a finite measure $\mu : \Sigma \to [0, \infty)$ by

$$\mu(A) := \sum_{n=1}^{\infty} a_n \frac{\lambda(B_n \cap A)}{\lambda(B_n)}, \qquad A \in \mathcal{B}(\mathbb{R}^2).$$

Note that $B_n = [-n, n]^2 \setminus B_{n-1}$ for every $n \geq 2$ and so $\lambda(B_n \cap B_k) = 0$ whenever $n \neq k$. Moreover, $\mathbb{R}^2 \setminus \{(0, 0)\} = \bigcup_{n=1}^{\infty} B_n$ with the union pairwise disjoint. Define functions $g_j : \mathbb{R}^2 \to \mathbb{C}$ for $j = 1, 2$ by

$$g_1(t, u) := 1 \qquad \text{and} \qquad g_2(t, u) := t + iu, \qquad (t, u) \in \mathbb{R}^2.$$

Clearly $g_1 \in L^2(\mu) \subseteq L^1(\mu)$. But, $g_2 \in L^1(\mu) \setminus L^2(\mu)$. In fact, since $\mu(B_n) = a_n$ for every $n \in \mathbb{N}$, it follows that

$$\int_{\mathbb{R}^2} |g_2| \, d\mu = \sum_{n=1}^{\infty} \int_{B_n} |g_2| \, d\mu \leq \sum_{n=1}^{\infty} \sqrt{2}\, n \, a_n = \sqrt{2} \sum_{n=1}^{\infty} n^{-2} < \infty$$

and that

$$\int_{\mathbb{R}^2} |g_2|^2 \, d\mu = \sum_{n=1}^{\infty} \int_{B_n} |g_2|^2 \, d\mu \geq \sum_{n=2}^{\infty} (n-1)^2 \, a_n = \sum_{n=2}^{\infty} (n-1)^2 / n^3 = \infty.$$

Now let $\Omega := \mathbb{R}^2$, $\Sigma := \mathcal{B}(\mathbb{R}^2)$ and $E := \mathbb{C}^2$ equipped with its usual unitary norm. Define a vector measure $\nu : \Sigma \to E$ by

$$\nu(A) := \Big(\int_A g_1 d\mu, \int_A g_2 d\mu \Big), \qquad A \in \Sigma.$$

Assume that $1 < p < 3/2$, in which case its adjoint index p' satisfies $3 < p' < \infty$. Our aim is to show that

$$L^p\big(|\langle \nu, \, x^* \rangle|\big) \not\subseteq L^1(\nu) \tag{5.25}$$

for every $x^* \in \mathbf{R}_\nu[E^*]$. First, we show that $E^* \setminus \{0\} = \mathbf{R}_\nu[E^*]$. It is clear that $\mathbf{R}_\nu[E^*] \subseteq E^* \setminus \{0\}$. To show the reverse inclusion, let $x^* \in E^* \setminus \{0\}$. With the usual identification $(\mathbb{C}^2)^* = \mathbb{C}^2$, we can uniquely express $x^* = (x_1^*, \, x_2^*)$ with $x_1^*, \, x_2^* \in \mathbb{C}$ in the canonical way. Thus,

$$\langle \nu, \, x^* \rangle(A) = x_1^* \int_A g_1 \, d\mu \; + \; x_2^* \int_A g_2 \, d\mu = \int_A \big(x_1^* g_1 \; + \; x_2^* g_2 \big) d\mu, \qquad A \in \Sigma.$$

Therefore we have, from [141, Theorem 6.13], that

$$\big|\langle \nu, \, x^* \rangle\big|(A) = \int_A \big| x_1^* g_1 \; + \; x_2^* g_2 \big| \, d\mu, \qquad A \in \Sigma. \tag{5.26}$$

If $x_2^* = 0$, then $\big|x_1^* g_1 + x_2^* g_2\big|(t, u) = |x_1^*| > 0$ for every $(t, u) \in \Omega$ and if $x_2^* \neq 0$, then $\big|x_1^* g_1 + x_2^* g_2\big|(t, u) = \big|x_1^* + (t + iu) x_2^*\big| > 0$ whenever $t + iu \neq -x_1^*/x_2^*$. So, (5.26) implies that the scalar measures $\big|\langle\nu,\ x^*\rangle\big|$ and μ are mutually absolutely continuous, that is, $x^* \in \mathbf{R}_\nu[E^*]$, which establishes the identity $E^*\setminus\{0\} = \mathbf{R}_\nu[E^*]$.

To identify $L^1(\nu)$, let $e_1^* := (1,0) \in E^*$ and $e_2^* := (0,1) \in E^*$. Then, directly from the definition of ν-integrability, we can see that a given Σ-measurable function $f : \Omega \to \mathbb{C}$ is ν-integrable if and only if it is $\langle\nu, e_j^*\rangle$-integrable for each $j = 1, 2$, in which case

$$\int_A f\,d\nu = \left(\int_A fg_1 d\mu,\ \int_A fg_2 d\mu\right), \qquad A \in \Sigma.$$

Therefore, we have that

$$\begin{aligned} L^1(\nu) &= L^1\big(|\langle\nu, e_1^*\rangle|\big) \cap L^1\big(|\langle\nu, e_2^*\rangle|\big) \\ &= L^1\big(|g_1|\,d\mu\big) \cap L^1\big(|g_2|\,d\mu\big) = L^1\big((|g_1| + |g_2|)\,d\mu\big). \end{aligned}$$

Moreover,

$$\int_\Omega |f|\,\big(|g_1|+|g_2|\big)\,d\mu = \int_\Omega |f|\,d|\langle\nu, e_1^*\rangle| + \int_\Omega |f|\,d|\langle\nu, e_2^*\rangle| \le 2\|f\|_{L^1(\nu)}, \quad f \in L^1(\nu). \tag{5.27}$$

To establish condition (5.25), assume the contrary, that is, there exists an element $x^* = (x_1^*, x_2^*) \in E^*\setminus\{0\}$ with

$$L^p\big(|\langle\nu, x^*\rangle|\big) \ \subseteq\ L^1(\nu).$$

Observe that both $L^p\big(|\langle\nu, x^*\rangle|\big)$ and $L^1(\nu)$ are B.f.s.' over (Ω, Σ, μ). Hence, the natural injection from $L^p\big(|\langle\nu, x^*\rangle|\big)$ into $L^1(\nu)$ is necessarily continuous; see Lemma 2.7 with $X(\mu) := L^p\big(|\langle\nu, x^*\rangle|\big)$ and $Y(\mu) := L^1(\nu)$. So, with C denoting the operator norm of this injection, we have from (5.27) that

$$\begin{aligned} \int_\Omega |f|\,\big(|g_1| + |g_2|\big)\,d\mu \le 2\,\|f\|_{L^1(\nu)} &\le 2C\left(\int_\Omega |f|^p\,d|\langle\nu,\ x^*\rangle|\right)^{1/p} \\ &< \infty, \quad f \in L^p\big(|\langle\nu,\ x^*\rangle|\big). \end{aligned} \tag{5.28}$$

Consider first the case when $x_2^* = 0$. Then the scalar measure $|\langle\nu,\ x^*\rangle|$ can be written as $d|\langle\nu,\ x^*\rangle| = |x_1^*|\,d\mu$ (see (5.26)) and hence, from (5.28), it follows that

$$\int_\Omega |f|\,\big(|g_1| + |g_2|\big)\,d\mu \le 2\|f\|_{L^1(\nu)} \le 2C\,|x_1^*|\left(\int_\Omega |f|^p\,d\mu\right)^{1/p} < \infty, \qquad f \in L^p(\mu).$$

This means that $\big(|g_1| + |g_2|\big) \in L^p(\mu)' = L^{p'}(\mu)$ and hence, also $(g_1 + g_2) \in L^{p'}(\mu)$. Since $g_1 = \chi_\Omega \in L^{p'}(\mu)$ and $p' \ge 3$, we have

$$g_2 \in L^{p'}(\mu) \subseteq L^2(\mu).$$

This contradicts the fact that $g_2 \notin L^2(\mu)$.

Next assume that $x_2^* \neq 0$ and let $\gamma := x_1^*/x_2^*$. Then (5.26) yields that

$$L^p\big(|\langle \nu,\ x^*\rangle|\big) = L^p\big(|x_1^* g_1 + x_2^* g_2|\, d\mu\big) = L^p\big(|\gamma g_1 + g_2|\, d\mu\big). \tag{5.29}$$

It follows from (5.28), for every $f \in L^p\big(|\langle \nu, x^*\rangle|\big)$, that

$$\int_\Omega \Big(|f|\cdot|\gamma g_1 + g_2|^{1/p}\Big)\frac{|g_1|+|g_2|}{|\gamma g_1 + g_2|^{1/p}}\, d\mu \le 2C\bigg(\int_\Omega |f|^p\, d|\langle \nu,\ x^*\rangle|\bigg)^{1/p} < \infty. \tag{5.30}$$

Observe, for every $h \in L^p(\mu)$, that there exists $f \in L^p\big(|\langle \nu, x^*\rangle|\big) = L^p\big(|\gamma g_1 + g_2| d\mu\big)$ satisfying $h = f \cdot \big|\gamma g_1 + g_2\big|^{1/p}$. Since $g_1 = \chi_\Omega$, we have $\big(|g_1| + |g_2|\big) \ge 1$ and so it follows from (5.30) that

$$\int_\Omega \frac{|h|}{|\gamma g_1 + g_2|^{1/p}}\, d\mu \le \int_\Omega \frac{|h|\,\big(|g_1|+|g_2|\big)}{|\gamma g_1 + g_2|^{1/p}}\, d\mu < \infty, \qquad h \in L^p(\mu).$$

This means that $\big|\gamma g_1 + g_2\big|^{-1/p} \in L^p(\mu)' = L^{p'}(\mu)$. In other words,

$$\big|\gamma + g_2\big|^{1-p'} = \Big(\big|\gamma + g_2\big|^{-1/p}\Big)^{p'} \in L^1(\mu). \tag{5.31}$$

To show that this is not the case, let $t_0 := -\mathrm{Re}\,\gamma$ and $u_0 := -\mathrm{Im}\,\gamma$. Choose $\varepsilon > 0$ and $N \in \mathbb{N}$ satisfying

$$A(\varepsilon, N) := \Big\{(t,u) \in \mathbb{R}^2 : t > t_0, u > u_0, \big(t - t_0\big)^2 + \big(u - u_0\big)^2 < \varepsilon^2\Big\} \subseteq B_N.$$

Define $h_0 := \big|\gamma + g_2\big|^{1-p'} \cdot \chi_{A(\varepsilon,N)}$. It then follows that

$$\begin{aligned}\int_\Omega h_0\, d\mu &= \frac{a_N}{\lambda(B_N)}\int_{A(\varepsilon,N)}\left(\frac{1}{\sqrt{(t-t_0)^2 + (u-u_0)^2}}\right)^{p'-1} d\lambda(t,u)\\ &= \frac{a_N}{\lambda(B_N)}\int_0^{\pi/2}\bigg(\int_0^\varepsilon \frac{r}{r^{p'-1}}\, dr\bigg) d\theta \ = \ \frac{a_N \pi}{2\lambda(B_N)}\int_0^\varepsilon \frac{1}{r^{p'-2}}\, dr \ = \ \infty\end{aligned}$$

because $p' > 3$. Since $|\gamma + g_2|^{1-p'} \ge h_0$, this contradicts (5.31). Thus, our assumption that $L^p\big(|\langle \nu, x^*\rangle|\big) \subseteq L^1(\nu)$ also gives a contradiction when $x_2^* \neq 0$. So, we have established (5.25) for all $x^* \in E^* \setminus \{0\} = \mathbf{R}_\nu[E^*]$.

(ii) Using the same notation and setting as in part (i), consider any σ-order continuous q-B.f.s. $X(\mu)$ over (Ω, Σ, μ) with $X(\mu) \subseteq L^p(\nu)$. Let $T : X(\mu) \to E$ denote the restriction of the integration operator $I_\nu : L^1(\nu) \to E$ to $X(\mu)$, so that $m_T = \nu$. The assumption $X(\mu) \subseteq L^p(\nu)$ implies that $T \in \mathcal{F}_{[p]}(X(\mu), E)$ via Theorem 5.7. However, by part (i), we have

$$X(\mu) \subseteq L^p(m_T) \subseteq L^p\big(|\langle m_T, x^*\rangle|\big) \not\subseteq L^1(m_T), \quad x^* \in E^* \setminus \{0\} = \mathbf{R}_\nu[E^*]. \quad \square$$

Now the final Question (C)! As for Question (B), an affirmative answer to (C) would allow us to factorize $T \in \mathcal{F}_{[p]}(X(\mu), E)$ and J_T via $L^p\big(|m_T|\big)$, again with the advantage that it is the L^p-space of a scalar measure, namely $|m_T|$. The following result provides an affirmative answer in terms of the Köthe dual $\big(X(\mu)_{[p]}\big)'$. Since always $L^p\big(|m_T|\big) \subseteq L^p(m_T)$, we note that $X(\mu) \subseteq L^p(m_T)$ whenever $X(\mu) \subseteq L^p\big(|m_T|\big)$.

Proposition 5.13. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. over (Ω, Σ, μ) and E be a Banach space. Suppose that $T : X(\mu) \to E$ is a μ-determined operator whose associated vector measure $m_T : \Sigma \to E$ has finite variation. Then, for each $1 \le p < \infty$, both $L^1\big(|m_T|\big)$ and its $(1/p)$-th power $L^p\big(|m_T|\big)$ are also B.f.s.' over (Ω, Σ, μ). Moreover, the inclusion $X(\mu) \subseteq L^p\big(|m_T|\big)$ holds if and only if the Radon–Nikodým derivative $\dfrac{d|m_T|}{d\mu}$ belongs to the Köthe dual $\big(X(\mu)_{[p]}\big)'$ of the q-B.f.s. $X(\mu)_{[p]}$. In this case, the restriction $I^{(p)}_{|m_T|}$ of the integration operator $I^{(p)}_{m_T} : L^p(m_T) \to E$ to $L^p\big(|m_T|\big) \subseteq L^p(m_T)$ is a continuous linear extension of T to $L^p\big(|m_T|\big)$ and consequently, T factorizes through $L^p(|m_T|)$.*

$$
\begin{tikzcd}
X(\mu) \arrow[r, "T"] \arrow[d] & E \\
L^p(|m_T|) \arrow[r] \arrow[ur, "I^{(p)}_{|m_T|}"] & L^p(m_T). \arrow[u, "I^{(p)}_{m_T}"']
\end{tikzcd}
$$

Proof. Since $L^1\big(|m_T|\big) = L^p\big(|m_T|\big)_{[p]}$, we have that

$$X(\mu) \subseteq L^p\big(|m_T|\big) \iff X(\mu)_{[p]} \subseteq L^1\big(|m_T|\big); \tag{5.32}$$

see Lemma 2.20(ii) with $Y(\mu) := L^p\big(|m_T|\big)$. Moreover, since m_T and $|m_T|$ have the same null sets, it follows from the μ-determinedness of T that $|m_T| \ll \mu$. Accordingly,

$$\int_\Omega |f| \frac{d|m_T|}{d\mu}\, d\mu = \int_\Omega |f|\, d|m_T|, \qquad f \in X(\mu)_{[p]}. \tag{5.33}$$

Suppose now that $X(\mu) \subseteq L^p\big(|m_T|\big)$, that is, $X(\mu)_{[p]} \subseteq L^1\big(|m_T|\big)$ by (5.32). By Lemma 2.7, with $X(\mu)_{[p]}$ in place of $X(\mu)$ and $L^1\big(|m_T|\big)$ in place of $Y(\mu)$, the natural injection from $X(\mu)_{[p]}$ into $L^1\big(|m_T|\big)$ is continuous. So, there is $C > 0$ satisfying

$$\int_\Omega |f|\, d|m_T| \le C \|f\|_{X(\mu)_{[p]}} < \infty, \qquad f \in X(\mu)_{[p]}.$$

This and (5.33) show that $\int_\Omega |f|\frac{d|m_T|}{d\mu}d\mu < \infty$ for every $f \in X(\mu)_{[p]}$. In other words, $\frac{d|m_T|}{d\mu} \in (X(\mu)_{[p]})'$.

Conversely, suppose that $\frac{d|m_T|}{d\mu} \in (X(\mu)_{[p]})'$. Then, again from (5.33), we have that $\int_\Omega |f|\, d|m_T| = \int_\Omega |f|\frac{d|m_T|}{d\mu}\, d\mu < \infty$ for every $f \in X(\mu)_{[p]}$, that is, the inclusion $X(\mu)_{[p]} \subseteq L^1\big(|m_T|\big)$ holds. This completes the proof. □

The following example exhibits a σ-order continuous q-B.f.s. $X(\mu)$, a Banach space E and a linear operator $T \in \mathcal{F}_{[p]}\big(X(\mu), E\big)$ such that $X(\mu) \nsubseteq L^p\big(|m_T|\big)$.

Example 5.14. Let $\Sigma := 2^{\mathbb{N}}$ and let $\mu : \Sigma \to [0,\infty)$ be a finite scalar measure with $\mu(\{n\}) > 0$ for every $n \in \mathbb{N}$. Define the positive function $\varphi : n \mapsto \mu(\{n\})$, for $n \in \mathbb{N}$, in which case $\varphi \in \ell^1$. For $0 < r < \infty$ the space $\ell^r(\mu) := L^r(\mu)$, equipped with the usual L^r-quasi-norm for the finite measure μ, is a σ-order continuous q-B.f.s. based on $(\mathbb{N}, \Sigma, \mu)$. Fix $1 < p < \infty$.

(i) The q-B.f.s. $\ell^r(\mu)$ is p-convex if and only if $r \geq p$; see (i) and (ii) of Example 2.73. In this case, the p-th power $\ell^r(\mu)_{[p]} = \ell^{r/p}(\mu)$ is a B.f.s with the property that its standard p-th power lattice norm $\|\cdot\|_{\ell^r(\mu)_{[p]}}$, as defined by (2.47) with $X(\mu) := \ell^r(\mu)$, coincides with the usual norm on $\ell^{r/p}(\mu)$. On the other hand, if $1 \leq r < p$, then the q-B.f.s. $\ell^r(\mu)_{[p]} = \ell^{r/p}(\mu)$ is not normable whereas $\ell^r(\mu)$ is a B.f.s.

(ii) Suppose that $\max\{1, r/p\} \leq q < \infty$. We construct a μ-determined, p-th power factorable operator T from the q-B.f.s. $\ell^r(\mu)$ into the Banach space ℓ^q. Observe that

$$f \in \ell^{r/p}(\mu) \quad \Longleftrightarrow \quad |f|^{r/p}\cdot\varphi \in \ell^1 \quad \Longleftrightarrow \quad |f|\cdot\varphi^{p/r} \in \ell^{r/p} \quad \Longrightarrow \quad |f|\cdot\varphi^{p/r} \in \ell^q,$$

where the last implication uses $(r/p) \leq q$. So, we can define the linear injection $S : f \mapsto f\cdot\varphi^{p/r}$ from $\ell^{r/p}(\mu)$ into ℓ^q. The map S is continuous because

$$\big\|S(f)\big\|_{\ell^q} \leq \big\|S(f)\big\|_{\ell^{r/p}} = \Big(\sum_{n=1}^{\infty}|f(n)|^{r/p}\cdot\varphi(n)\Big)^{p/r} = \|f\|_{\ell^{r/p}(\mu)}, \quad f \in \ell^{r/p}(\mu).$$

Now let $X(\mu) := \ell^r(\mu) = L^r(\mu)$ and let $E := \ell^q$. The natural inclusion map $i_{[p]} : X(\mu) \to X(\mu)_{[p]} = \ell^{r/p}(\mu) = L^{r/p}(\mu)$ is continuous because $(r/p) < r$. So, the continuous linear injection $T := S \circ i_{[p]}$ from $X(\mu)$ into E is μ-determined via Lemma 4.5(ii). The p-th power factorability of T is clear because $T_{[p]} := S$ is a continuous linear extension of T to the larger domain space $X(\mu)_{[p]} = \ell^{r/p}(\mu)$.

(iii) Under the same assumption as in part (ii), the associated vector measure $m_T : \Sigma \to E = \ell^q$ is given by $m_T(A) = T(\chi_A) = \chi_A\cdot\varphi^{p/r}$ for $A \in \Sigma$. Therefore,

$$L^1(m_T) = \big\{f \in \mathbb{C}^{\mathbb{N}} : f\cdot\varphi^{p/r} \in \ell^q\big\} \tag{5.34}$$

and $I_{m_T}(f) = f\cdot\varphi^{p/r}$ for every $f \in L^1(m_T)$, which can be verified by applying Theorem 3.5 with $\nu := m_T$. Accordingly,

$$L^p(m_T) = \big\{f \in \mathbb{C}^{\mathbb{N}} : f\cdot\varphi^{1/r} \in \ell^{pq}\big\}.$$

(iv) It follows, from Lemma 3.20(i) applied to the purely atomic vector measure $\nu := m_T$, that $|m_T|(\{n\}) = \big\|m_T(\{n\})\big\|_E = \varphi^{p/r}(n)$ for $n \in \mathbb{N}$ and hence, $|m_T|(A) = \sum_{n\in A} \varphi^{p/r}(n)$ for $A \in \Sigma$. Thus,

$$L^1\big(|m_T|\big) \;=\; \big\{f \in \mathbb{C}^{\mathbb{N}} : f \cdot \varphi^{p/r} \in \ell^1\big\}, \tag{5.35}$$

which is independent of q. Since $1 < p < \infty$ and $q \geq 1$ by assumption, it follows from (5.34) and (5.35) that $L^1\big(|m_T|\big) \neq L^1(m_T)$.

According to (5.35) we have that

$$L^p(|m_T|) \;=\; \big\{f \in \mathbb{C}^{\mathbb{N}} : |f|^p \cdot \varphi^{p/r} \in \ell^1\big\} \;=\; \big\{f \in \mathbb{C}^{\mathbb{N}} : f \cdot \varphi^{1/r} \in \ell^p\big\}.$$

This identity, together with $X(\mu) = \ell^r(\mu) = \big\{f \in \mathbb{C}^{\mathbb{N}} : f \cdot \varphi^{1/r} \in \ell^r\big\}$, show that $X(\mu) \subseteq L^p\big(|m_T|\big)$ if and only if $r \leq p$. In this case $\varphi \in \ell^1 \subseteq \ell^{p/r}$ and so $|m_T|$ is a finite measure. Consequently, if $1 < p < r$, then $X(\mu) = \ell^r(\mu) \nsubseteq L^p\big(|m_T|\big)$. $\square$

5.4 Compactness criteria

For $1 < p < \infty$, every operator $T \in \mathcal{F}_{[p]}(X(\mu), E)$ factorizes through $L^p(m_T)$ and satisfies $T = I^{(p)}_{m_T} \circ J^{(p)}_T$; see Remark 5.8(II)(i). This enables us to determine compactness properties of T by applying the relevant results from Chapter 3.

Proposition 5.15. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. over (Ω, Σ, μ) and E be a Banach space. Let $1 < p < \infty$. The following assertions hold for every operator $T \in \mathcal{F}_{[p]}(X(\mu), E)$.*

- (i) *The operator T is weakly compact.*
- (ii) *The following conditions are equivalent.*
  - (a) *The operator T is compact.*
  - (b) *The range $\mathcal{R}(m_T) = \{T(\chi_A) : A \in \Sigma\}$ of the associated vector measure $m_T : \Sigma \to E$ of T is a relatively compact set in E.*
- (iii) *If, in addition, the variation measure $|m_T| : \Sigma \to [0, \infty]$ of m_T is σ-finite, then conditions* (a) *and* (b) *in part* (ii) *are equivalent to any of the following conditions.*
  - (c) *The restriction $I_{|m_T|}$ of the integration operator $I_{m_T} : L^1(m_T) \to E$ to $L^1\big(|m_T|\big) \subseteq L^1(m_T)$ is completely continuous.*
  - (d) *For every set $A \in \Sigma$ with $|m_T|(A) < \infty$, the subset of E given by $m_T(\Sigma \cap A) = \{T(\chi_A) : B \in \Sigma \cap A\}$ is relatively compact.*

Proof. (i) Recall, in the notation of Remark 5.8II(i), that $T = I^{(p)}_{m_T} \circ J^{(p)}_T$. Now Remark 3.62(i), with $\nu := m_T$, yields that $I^{(p)}_{m_T}$ is weakly compact and hence, so is T.

For parts (ii) and (iii), apply Proposition 3.56 with $\nu := m_T$. $\square$

The following two corollaries of Proposition 5.15 provide useful criteria for an operator $T \in \mathcal{F}_{[p]}(X(\mu), E)$, with $1 < p < \infty$, to be compact.

Corollary 5.16. *Let $\mu : \Sigma \to [0, \infty)$ be a purely atomic measure, $X(\mu)$ be a σ-order continuous q-B.f.s. over (Ω, Σ, μ) and $1 < p < \infty$. Given a Banach space E, every operator $T \in \mathcal{F}_{[p]}(X(\mu), E)$ is compact.*

Proof. Since T is μ-determined, $\mathcal{N}_0(m_T) = \mathcal{N}_0(\mu)$; see Lemma 4.5(i). So, m_T is also purely atomic and consequently, has compact range, [78, Theorem 10]. Then Proposition 5.15(ii) implies that T is compact. $\square$

If μ is non-atomic, then we can easily find a q.B.f.s. $X(\mu)$, a Banach space E and a non-compact operator $T \in \mathcal{F}_{[p]}(X(\mu), E)$.

Example 5.17. Let $\mu : \Sigma \to [0, \infty)$ be a non-atomic measure and $X(\mu) := L^4(\mu)$. With $E := L^2(\mu)$, consider the inclusion map $T : X(\mu) \to E$. Then $T \in \mathcal{F}_{[p]}(X(\mu), E)$ whenever $1 < p \leq 2$; just note that the inclusion $T_{[p]}$ of $X(\mu)_{[p]} = L^{4/p}(\mu)$ into E is an extension of T. Moreover, $\{T(\chi_A) : A \in \Sigma\} = \{\chi_A : A \in \Sigma\}$ is not relatively compact in E via Lemma 3.21, where we have used the fact that $L^1(m_T) = L^4(\mu) \subseteq E$ (cf. Corollary 3.66(ii). Then Proposition 5.15(ii) implies that T is not compact. $\square$

Corollary 5.18. *Let $X(\mu)$ be a σ-order continuous B.f.s. over (Ω, Σ, μ) and E be a Banach space. Suppose that $T \in \mathcal{F}_{[p]}(X(\mu), E)$ for some $1 < p < \infty$ and that the associated vector measure $m_T : \Sigma \to E$ of T has σ-finite variation. The following assertions hold.*

(i) *If the restricted integration operator $I_{|m_T|} : L^1(|m_T|) \to E$ is weakly compact, then T is compact.*

(ii) *If the integration operator $I_{m_T} : L^1(m_T) \to E$ is weakly compact, then T is compact.*

(iii) *If the Banach space E is reflexive, then T is compact.*

Proof. (i) Given $A \in \Sigma$ with $|m_T|(A) < \infty$, let $|m_T|_A : \Sigma \cap A \to [0, \infty)$ denote the restriction of $|m_T|$ to $\Sigma \cap A$. Then $L^1(|m_T|_A)$ can naturally be regarded as a closed subspace of $L^1(|m_T|)$. So, the restriction T_A of the weakly compact operator T to $L^1(|m_T|_A)$ is also weakly compact. The Dunford-Pettis property of $L^1(|m_T|_A)$, [42, Ch. III, Corollary 2.14], guarantees that T_A is completely continuous and hence, $\{T(\chi_B) : B \in \Sigma \cap A\} = \{T_A(\chi_B) : B \in \Sigma \cap A\}$ is relatively compact via Corollary 2.42, with $|m_T|_A$ in place of μ and T_A in place of T. Then Proposition 5.15(iii) implies that T is compact.

(ii) Let $j_1 : L^1(|m_T|) \to L^1(m_T)$ denote the natural inclusion map. By the assumption on I_{m_T}, we see that $I_{|m_T|} = I_{m_T} \circ j_1$ is weakly compact. So, part (ii) holds via (i).

(iii) The given assumption implies the weak compactness of I_{m_T}. So, part (iii) follows from (ii). $\square$

Example 5.19. (i) The σ-finiteness assumption in Corollary 5.18 is not necessary. To see this, adopt the notation of Example 4.11 and suppose further that $1 < q < r < \infty$ and $g \in L^q(\mathbb{T}) \setminus L^r(\mathbb{T})$. With $X(\mu) := L^r(\mathbb{T})$ and $E := L^r(\mathbb{T})$, let T denote the convolution operator $C_g^{(r)} := L^r(\mathbb{T}) \to L^r(\mathbb{T})$. Choose u according to $q^{-1} + u^{-1} = r^{-1} + 1$, so that T can be extended to the larger domain space $L^u(\mathbb{T})$. Hence, $T \in \mathcal{F}_{[p]}(X(\mu), E)$ whenever $1 < p < (r/u)$ because we already know that T is μ-determined; see Example 4.11. The operator T is compact, [48, Corollary 6]. However, the variation measure $|m_T|$ associated with T is totally infinite, [123, Theorem 1.2]. In short, the operator $T \in \mathcal{F}_{[p]}(X(\mu), E)$ is compact and E is reflexive whereas the variation measure $|m_T|$ is totally infinite. For more precise details and further examples along these lines we refer to Chapter 7.

(ii) We present a non-compact, μ-determined operator T from a q-B.f.s. $X(\mu)$ into a reflexive Banach space E such that

(a) $|m_T|(\Omega) < \infty$,

(b) $\mathcal{R}(m_T) = \{T(\chi_A) : A \in \Sigma\}$ is relatively compact in E, and

(c) $T \notin \mathcal{F}_{[p]}(X(\mu), E)$ for every $1 < p < \infty$.

Let $\Omega := [0, 1]$, $\Sigma := \mathcal{B}([0, 1])$ and $\mu : \Sigma \to [0, \infty)$ be Lebesgue measure. With $1 < r < \infty$ and $E := L^r([0, 1])$, let $\nu_r : \Sigma \to E$ denote the Volterra measure of order r. Let T denote the restricted integration operator $I_{|\nu_r|} : L^1(|\nu_r|) \to E$ and let $X(\mu) := L^1(|\nu_r|)$, in which case $m_T = \nu_r$. We claim that T is μ-determined. In fact, since the Volterra operator $V_r : L^r([0, 1]) \to L^r([0, 1])$ is μ-determined (see Example 4.9), we have $\mathcal{N}_0(\nu_r) = \mathcal{N}_0(\mu)$. The fact that $m_T = \nu_r$ yields $\mathcal{N}_0(m_T) = \mathcal{N}_0(\nu_r) = \mathcal{N}_0(\mu)$ and hence, T is μ-determined; see Lemma 4.5(i).

Next, it follows from [129, Proposition 5.2(ii)] that T is not compact.

Condition (a) holds because $d|\nu_r|(t) = (1 - t)^{1/r} dt$; see Example 3.26. Moreover, condition (b) has already been established just prior to Lemma 3.50 in Chapter 3. The remaining condition (c) is a special case of Proposition 5.22 below (as will be confirmed in Remark 5.23(i)). □

An immediate implication of Example 5.19(ii) is that we cannot remove the assumption in Proposition 5.15 that $T \in \mathcal{F}_{[p]}(X(\mu), E)$.

Let us now present another consequence of Proposition 5.15, which shows that if $T \in \mathcal{F}_{[p]}(X(\mu), E)$ is compact, with $1 < p < \infty$, then so is its extension $T_{[r]} : X(\mu)_{[r]} \to E$ whenever $1 < r < p$. This is not always so for the linear operator $T_{[p]} : X(\mu)_{[p]} \to E$, in general; see Example 5.21 below.

Corollary 5.20. *Let $X(\mu)$ be a σ-order continuous B.f.s. over (Ω, Σ, μ) and E be a Banach space. Let $1 < p < \infty$ and $T \in \mathcal{F}_{[p]}(X(\mu), E)$. Then T is compact if and only if its extension $T_{[r]} : X(\mu)_{[r]} \to E$ is compact for all/some $1 \leq r < p$.*

Proof. Fix $1 \leq r < p$. Recall that $X(\mu) \subseteq X(\mu)_{[r]} \subseteq X(\mu)_{[p]}$ (see Lemma 2.21(iv)). Let $T_{[r]} : X(\mu)_{[r]} \to E$ denote the restriction of $T_{[p]}$ to $X(\mu)_{[r]}$. Then

$T_{[p]} : X(\mu)_{[p]} \to E$ is a continuous linear extension of $T_{[r]}$ to $X(\mu)_{[p]}$ with

$$T_{[r]} \in \mathcal{F}_{[p/r]}\big(X(\mu)_{[r]}, E\big) \qquad \text{and} \qquad \big(T_{[r]}\big)_{[p/r]} \;=\; T_{[p]},$$

where we have also used the fact that $(X(\mu)_{[r]})_{[p/r]} = X(\mu)_{[p]}$; see Lemma 2.20(i).

Assume now that T is compact. From the identity $m_T = m_{T_{[r]}}$, we see that $\mathcal{R}(m_{T_{[r]}})$ equals $\mathcal{R}(m_T)$ with $\mathcal{R}(m_T)$ relatively compact in E by the compactness of T. So, apply Proposition 5.15(ii), with $T_{[r]}$ in place of T and (p/r) in place of p, to conclude that $T_{[r]}$ is compact.

Conversely, assume that $T_{[r]}$ is compact for some $1 \le r < p$. Then, the compactness of T follows from the identity $T = T_{[r]} \circ i_{[r]}$ with $i_{[r]} : X(\mu) \to X(\mu)_{[r]}$ denoting the natural inclusion. □

Example 5.21. Let $\Omega := [0,1]$, $\Sigma := \mathcal{B}([0,1])$ and $\mu : \Sigma \to [0,\infty)$ be Lebesgue measure. With $X(\mu) := L^p\big((1-t)^{1/p}dt\big)$ for $1 < p < \infty$ and $E := L^p([0,1])$, define $T : X(\mu) \to E$ to be the unique linear extension of the Volterra operator $V_p : L^p([0,1]) \to E$ to $L^p\big((1-t)^{1/p}dt\big)$. Such an extension is possible because

$$L^p([0,1]) \subseteq L^p\big((1-t)^{1/p}dt\big) \subseteq L^1\big((1-t)^{1/p}dt\big) \;=\; L^1(|\nu_p|) \subseteq L^1(\nu_p)$$

with ν_p denoting the Volterra measure of order p; see Example 3.26. Then $T \in \mathcal{F}_{[p]}(X(\mu), E)$. In fact, from the definition we have $X(\mu)_{[p]} = L^1\big((1-t)^{1/p}dt\big)$ on which the restricted integration operator $I_{|\nu_p|}$ is an extension of T. That is, $T_{[p]} = I_{|\nu_p|}$. Since $\mathcal{R}(m_T) = \mathcal{R}(\nu_p)$ and $\mathcal{R}(\nu_p)$ is relatively compact (by compactness of V_p), Proposition 5.15(ii) yields that the operator $T \in \mathcal{F}_{[p]}(X(\mu), E)$ is compact. On the other hand, $I_{|\nu_p|}$ is not compact; see Example 5.19(ii) (with p in place of r). In other words, the extension $T_{[p]}$ of T is not compact whereas T is compact. □

If the domain space of a μ-determined operator T is already equal to its o.c. optimal domain $L^1(m_T)$ then, of course, T is not p-th power factorable. The following proposition provides a general method for creating μ-determined, non-p-th power factorable operators whose domain space is *strictly smaller* than their o.c. optimal domain.

Proposition 5.22. *Let $1 < p < \infty$ and $\nu : \Sigma \to E$ be a Banach-space-valued vector measure whose variation measure $|\nu|$ is finite. Suppose that Σ is σ-decomposable relative to ν and $\mu : \Sigma \to [0,\infty)$ is any control measure for ν. Then*

$$L^1(|\nu|)_{[p]} \not\subseteq L^1(\nu) \tag{5.36}$$

and the restriction $I_{|\nu|} : L^1(|\nu|) \to E$ of the integration operator I_ν to $L^1(|\nu|)$ is μ-determined but, not p-th power factorable.

Proof. It follows from Theorem 3.7 that $L^1(\nu)$ is a B.f.s. over (Ω, Σ, μ). Since the finite measures $|\nu|$ and μ are mutually absolutely continuous, the space $L^1(|\nu|)$ is also a B.f.s. over (Ω, Σ, μ) and hence, its p-th power $L^{1/p}(|\nu|) = L^1(|\nu|)_{[p]}$ is a q-B.f.s. over (Ω, Σ, μ); see Proposition 2.22 with $X(\mu) := L^1(|\nu|)$. To prove (5.36) assume the contrary, that is,

$$L^1(|\nu|)_{[p]} \subseteq L^1(\nu). \tag{5.37}$$

Then the natural inclusion map $j_{[p]} : L^1(|\nu|)_{[p]} \to L^1(\nu)$ is continuous via Lemma 2.7, with $X(\mu) := L^1(|\nu|)_{[p]}$ and $Y(\mu) := L^1(\nu)$ there.

First consider the case when ν is non-atomic. Since $j_{[p]} : L^1(|\nu|)_{[p]} \to L^1(\nu)$ is a non-zero continuous linear operator and the dual space of the Banach space $L^1(\nu)$ is non-trivial, the q-B.f.s. $L^1(|\nu|)_{[p]}$ would have a non-trivial dual. This contradicts the fact that $\big(L^1(|\nu|)_{[p]}\big)^* = L^{1/p}(|\nu|)^* = \{0\}$; see Example 2.10. Therefore, (5.37) cannot hold in this case.

Next, consider the case when ν is purely atomic. Since ν is σ-decomposable, the space $L^1(|\nu|)$ is infinite-dimensional and so ν has infinitely countably many atoms (see Lemma 3.20(ii)). So, we may assume that $\Omega = \mathbb{N}$ and $\Sigma = 2^{\mathbb{N}}$ and that each singleton $\{n\} \in 2^{\mathbb{N}}$ is an atom of ν. Let

$$g(n) := |\nu|(\{n\}), \qquad n \in \mathbb{N},$$

in which case

$$L^1(|\nu|) = (1/g) \cdot \ell^1 = \{f/g : f \in \ell^1\}.$$

Given $n \in \mathbb{N}$, let $f_n := \big(g(n)\big)^{-p} \chi_{\{n\}}$. Then

$$\|f_n\|_{L^1(|\nu|)_{[p]}} = \|f_n\|_{L^{1/p}(|\nu|)} = \big(g(n)\big)^{-p} \big(|\nu|(\{n\})\big)^p = 1, \qquad n \in \mathbb{N},$$

and hence, $\{f_n\}_{n=1}^\infty \subseteq \mathbf{B}[L^1(|\nu|)_{[p]}]$, that is,

$$\sup_{n\in\mathbb{N}} \big\|j_{[p]}(f_n)\big\|_{L^1(\nu)} \le \big\|j_{[p]}\big\| < \infty. \tag{5.38}$$

On the other hand, since $\|\nu\|(\{n\}) = |\nu|(\{n\})$ (see Lemma 3.20(i)), it follows for $n \to \infty$ that

$$\begin{aligned} \big\|j_{[p]}(f_n)\big\|_{L^1(\nu)} &= \big(g(n)\big)^{-p} \big\|\chi_{\{n\}}\big\|_{L^1(\nu)} = \big(g(n)\big)^{-p} \|\nu\|(\{n\}) \\ &= \big(g(n)\big)^{-p} |\nu|(\{n\}) = \big(g(n)\big)^{1-p} \quad \longrightarrow \infty, \end{aligned}$$

because $\sum_{n=1}^\infty |\nu|(\{n\}) < \infty$ implies that $\lim_{n\to\infty} g(n) = \lim_{n\to\infty} |\nu|(\{n\}) = 0$. This contradicts (5.38) and so again (5.37) cannot hold.

Finally consider the general case. In other words, ν admits an atomic part Ω_{a}, which is the union of the countably many atoms of ν (by Lemma 3.20(ii)), and a non-atomic part $\Omega_{\mathrm{na}} = \Omega \setminus \Omega_{\mathrm{a}}$, such that both Ω_{a} and Ω_{na} are non-ν-null sets. Let ν_{na} denote the restriction of ν to the measurable space $\big(\Omega_{\mathrm{na}}, \Sigma \cap \Omega_{\mathrm{na}}\big)$. Then (5.37) implies that

$$L^1\big(|\nu_{\mathrm{na}}|\big)_{[p]} \subseteq L^1(\nu_{\mathrm{na}}).$$

But, as already argued above, this is not the case because ν_{na} is purely non-atomic. So, again (5.37) cannot hold. Since all possibilities have been considered, we can conclude that (5.36) holds

Now let $T := I_{|\nu|}$. Then the associated measure m_T equals ν. Hence, we have $\mathcal{N}_0(m_T) = \mathcal{N}_0(\nu) = \mathcal{N}_0(\mu)$, that is, $T = I_{|\nu|}$ is μ-determined. That T is not p-th power factorable follows from (5.36). $\square$

Remark 5.23. Let the notation be as in Proposition 5.22.

(i) Proposition 5.22 is only of interest if $L^1(|\nu|) \neq L^1(\nu)$. We recall that $L^1(|\nu|) = L^1(\nu)$ if and only if $L^1(\nu)$ is lattice isomorphic to an abstract L^1-space; see Lemma 3.14(iii).

An example of a vector measure ν for which $L^1(|\nu|) \neq L^1(\nu)$ is the Volterra measure ν_r of order r for $1 < r < \infty$; see Example 3.26(ii). Consequently, the restricted integration operator $I_{|\nu_r|} : L^1(|\nu_r|) \to L^r([0,1])$ is not p-th power factorable whenever $1 < p < \infty$.

(ii) Since $|\nu|$ and any control measure μ for ν are mutually absolutely continuous, the spaces $L^1(\nu)$ and $L^1(|\nu|)$ are both B.f.s' over the finite measure space $(\Omega, \Sigma, |\nu|)$. Accordingly, we may use $|\nu|$ instead of a general control measure μ to conclude that the restricted integration operator $I_{|\nu|} : L^1(|\nu|) \to E$ of the integration operator I_ν to $L^1(|\nu|)$ is $|\nu|$-determined but, not p-th power factorable. $\square$

For E a Banach lattice, the positive operators which belong to the class $\mathcal{F}_{[p]}(X(\mu), E)$, with $1 \le p < \infty$, possess an additional desirable property.

Proposition 5.24. *Let $1 \le p < \infty$ and $X(\mu)$ be a σ-order continuous q-B.f.s. over (Ω, Σ, μ) and E be a Banach lattice. If a positive operator $T : X(\mu) \to E$ belongs to the class $\mathcal{F}_{[p]}\big(X(\mu), E\big)$, then T is p-convex.*

Proof. By Theorem 5.7, the operator T is factorized through $L^p(m_T)$ such that $T = I^{(p)}_{m_T} \circ J^{(p)}_T$ (in the notation of Remark 5.8(II)). The vector measure m_T is positive thanks to the positivity of T. Accordingly, the restricted integration operator $I^{(p)}_{m_T} : L^p(m_T) \to E$ is also positive and, in particular, $I^{(p)}_{m_T} \in \Lambda_p(L^p(m_T), E)$. On the other hand, the canonical map $J^{(p)}_T : X(\mu) \to L^p(m_T)$ is p-convex; see Remark 5.8(II)(iii). So, the composition $T = I^{(p)}_{m_T} \circ J^{(p)}_T$ is also p-convex, via Proposition 2.63(ii). $\square$

We have applied Proposition 2.63(ii) to establish Proposition 5.24, for which the order of the composition $J^{(p)}_T \circ I^{(p)}_{m_T}$ is crucial. As seen from the proof of Proposition 5.24, if $I^{(p)}_{m_T} \in \Lambda_p(L^p(m_T), E)$, then T is p-convex. Do we really need such an assumption? To be precise, let $X(\mu)$ be a σ-order continuous q-B.f.s and E be a Banach lattice. Given $1 \le p < \infty$, do we always have the inclusion

$$\mathcal{F}_{[p]}(X(\mu), E) \subseteq \mathcal{K}^{(p)}(X(\mu), E)\,?$$

The answer is affirmative for $p = 1, 2$, while it seems to be open for the remaining cases. The case $p = 1$ is easy. Indeed, let $T : X(\mu) \to E$ be a μ-determined operator, which is equivalent to saying that $T \in \mathcal{F}_{[1]}(X(\mu), E)$ because T is 1-th power factorable by definition. Now, E is 1-convex, which can be seen directly from the definition (or derived from the 1-convexity of id_E via Lemma 2.50). So, it follows from Corollary 2.64 that T is 1-convex.

The affirmative answer for $p = 2$ is the following result.

Proposition 5.25. *Let $X(\mu)$ be a σ-order continuous q-B.f.s over (Ω, Σ, μ) and E be a Banach lattice. Then*

$$\mathcal{F}_{[2]}\big(X(\mu), E\big) \subseteq \mathcal{K}^{(2)}\big(X(\mu), E\big).$$

Proof. Consider any operator $T \in \mathcal{F}_{[2]}\big(X(\mu), E\big)$. Then the canonical inclusion map $J_T^{(2)} : X(\mu) \to L^2(m_T)$ is 2-convex; see Remark 5.8(II)(iii) with $p := 2$. Hence, the composition $T = I_{m_T}^{(2)} \circ J_T^{(2)}$ (which exists by Remark 5.8(II)(i) with $p := 2$) is also 2-convex via Proposition 2.63(iii). $\square$

Chapter 6

Factorization of p-th Power Factorable Operators through L^q-spaces

Maurey–Rosenthal factorization theory establishes a clear link between norm inequalities for operators, geometrical properties of Banach lattices, and factorizations through L^q-spaces. This theory essentially has its roots in the work of Rosenthal, Krivine and Maurey (see [92], [99], [106], [137]). They developed these ideas in the late 1960s and during the 1970s for purposes related to the study of the structure of Banach lattices and operators. In the 1980s, the contributions of García Cuerva and Rubio de Francia provided powerful new tools for this theory in the context of harmonic analysis, [61] [62], [138]. Currently, this factorization theory has become a keystone in several areas of functional analysis with varied applications, for instance, to interpolation theory of Banach spaces, operator ideal theory, [31], [41], the study of almost everywhere convergence of series in function spaces, [163, III.H] and to the factorization of inequalities, [14], [61], [62]. Further contributions to the Maurey–Rosenthal theory have recently been developed in the context of the geometry of Banach lattices (see for instance [30], [32], [56]).

The aim of this chapter is to relate the factorization results for p-th power factorable operators as presented in Chapter 5 with Maurey–Rosenthal factorization theory. As we will see, the combination of these two theories leads to a meaningful improvement of the factorization results through L^p-spaces which are obtained when using each theory individually. The abstract version of the Maurey–Rosenthal theorems relates the p-concavity of an operator T (assuming certain p-convexity conditions on the domain space of T) to its factorization through an L^p-space via a multiplication operator. If we focus our attention on T being *p-th power factorable*, then the required assumptions can be weakened, thereby obtaining better factorizations for such operators. The reason is, roughly speaking, that a p-th power factorable operator T always factorizes through the p-convex Banach lattice $L^p(m_T)$. In this chapter we extend these results further, under suitable assumptions, to show that it is possible to factorize each p-th power factorable operator through an appropriate L^q-space (of some finite positive scalar measure). Consequently, for this class of operators we provide two canonical factorization procedures, called the upper scheme and the lower scheme; the first one factorizes through an L^q-space and the other one through the corresponding $L^{q/p}$-space with $(q/p) \leq q$. In fact, these schemes preserve and *improve* the structure of both the canonical factorization procedures for p-th power factorable operators which were presented in Chapter 5.

To be more precise, suppose that (Ω, Σ, μ) is a positive, finite measure space. Let $1 \leq p < \infty$ and $T \in \mathcal{F}_{[p]}\big(X(\mu), E\big)$ with $X(\mu)$ a σ-order continuous q-B.f.s. and E a Banach space. Then, according to Chapter 5, the operator T factorizes through $L^p(m_T)$. Although $L^p(m_T)$ is always a p-convex B.f.s., the factorization through a classical L^p-space (of a scalar measure) would be preferable, whenever available. Under certain conditions (e.g., $|m_T|(\Omega) < \infty$), such an operator T can indeed be factored through a classical L^p-space, namely through $L^p\big(|m_T|\big)$ (see Proposition 5.13), which then provides an additional tool for analyzing T. However, many interesting p-th power factorable operators fail to factorize in this way. This suggests that we should seek a "different type" of factorization, maybe still through the L^q-space of a scalar measure but, allowing $0 < q < \infty$.

The Maurey–Rosenthal theory provides such a factorization, modulo certain assumptions. One version can be formulated as follows. Let $1 \leq q < \infty$ and let $X(\mu)$ be a σ-order continuous *q-convex* B.f.s. Consider a Banach space E and a *q-concave* linear operator $T : X(\mu) \to E$. Then there exists $g \in L^1(\mu)$ satisfying

$$\sup_{\|f\| \leq 1} \Big(\int_\Omega |f|^q g \, d\mu \Big)^{1/q} \leq \mathbf{M}_{(q)}[T] \cdot \mathbf{M}^{(q)}[X(\mu)]$$

and

$$\big\|T(f)\big\|_E \leq \Big(\int_\Omega |f|^q g \, d\mu \Big)^{1/q}$$

for every $f \in X(\mu)$. There is also a factorization version of this result. Namely, under the above assumptions, T factorizes as

$$\begin{tikzcd}
X(\mu) \arrow[rr, "T"] \arrow[dr, "M_{g^{1/q}}"'] & & E \\
& L^q(\mu) \arrow[ur, "S"'] &
\end{tikzcd}$$

where S is a continuous linear operator, $M_{g^{1/q}}$ is the continuous operator of multiplication by $g^{1/q}$ and $\|M_{g^{1/q}}\| \cdot \|S\| \leq \mathbf{M}_{(q)}[T] \cdot \mathbf{M}^{(q)}[X(\mu)]$. This is part of Corollary

6.17 below which is motivated by an earlier version of the Maurey–Rosenthal type theorem, [30, Corollary 5], with slightly different settings. By using different arguments, a similar factorization result can also be found in [62, Ch. VI, Theorem 4.5]. In fact, what is traditionally known as the Maurey–Rosenthal Theorem is a result regarding factorizations through $L^2(\mu)$ for operators taking their values in $L^1(\mu)$, [41, Theorem 12.30]. Although the arguments given there are related to ours, this version does not correspond to the one that we develop here, where the B.f.s. is the domain space of the operator.

One of the main aims of this chapter is to isolate a class of operators $T \in \mathcal{L}(X(\mu), E)$ which admit factorizations through *both* $L^p(m_T)$ and $L^q(\mu)$ simultaneously (for some $1 \le p < \infty$ and $0 < q < \infty$) *without* any convexity assumption on $X(\mu)$. Our main result (Theorem 6.9) provides equivalent factorizations for such an operator T. Moreover, we "recover" the above version of the Maurey–Rosenthal Theorem as a consequence of Theorem 6.9; see Corollary 6.17.

6.1 Basic results

Given $1 \le p < \infty$ and $0 < q < \infty$, recall the definition of the seminorm $\|\cdot\|_{\mathrm{b},\mathrm{X}(\mu)_{[\mathrm{q}]}}$; see (2.25) with $X(\mu)_{[q]}$ in place of $X(\mu)$. Observe (via (2.46) with (q/p) in place of p and Lemma 2.21(iv)) that

$$|f|^{q/p} \in X(\mu)_{[q/p]} \subseteq X(\mu)_{[q]}, \qquad f \in X(\mu). \tag{6.1}$$

Definition 6.1. Let $1 \le p < \infty$ and $0 < q < \infty$. We say that a continuous linear operator $T : X(\mu) \to E$ from a q-B.f.s. $X(\mu)$ into a Banach space E is *bidual (p,q)-power-concave* if there exists a constant $C_1 > 0$ such that

$$\sum_{j=1}^{n} \|T(f_j)\|_E^{q/p} \le C_1 \Big\| \sum_{j=1}^{n} |f_j|^{q/p} \Big\|_{\mathrm{b},X(\mu)_{[q]}}, \qquad f_1, \dots, f_n \in X(\mu), \quad n \in \mathbb{N}. \tag{6.2}$$

The class of all such operators is denoted by $\mathcal{A}_{p,q}(X(\mu), E)$. For $p := 1$, every bidual $(1,q)$-power-concave operator T will be called simply *bidual q-concave*, in which case (6.2) can be written as

$$\Big(\sum_{j=1}^{n} \|T(f_j)\|_E^{q} \Big)^{1/q} \le C_1^{1/q} \Big\| \sum_{j=1}^{n} |f_j|^{q} \Big\|_{\mathrm{b},X(\mu)_{[q]}}^{1/q}, \qquad f_1, \dots, f_n \in X(\mu), \quad n \in \mathbb{N}. \tag{6.3}$$

In order to investigate some basic properties of $\mathcal{A}_{p,q}(X(\mu), E)$, let us define another class of operators. Under the same assumptions on p and q as in Definition 6.1, we say that a continuous linear operator $T : X(\mu) \to E$ is *(p,q)-power-concave* if there is a constant $C_2 > 0$ such that, for all choices of $n \in \mathbb{N}$ and

$f_1, \dots, f_n \in X(\mu)$,

$$\sum_{j=1}^{n} \big\|T(f_j)\big\|_E^{q/p} \le C_2 \Big\| \sum_{j=1}^{n} |f_j|^{q/p} \Big\|_{X(\mu)_{[q]}} \tag{6.4}$$

or equivalently, from the definition of the quasi-norm $\|\cdot\|_{X(\mu)_{[q]}}$, that

$$\begin{aligned} \Big(\sum_{j=1}^{n} \big\|T(f_j)\big\|_E^{q/p} \Big)^{p/q} &\le \Big(C_2 \Big\| \sum_{j=1}^{n} |f_j|^{q/p} \Big\|_{X(\mu)_{[q]}} \Big)^{p/q} \\ &= \big(C_2\big)^{p/q} \Big\| \Big(\sum_{j=1}^{n} |f_j|^{q/p} \Big)^{1/q} \Big\|_{X(\mu)}^{p}. \end{aligned} \tag{6.5}$$

We shall write $\mathcal{B}_{p,q}(X(\mu), E)$ for the class of all such operators. Operators in this class are somewhat more tractable than those in $\mathcal{A}_{p,q}(X(\mu), E)$ because the right-hand side of (6.4) involves the quasi-norm $\|\cdot\|_{X(\mu)_{[q]}}$ which is, at least formally, easier to deal with than the seminorm $\|\cdot\|_{\mathrm{b},X(\mu)_{[q]}}$ which involves the bidual $(X(\mu)_{[q]})^{**}$.

It is clear from (2.104) and (6.5) that the $(1, q)$-power-concave operators and the q-concave operators (from $X(\mu)$ into E) are the same. That is,

$$\mathcal{B}_{1,q}(X(\mu), E) = \mathcal{K}_{(q)}(X(\mu), E). \tag{6.6}$$

As will be seen in Theorem 6.9, operators from $\mathcal{A}_{p,q}(X(\mu), E)$ admit various factorizations whereas this is *not*, in general, the case for elements of $\mathcal{B}_{p,q}(X(\mu), E)$. The usefulness of $\mathcal{B}_{p,q}(X(\mu), E)$ is that it provides a tool which helps to identify various properties of $\mathcal{A}_{p,q}(X(\mu), E)$. To motivate the discussion concerning the relationship between these concepts let us present some basic facts.

Proposition 6.2. *Let $1 \le p < \infty$ and $0 < q < \infty$. The following assertions hold for any σ-order continuous q-B.f.s. $X(\mu)$ and any Banach space E.*

(i) $\mathcal{A}_{p,q}(X(\mu), E) \subseteq \mathcal{B}_{p,q}(X(\mu), E) \subseteq \mathcal{K}_{(q/p)}(X(\mu), E)$.

(ii) *Every (p, q)-power-concave operator from $X(\mu)$ into E is p-th power factorable.*

(iii) *Suppose that $X(\mu)$ is q-concave. Then a linear operator $T \in \mathcal{L}(X(\mu), E)$ is (p, q)-power-concave if and only if it is p-th power factorable.*

(iv) *If $X(\mu)$ is q-convex, then $\mathcal{A}_{p,q}(X(\mu), E) = \mathcal{B}_{p,q}(X(\mu), E)$.*

(v) *Suppose that $X(\mu)$ is both q-convex and q-concave. Then a continuous linear operator $T : X(\mu) \to E$ is bidual (p, q)-power-concave if and only if it is (p, q)-power concave if and only if it is p-th power factorable.*

(vi) *Let Z be a Banach space. Given $T \in \mathcal{A}_{p,q}(X(\mu), E)$ and $S \in \mathcal{L}(E, Z)$, the composition $S \circ T : X(\mu) \to Z$ is again a bidual (p, q)-power-concave operator.*

Proof. (i) The first inclusion follows from (6.2) and (6.4) because of the inequality $\|\cdot\|_{\mathrm{b},X(\mu)_{[q]}} \leq \|\cdot\|_{X(\mu)_{[q]}}$; see (2.27) with $X(\mu)_{[q]}$ in place of $X(\mu)$. Concerning the second inclusion, fix $T \in \mathcal{B}_{p,q}(X(\mu), E)$. To prove that T is (q/p)-concave, we need to exhibit a constant $C_0 > 0$ satisfying

$$\Big(\sum_{j=1}^{n} \|T(f_j)\|_E^{q/p}\Big)^{p/q} \leq C_0 \Big\| \Big(\sum_{j=1}^{n} |f_j|^{q/p}\Big)^{p/q} \Big\|_{X(\mu)} \tag{6.7}$$

whenever $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X(\mu)$. Since T is (p,q)-power-concave, select a constant $C_2 > 0$ satisfying (6.5). The continuous inclusion $X(\mu)_{[1/p]} \subseteq X(\mu)$ is guaranteed by Lemma 2.21(iv). So, there is a constant $C_3 > 0$ such that

$$\|g\|_{X(\mu)} \leq C_3 \|g\|_{X(\mu)_{[1/p]}} = C_3 \big\| |g|^p \big\|_{X(\mu)}^{1/p}, \qquad g \in X(\mu)_{[1/p]}. \tag{6.8}$$

Now, (6.5) and (6.8) with $g := \big(\sum_{j=1}^{n} |f_j|^{q/p}\big)^{1/q} \in X(\mu)_{[1/p]}$ yield that

$$\Big(\sum_{j=1}^{n} \|T(f_j)\|_E^{q/p}\Big)^{p/q} \leq (C_2)^{p/q} (C_3)^p \Big\| \Big(\sum_{j=1}^{n} |f_j|^{q/p}\Big)^{p/q} \Big\|_{X(\mu)},$$

that is, (6.7) holds with $C_0 := (C_2)^{p/q} (C_3)^p$. So, T is indeed (q/p)-concave and hence, the second inclusion in part (i) is established.

(ii) Let $T \in \mathcal{B}_{p,q}\big(X(\mu), E\big)$. Take a constant $C_2 > 0$ satisfying (6.4). Apply (6.4) to a single function $f \in X(\mu)$ to deduce that

$$\big\|T(f)\big\|_E = \Big(\|T(f)\|_E^{q/p}\Big)^{p/q} \leq \Big(C_2 \big\| |f|^{q/p} \big\|_{X(\mu)_{[q]}}\Big)^{p/q} = (C_2)^{p/q} \big\| |f|^{1/p} \big\|_{X(\mu)}^p,$$

which implies that T is p-th power factorable via Lemma 5.3.

(iii) Fix any $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X(\mu)$. With $i_{[p]} : X(\mu) \to X(\mu)_{[p]}$ denoting the canonical injection as given in (5.1), we claim that

$$\sum_{j=1}^{n} \|i_{[p]}(f_j)\|_{X(\mu)_{[p]}}^{q/p} \leq \Big(\mathbf{M}_{(q)}[X(\mu)]\Big)^q \Big\| \sum_{j=1}^{n} |f_j|^{q/p} \Big\|_{X(\mu)_{[q]}}. \tag{6.9}$$

Note, according to (6.1), that the right-hand side of (6.9) is meaningful. Now, observe that $|f_j|^{1/p} \in X(\mu)_{[1/p]} \subseteq X(\mu)$ for each $j = 1, \dots, n$. This and the q-concavity of $X(\mu)$ yield that

$$\begin{aligned}\Big(\sum_{j=1}^{n} \|i_{[p]}(f_j)\|_{X(\mu)_{[p]}}^{q/p}\Big)^{1/q} &= \Big(\sum_{j=1}^{n} \big\| |f_j|^{1/p} \big\|_{X(\mu)}^q\Big)^{1/q} \\ &\leq \Big(\mathbf{M}_{(q)}[X(\mu)]\Big) \Big\| \Big(\sum_{j=1}^{n} |f_j|^{q/p}\Big)^{1/q} \Big\|_{X(\mu)} = \Big(\mathbf{M}_{(q)}[X(\mu)]\Big) \Big\| \sum_{j=1}^{n} |f_j|^{q/p} \Big\|_{X(\mu)_{[q]}}^{1/q},\end{aligned}$$

from which (6.9) follows.

Now let $T \in \mathcal{L}(X(\mu), E)$ be a p-th power factorable operator. Its continuous linear extension $T_{[p]} : X(\mu)_{[p]} \to E$ satisfies $T = T_{[p]} \circ i_{[p]}$ as in (5.2). It follows from (6.9) that

$$\begin{aligned}\sum_{j=1}^{n} \big\|T(f_j)\big\|_E^{q/p} &= \sum_{j=1}^{n} \big\|\big(T_{[p]} \circ i_{[p]}\big)(f_j)\big\|_E^{q/p} \leq \big\|T_{[p]}\big\|^{q/p} \sum_{j=1}^{n} \big\|i_{[p]}(f_j)\big\|_{X(\mu)_{[p]}}^{q/p} \\ &\leq \big\|T_{[p]}\big\|^{q/p} \Big(\mathbf{M}_{(q)}[X(\mu)]\Big)^q \Big\| \sum_{j=1}^{n} |f_j|^{q/p} \Big\|_{X(\mu)_{[q]}}.\end{aligned}$$

Therefore, T is (p, q)-power-concave.

Conversely, every (p, q)-power-concave operator from $X(\mu)$ into E is p-th power factorable via part (ii).

(iv) The q-convex q-B.f.s. $X(\mu)$ admits an equivalent lattice *norm* via Proposition 2.23(iv). Since $X(\mu)_{[q]}$ is also σ-o.c. (see Lemma 2.21(iii)), Proposition 2.13(vi) implies that $\|\cdot\|_{X(\mu)_{[q]}}$ and $\|\cdot\|_{\mathrm{b},X(\mu)_{[q]}}$ are equivalent. An examination of (6.2) and (6.4) then yields the desired equality.

(v) This follows from parts (iii) and (iv).

(vi) Take any constant $C_1 > 0$ satisfying (6.2). Then we have

$$\sum_{j=1}^{n} \big\|(S \circ T)(f_j)\big\|_Z^{q/p} \leq \|S\|^{q/p} \sum_{j=1}^{n} \big\|T(f_j)\big\|_E^{q/p} \leq C_1 \|S\|^{q/p} \Big\| \sum_{j=1}^{n} \big|f_j\big|^{q/p} \Big\|_{\mathrm{b},X(\mu)_{[q]}}$$

whenever $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X(\mu)$. Therefore, $S \circ T \in \mathcal{A}_{p,q}(X(\mu), Z)$. □

Corollary 6.3. *Let $1 \leq p < \infty$ and $0 < q < \infty$. Given are a positive, finite measure space (Ω, Σ, μ), a Banach space E and a function $g \in L^1(\mu)^+$.*

(i) *An E-valued continuous linear operator defined on $L^q(g\,d\mu)$ is bidual (p, q)-power-concave if and only if it is (p, q)-power-concave if and only if it is p-th power factorable.*

(ii) $\mathcal{A}_{p,q}\big(L^q(g\ d\mu), E\big) = \mathcal{B}_{p,q}\big(L^q(g\ d\mu), E\big) = \mathcal{L}\big(L^{q/p}(g\ d\mu),\ E\big) \circ i_{[p]}$ *where* $i_{[p]} : L^q(g\,d\mu) \to L^{q/p}(g\,d\mu) = L^q(g\,d\mu)_{[p]}$ *denotes the canonical injection.*

(iii) *For $p = 1$ we have* $\mathcal{A}_{1,q}\big(L^q(g\,d\mu), E\big) = \mathcal{B}_{1,q}\big(L^q(g\,d\mu), E\big) = \mathcal{L}\big(L^q(g\,d\mu),\ E\big)$.

Proof. Since $L^q(g\,d\mu)$ is both q-convex and q-concave (see Example 2.73(i)), part (i) is an immediate consequence of Proposition 6.2(v).

Part (ii) follows from (i) after recalling that the set $\mathcal{L}\big(L^{q/p}(g\ d\mu),\ E\big) \circ i_{[p]}$ consists exactly of all E-valued, p-th power factorable operators defined on $L^q(g\,d\mu)$; see (5.3) with $X(\mu) := L^q(g\,d\mu)$ and note that $X(\mu)_{[p]} = L^q(g\,d\mu)_{[p]} = L^{q/p}(g\,d\mu)$.

Finally, part (iii) is a special case of (ii) with $p := 1$ (as $i_{[1]} : L^q(g\,d\mu) \to L^q(g\,d\mu)$ is the identity operator). □

Corollary 6.3 provides a canonical example of bidual (p,q)-power-concave operators because a general bidual (p,q)-power-concave operator defined on $X(\mu)$ can always be factorized through $L^q(g\,d\mu)$ for some suitable function $g \in L^1(\mu)$; see Proposition 6.27(iii) below.

It is time to present some examples and comments concerning Proposition 6.2.

Example 6.4. Let (Ω, Σ, μ) be a non-atomic, positive, finite measure space and consider the B.f.s. $X(\mu) := L^r(\mu)$ for any fixed $1 \le r < \infty$. As usual, E is any Banach space.

(i) If $r < p < \infty$ then, for every $0 < q < \infty$, we have that

$$\mathcal{A}_{p,q}\big(X(\mu), E\big) = \mathcal{B}_{p,q}\big(X(\mu), E\big) = \{0\}.$$

This follows from the facts that $\mathcal{A}_{p,q}(X(\mu), E) \subseteq \mathcal{B}_{p,q}(X(\mu), E)$, that every operator in $\mathcal{B}_{p,q}\big(X(\mu), E\big)$ is p-th power factorable (via parts (i) and (ii) of Proposition 6.2) and because every E-valued p-th power factorable operator defined on $X(\mu)$ is necessarily the zero operator (see Example 5.6(ii)).

(ii) If $r < q < \infty$, then $\mathcal{A}_{p,q}\big(L^r(\mu), E\big) = \{0\}$ for every $1 \le p < \infty$. In fact, $X(\mu)_{[q]} = L^{r/q}(\mu)$ has trivial dual (see Example 2.10) and hence, Proposition 2.13(iii) (with $X(\mu)_{[q]}$ in place of $X(\mu)$) implies that $\|\cdot\|_{\mathrm{b},X(\mu)_{[q]}}$ is identically zero. So, every bidual (p,q)-power-concave operator must be zero in view of Definition 6.1.

(iii) Assume that $1 \le p \le r < q < \infty$. Part (ii) gives $\mathcal{A}_{p,q}\big(X(\mu), E\big) = \{0\}$. On the other hand, the set of all E-valued, p-th power factorable operators is non-trivial; see Example 5.6(iii). Moreover, since $X(\mu) = L^r(\mu)$ is q-concave (see Example 2.73(i)), this set equals $\mathcal{B}_{p,q}(X(\mu), E)$ via Proposition 6.2(iii). In particular,

$$\mathcal{A}_{p,q}\big(X(\mu), E\big) \ne \mathcal{B}_{p,q}\big(X(\mu), E\big),$$

so that the first inclusion in part (i) of Proposition 6.2 is strict. □

The second inclusion in part (i) of Proposition 6.2 may also be strict, in general.

Example 6.5. Suppose that (Ω, Σ, μ) is as in the previous Example 6.4 and $E \ne \{0\}$ is a Banach space. Let p, q and r be positive numbers satisfying

$$1 \le r < p < \infty \qquad \text{and} \qquad p\,r \le q < \infty.$$

It follows from Example 6.4(i) that $\mathcal{B}_{p,q}\big(L^r(\mu), E\big) = \{0\}$. On the other hand,

$$\mathcal{K}_{(q/p)}\big(L^r(\mu), E\big) = \mathcal{L}(L^r(\mu), E) \tag{6.10}$$

via Corollary 2.69 because $(q/p) \ge r$ gives that $L^r(\mu)$ is (q/p)-concave (see Example 2.73(i-b)). But, $\mathcal{L}(L^r(\mu), E) \ne \{0\}$; for instance, for any $e \in E \setminus \{0\}$, the non-zero (even μ-determined) operator $f \mapsto \big(\int_\Omega f\,d\mu\big)e$ belongs to $\mathcal{L}(L^r(\mu), E)$. So, we can conclude that

$$\mathcal{B}_{p,q}\big(L^r(\mu), E\big) \ne \mathcal{K}_{(q/p)}\big(L^r(\mu), E\big). \qquad \square$$

We shall present a characterization of (p,q)-power-concave operators in Proposition 6.34 (see Section 6.3). This characterization will then enable us to exhibit a p-th power factorable operator which is not (p,q)-power-concave (see Example 6.35) and hence, the converse statement of part (ii) of Proposition 6.2 is not valid, in general. Such an operator has to be defined on a non-q-concave q-B.f.s. and hence, we cannot remove the assumption that $X(\mu)$ is q-concave in part (iii) of Proposition 6.2. Nevertheless, the arguments in the proof of part (iii) of Proposition 6.2 will now be used to characterize q-concavity of a general σ-order continuous q-B.f.s. $X(\mu)$ in terms of the natural inclusion $i_{[p]} : X(\mu) \to X(\mu)_{[p]}$ for $1 \le p < \infty$.

Remark 6.6. Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ) and E be a Banach space. Suppose that $0 < q < \infty$.

(i) The following three conditions are equivalent.

(a) The q-B.f.s. $X(\mu)$ is q-concave.

(b) For every $1 \le p < \infty$, there exists a constant $C > 0$ such that the natural inclusion $i_{[p]} : X(\mu) \to X(\mu)_{[p]}$ satisfies

$$\sum_{j=1}^{n} \big\| i_{[p]}(f_j) \big\|_{X(\mu)_{[p]}}^{q/p} \le C \Big\| \sum_{j=1}^{n} \big|f_j\big|^{q/p} \Big\|_{X(\mu)_{[q]}}, \quad f_1, \dots, f_n \in X(\mu), \quad n \in \mathbb{N}. \tag{6.11}$$

(c) For some $1 \le p < \infty$ there exists $C > 0$ such that $i_{[p]}$ satisfies (6.11).

Let us verify this. The implication (a) $\Rightarrow$ (b), with $C := \big(\mathbf{M}_{(q)}[X(\mu)]\big)^q$, is already established in the proof of part (iii) of Proposition 6.2; see (6.9). There it was also noted that the right-hand side of (6.11) is always finite, that is, $\sum_{j=1}^{n} \big|f_j\big|^{q/p} \in X(\mu)_{[q]}$.

Since the implication (b) $\Rightarrow$ (c) is clear, it remains to prove the implication (c) $\Rightarrow$ (a). So, let p and $C > 0$ be as in part (c). Given $n \in \mathbb{N}$ and arbitrary $s_1, \dots, s_n \in \operatorname{sim} \Sigma$, we first claim that

$$\Big(\sum_{j=1}^{n} \|s_j\|_{X(\mu)}^{q} \Big)^{1/q} \le C^{1/q} \Big\| \Big(\sum_{j=1}^{n} |s_j|^q \Big)^{1/q} \Big\|_{X(\mu)}. \tag{6.12}$$

In fact, since $|s_j|^p \in X(\mu) \subseteq X(\mu)_{[p]}$ for each $j = 1, \dots, n$ (see Lemma 2.21(iv)), condition (c) implies via (6.11) that

$$\begin{aligned} \Big(\sum_{j=1}^{n} \|s_j\|_{X(\mu)}^{q} \Big)^{1/q} &= \Big(\sum_{j=1}^{n} \big\| |s_j|^p \big\|_{X(\mu)_{[p]}}^{q/p} \Big)^{1/q} = \Big(\sum_{j=1}^{n} \big\| i_{[p]}(|s_j|^p) \big\|_{X(\mu)_{[p]}}^{q/p} \Big)^{1/q} \\ &\le \Big(C \Big\| \sum_{j=1}^{n} \big(|s_j|^p\big)^{q/p} \Big\|_{X(\mu)_{[q]}} \Big)^{1/q} = C^{1/q} \Big\| \Big(\sum_{j=1}^{n} |s_j|^q \Big)^{1/q} \Big\|_{X(\mu)}. \end{aligned}$$

Thus, (6.12) holds.

Now, consider arbitrary functions $f_1, \dots, f_n \in X(\mu)$ with $n \in \mathbb{N}$. Let $K \geq 1$ be any constant appearing in (Q3) of Chapter 2 for the quasi-norm $\|\cdot\|_{X(\mu)}$ and let $r > 0$ be the number determined by $2^r = K$ (see the discussion prior to Proposition 2.2). To establish (a) it suffices to show that

$$\Big(\sum_{j=1}^{n} \|f_j\|_{X(\mu)}^q\Big)^{1/q} \leq 16^{1/r} C^{1/q} \Big\| \Big(\sum_{j=1}^{n} |f_j|^q\Big)^{1/q} \Big\|_{X(\mu)}, \tag{6.13}$$

where we note that the right-hand side of (6.13) is finite because

$$\Big\| \Big(\sum_{j=1}^{n} |f_j|^q\Big)^{1/q} \Big\|_{X(\mu)} = \Big\| \sum_{j=1}^{n} |f_j|^q \Big\|_{X(\mu)_{[q]}}^{1/q}$$

with $|f_j|^q \in X(\mu)_{[q]}$ for $j = 1, \dots, n$. To this end, we may assume that $f_j \geq 0$ for $j = 1, \dots, n$ because $\|\cdot\|_{X(\mu)}$ is a lattice quasi-norm. Given any fixed $j = 1, \dots, n$, we can select a sequence $\{s_j^{(k)}\}_{k=1}^{\infty} \subseteq (\operatorname{sim}\Sigma)^+$ such that $s_j^{(k)} \uparrow f_j$ pointwise (relative to k). So, $\lim_{k\to\infty} \|s_j^{(k)}\|_{X(\mu)} \leq \|f_j\|_{X(\mu)} < \infty$. Moreover, since $X(\mu)$ is σ-o.c., we have $\lim_{k\to\infty} s_j^{(k)} = f_j$ in the q-B.f.s. $X(\mu)$. Then Proposition 2.2(vi) implies that $\|f_j\|_{X(\mu)} \leq 4^{1/r} \lim_{k\to\infty} \|s_j^{(k)}\|_{X(\mu)}$. Thus

$$\Big(\sum_{j=1}^{n} \|f_j\|_{X(\mu)}^q\Big)^{1/q} \leq 4^{1/r} \lim_{k\to\infty} \Big(\sum_{j=1}^{n} \|s_j^{(k)}\|_{X(\mu)}^q\Big)^{1/q}. \tag{6.14}$$

Next, since $\big(\sum_{j=1}^{n} |s_j^{(k)}|^q\big)^{1/q} \uparrow \big(\sum_{j=1}^{n} |f_j|^q\big)^{1/q}$ pointwise (relative to k) it follows, again because $X(\mu)$ is σ-o.c. with $\big(\sum_{j=1}^{n} |f_j|^q\big)^{1/q} \in X(\mu)$, that

$$\lim_{k\to\infty} \Big(\sum_{j=1}^{n} |s_j^{(k)}|^q\Big)^{1/q} = \Big(\sum_{j=1}^{n} |f_j|^q\Big)^{1/q}$$

in the topology of the q-B.f.s. $X(\mu)$. So, Proposition 2.2(vi) once more implies that

$$\lim_{k\to\infty} \Big\| \Big(\sum_{j=1}^{n} |s_j^{(k)}|^q\Big)^{1/q} \Big\|_{X(\mu)} \leq 4^{1/r} \Big\| \Big(\sum_{j=1}^{n} |f_j|^q\Big)^{1/q} \Big\|_{X(\mu)}. \tag{6.15}$$

Since (6.12) holds for $\{s_j^{(k)}\}_{j=1}^{n} \subseteq \operatorname{sim}\Sigma$, for each $k \in \mathbb{N}$, it follows from (6.14) and (6.15) that

$$\begin{aligned}\Big(\sum_{j=1}^{n} \|f_j\|_{X(\mu)}^q\Big)^{1/q} &\leq 4^{1/r} \lim_{k\to\infty} \Big(\sum_{j=1}^{n} \|s_j^{(k)}\|_{X(\mu)}^q\Big)^{1/q} \\ &\leq 4^{1/r} C^{1/q} \lim_{k\to\infty} \Big\| \Big(\sum_{j=1}^{n} |s_j^{(k)}|^q\Big)^{1/q} \Big\|_{X(\mu)} \leq 16^{1/r} C^{1/q} \Big\| \Big(\sum_{j=1}^{n} |f_j|^q\Big)^{1/q} \Big\|_{X(\mu)}.\end{aligned}$$

So, (6.13) holds and hence, $X(\mu)$ is q-concave.

(ii) We have not yet defined (p,q)-power-concavity for a linear operator T on $X(\mu)$ taking values in a quasi-Banach space $(E, \|\cdot\|_E)$. However, we can easily extend the previous definition (given for a Banach space E) to this setting. The terms $\|T(f_j)\|_E$ in the left-hand side of (6.4) now denote the quasi-norm $\|\cdot\|_E$ of E applied to each vector $T(f_j) \in E$ for $j = 1, \ldots, n$. It then makes sense to speak about the (p,q)-power-concavity of $i_{[p]} : X(\mu) \to X(\mu)_{[p]}$, even when $X(\mu)_{[p]}$ is not a B.f.s. but only a q-B.f.s. Part (i) above can then be rephrased as follows: statement (a) is equivalent to each of the assertions

(b′) $i_{[p]} : X(\mu) \to X(\mu)_{[p]}$ is (p,q)-power-concave for every $1 \le p < \infty$, and

(c′) $i_{[p]} : X(\mu) \to X(\mu)_{[p]}$ is (p,q)-power-concave for some $1 \le p < \infty$,

respectively. □

Remark 6.7. Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ) and E be a Banach space. Suppose that $1 \le p_1 \le p_2$ and $0 < q < \infty$.

(i) The inclusion $\mathcal{A}_{p_2,q}(X(\mu), E) \subseteq \mathcal{A}_{p_1,q}(X(\mu), E)$ always holds (see Corollary 6.16 below).

(ii) If $q \le r < \infty$, then the inclusion $\mathcal{B}_{p_2,q}(X(\mu), E) \subseteq \mathcal{B}_{p_1,r}(X(\mu), E)$ always holds (see Corollary 6.38 below). However, the corresponding containment $\mathcal{A}_{p_2,q}(X(\mu), E) \subseteq \mathcal{A}_{p_1,r}(X(\mu), E)$ need not hold, in general, even when $p_1 = p_2$, as we show now.

(iii) Assume that μ is non-atomic and $E \neq \{0\}$. Let $1 \le p \le q < r$ and $X(\mu) := L^q(\mu)$. Then $\mathcal{A}_{p,r}(X(\mu), E) = \{0\}$ by Example 6.4(ii) (with q and r interchanged). However, it follows from Proposition 6.2(v) that $\mathcal{A}_{p,q}(X(\mu), E)$ is equal to the class of all p-th power factorable operators from $X(\mu)$ into E (because $X(\mu) = L^q(\mu)$ is both q-convex and q-concave; see Example 2.73(i)). This latter class is always non-trivial because it contains $\mathcal{F}_{[p]}(L^q(\mu), E)$ and $p \le q$ (see Example 5.6(iii)). □

According to the following remark, a *non-μ-determined*, bidual (p,q)-power-concave operator T can always be treated as if it were μ-determined; one simply restricts T to its essential carrier.

Remark 6.8. Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ) and E be a Banach space. Given $T \in \mathcal{L}(X(\mu), E)$, let Ω_1 be an essential carrier of T as defined after Proposition 4.28. By setting $\mu_1 := \mu|_{\Sigma \cap \Omega_1}$, the complemented subspace

$$X(\mu_1) = \chi_{\Omega_1} \cdot X(\mu) = \{f\chi_{\Omega_1} : f \in X(\mu)\}$$

of $X(\mu)$ can be considered, in a natural way, as a q-B.f.s. over the finite measure space $(\Omega_1, \Sigma \cap \Omega_1, \mu_1)$ with

$$X(\mu_1)_{[q]} = \chi_{\Omega_1} \cdot X(\mu)_{[q]}, \qquad 0 < q < \infty.$$

Moreover, the restriction T_1 of T to $X(\mu_1)$ is μ_1-determined (and still E-valued); see Proposition 4.28.

Suppose that $1 \le p < \infty$ and $0 < q < \infty$. Let us show that $T \in \mathcal{A}_{p,q}(X(\mu), E)$ if and only if $T_1 \in \mathcal{A}_{p,q}(X(\mu_1), E)$. First, given $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in X(\mu)$, recall that $|f_j|^{q/p} \in X(\mu)_{[q]}$ for each $j = 1, \ldots, n$; see (6.1). Moreover,

$$\sum_{j=1}^{n} \big|f_j \chi_{\Omega_1}\big|^{q/p} = \chi_{\Omega_1}\Big(\sum_{j=1}^{n} |f_j|^{q/p}\Big) \in \chi_{\Omega_1} \cdot X(\mu)_{[q]} = X(\mu_1)_{[q]}$$

and hence, (2.36) with $X(\mu)_{[q]}$ in place of $X(\mu)$ and $A := \Omega_1$ implies that

$$\Big\|\sum_{j=1}^{n} \big|f_j \chi_{\Omega_1}\big|^{q/p}\Big\|_{\mathrm{b},X(\mu_1)_{[q]}} = \Big\|\sum_{j=1}^{n} \big|f_j \chi_{\Omega_1}\big|^{q/p}\Big\|_{\mathrm{b},X(\mu)_{[q]}}. \tag{6.16}$$

Now, assume that $T \in \mathcal{A}_{p,q}\big(X(\mu), E\big)$ and take any constant $C_1 > 0$ satisfying (6.2). Fix $n \in \mathbb{N}$ and $g_1, \ldots, g_n \in X(\mu_1)$, in which case $g_j = f_j \chi_{\Omega_1}$ for some $f_j \in X(\mu)$ and $j = 1, \ldots, n$. From (6.2) and (6.16), with $f_j \chi_{\Omega_1}$ in place of f_j for $j = 1, \ldots, n$, it follows that

$$\begin{aligned}\sum_{j=1}^{n} \big\|T_1(g_j)\big\|_E^{q/p} &= \sum_{j=1}^{n} \big\|T(f_j \chi_{\Omega_1})\big\|_E^{q/p} \\ &\le C_1 \Big\|\sum_{j=1}^{n} \big|f_j \chi_{\Omega_1}\big|^{q/p}\Big\|_{\mathrm{b},X(\mu)_{[q]}} = C_1 \Big\|\sum_{j=1}^{n} \big|g_j\big|^{q/p}\Big\|_{\mathrm{b},X(\mu_1)_{[q]}}.\end{aligned}$$

So, $T_1 \in \mathcal{A}_{p,q}(X(\mu_1), E)$.

Conversely, assume that $T_1 \in \mathcal{A}_{p,q}\big(X(\mu_1), E\big)$. Then there exists a constant $C > 0$ such that

$$\sum_{j=1}^{n} \big\|T_1(g_j)\big\|_E^{q/p} \le C \Big\|\sum_{j=1}^{n} \big|g_j\big|^{q/p}\Big\|_{\mathrm{b},X(\mu_1)_{[q]}}$$

for all $n \in \mathbb{N}$ and $g_1, \ldots, g_n \in X(\mu_1)$. Given $f_1, \ldots, f_n \in X(\mu)$, we have that $g_j := f_j \chi_{\Omega_1} \in X(\mu_1)$ for each $j = 1, \ldots, n$. Then the previous inequality and (6.16) imply that

$$\begin{aligned}\sum_{j=1}^{n} \big\|T(f_j)\big\|_E^{q/p} &= \sum_{j=1}^{n} \big\|T(g_j)\big\|_E^{q/p} = \sum_{j=1}^{n} \big\|T_1(g_j)\big\|_E^{q/p} \le C \Big\|\sum_{j=1}^{n} \big|g_j\big|^{q/p}\Big\|_{\mathrm{b},X(\mu_1)_{[q]}} \\ &= C \Big\|\sum_{j=1}^{n} \big|f_j \chi_{\Omega_1}\big|^{q/p}\Big\|_{\mathrm{b},X(\mu)_{[q]}} \le C \Big\|\sum_{j=1}^{n} \big|f_j\big|^{q/p}\Big\|_{\mathrm{b},X(\mu)_{[q]}},\end{aligned}$$

where we have used the facts that $T(f_j) = T(f_j \chi_{\Omega_1})$ for $j = 1, \ldots, n$ (see Proposition 4.28(i)) and that $\|\cdot\|_{\mathrm{b},X(\mu)_{[q]}}$ is a lattice seminorm via Proposition 2.13(ii). Therefore, $T \in \mathcal{A}_{p,q}(X(\mu), E)$. $\square$

6.2 Generalized Maurey–Rosenthal factorization theorems

In this section we present a generalized Maurey–Rosenthal Factorization Theorem for p-th power factorable operators. We show that the vector norm inequality (6.2), as given in the definition of bidual (p,q)-power-concave operators, is equivalent to two factorization schemes (of a quite different nature) which, in turn, can also be formulated as inclusions between appropriate spaces. The first scheme is related to the known factorization through $L^p(m_T)$ for p-th power factorable operators $T : X(\mu) \to E$ as presented in Chapter 5, whereas the second one is based on factorization of the operator through $X(\mu)_{[p]}$.

Let us motivate the result with an example.

Let E be a Banach space, $1 < p \leq q < \infty$, $X(\mu) := L^q(\mu)$ for a positive, finite measure μ and $T \in \mathcal{F}_{[p]}\big(X(\mu), E\big)$. Then $X(\mu)_{[p]} = L^{q/p}(\mu)$ and the following diagram commutes

$$\begin{tikzcd} X(\mu) = L^q(\mu) \arrow[rr] \arrow[d] \arrow[drr, "J_T"] & & L^p(m_T) \arrow[d] \\ X(\mu)_{[p]} = L^{q/p}(\mu) \arrow[rr] & & L^1(m_T) \end{tikzcd}$$

with each unlabelled arrow indicating the respective inclusion map and J_T denoting the inclusion of the domain space $X(\mu)$ of T into its optimal domain $L^1(m_T)$; see Theorem 5.7. Both factorizations in the diagram are, in a sense, natural for this particular setting.

The following result not only provides a generalized Maurey–Rosenthal type theorem but also, for a μ-determined operator T on a σ-order continuous q-B.f.s. $X(\mu)$, tells us precisely when the inclusion map J_T from $X(\mu)$ into $L^1(m_T)$ factors through a scheme analogous to the one above (just by replacing μ with a suitable measure arising as an indefinite integral $\mu_g : A \mapsto \int_A g\, d\mu$ for $A \in \Sigma$). The requirement that T is μ-determined is assumed throughout this section. This is not a genuine restriction. Indeed, if T fails to satisfy this condition, then we can instead consider the restriction of T to its essential carrier as noted in Remark 6.8.

Theorem 6.9. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ). Let E be a Banach space and $T \in \mathcal{L}(X(\mu), E)$ be μ-determined. For any $1 \leq p < \infty$ and $0 < q < \infty$, the following assertions are equivalent.*

(i) *There exists a constant* $C > 0$ *such that*

$$\Big(\sum_{j=1}^{n} \|T(f_j)\|_E^{q/p}\Big)^{1/q} \le C \Big\| \sum_{j=1}^{n} |f_j|^{q/p} \Big\|_{\mathrm{b},X(\mu)_{[q]}}^{1/q}$$

for all $n \in \mathbb{N}$ *and* $f_1, \dots, f_n \in X(\mu)$*; namely,* T *is bidual* (p,q)*-power-concave.*

(ii) *There exists a function* $g \in \big(X(\mu)_{[q]}\big)'$ *with* $g > 0$ (μ*-a.e.) such that*

$$\|T(f)\|_E \le \Big(\int_\Omega |f|^{q/p} g \, d\mu\Big)^{p/q} < \infty, \qquad f \in X(\mu). \tag{6.17}$$

(iii) *There exists* $g \in L^0(\mu)$ *with* $g > 0$ (μ*-a.e.) such that the inclusions*

$$X(\mu) \subseteq L^q(g\,d\mu) \subseteq L^p(m_T)$$

hold and are continuous.

(iv) *There exists* $g \in L^0(\mu)$ *with* $g > 0$ (μ*-a.e.) such that the inclusions*

$$X(\mu)_{[p]} \subseteq L^{q/p}(g\,d\mu) \subseteq L^1(m_T)$$

hold and are continuous.

(v) $T \in \mathcal{F}_{[p]}(X(\mu), E)$ *and there exist a* Σ*-measurable function* $g > 0$ (μ*-a.e.) such that* $g^{p/q} \in \mathcal{M}\big(X(\mu)_{[p]}, L^{q/p}(\mu)\big)$ *and a* μ*-determined operator* $S \in \mathcal{L}(L^{q/p}(\mu), E)$ *satisfying* $T_{[p]} = S \circ \mathrm{M}_{g^{p/q}}$*. That is, the following diagram commutes:*

$$\begin{tikzcd}
X(\mu) \arrow[r, "T"] \arrow[d, "i_{[p]}"'] & E \\
X(\mu)_{[p]} \arrow[r, "M_{g^{p/q}}"] \arrow[ur, "T_{[p]}"] & L^{q/p}(\mu). \arrow[u, "S"']
\end{tikzcd}$$

(vi) $T \in \mathcal{F}_{[p]}\big(X(\mu), E\big)$ *and the natural inclusion map* $J_T^{(p)} : X(\mu) \to L^p(m_T)$ *is bidual* q*-concave.*

(vii) $T \in \mathcal{F}_{[p]}\big(X(\mu), E\big)$ *and the natural inclusion map* $\beta_{[p]} : X(\mu)_{[p]} \to L^1(m_T)$ *is bidual* (q/p)*-concave.*

(viii) $T \in \mathcal{F}_{[p]}\big(X(\mu), E\big)$ *and the extension* $T_{[p]} : X(\mu)_{[p]} \to E$ *of* T *is bidual* (q/p)*-concave.*

Let us make some comments concerning Theorem 6.9.

Remark 6.10. (I) If a μ-determined operator $T \in \mathcal{L}(X(\mu), E)$ satisfies the equivalent conditions (iii) and (iv), then the following diagram commutes

$$\begin{tikzcd}
X(\mu) \arrow[r] \arrow[d] \arrow[rrd, "J_T"] & L^q(g\,d\mu) \arrow[r] & L^p(m_T) \arrow[d] \\
X(\mu)_{[p]} \arrow[r] & L^{q/p}(g\,d\mu) \arrow[r] & L^1(m_T)
\end{tikzcd}$$

with each unlabeled arrow indicating the respective inclusion map. So we have two factorizations. Namely, the upper diagram $X(\mu) \to L^q(g\ d\mu) \to L^p(m_T)$ which provides a factorization of the inclusion $X(\mu) \subseteq L^p(m_T)$, that is, of the map $J_T^{(p)}$, and the lower diagram $X(\mu)_{[p]} \to L^{q/p}(g\,d\mu) \to L^1(m_T)$ which provides a factorization of the inclusion $X(\mu)_{[p]} \subseteq L^1(m_T)$. Statement (iii) of Theorem 6.9 corresponds exactly to the upper scheme, while statement (v) gives the lower one. This creates further possibilities for determining different properties of T (by using either one of the two factorizations). Note that the four corner spaces in the above diagram are already known to us for p-th power factorable operators (see Remark 5.8(II)(ii)). A new feature is the factorizations through the additional spaces $L^q(g d\mu)$ and $L^{q/p}(g\,d\mu)$, for some appropriate g. By inserting $f = \chi_\Omega$ into (6.17) we see that necessarily $g \in L^1(\mu)$. If $0 < q < 1$, then both $L^q(g\,d\mu)$ and $L^{q/p}(g\,d\mu)$ are, of course, q-B.f.s.'

(II) For every function $g \in L^0(\mu)$ with $g > 0$ (μ-a.e.), the multiplication operator $M_{g^{1/q}} : f \mapsto g^{1/q}f$, for $f \in L^q(g\ d\mu)$, is a surjective linear map from $L^q(g\,d\mu)$ onto $L^q(\mu)$ which satisfies $\|M_{g^{1/q}}f\|_{L^q(\mu)} = \|f\|_{L^q(g\,d\mu)}$, for $f \in L^q(g\,d\mu)$. Here $\|\cdot\|_{L^q(\mu)}$ and $\|\cdot\|_{L^q(gd\mu)}$ denote the usual quasi-norms in L^q-spaces for $0 < q < \infty$. In particular, $M_{g^{1/q}}$ is a bicontinuous isomorphism of $L^q(g\,d\mu)$ onto $L^q(\mu)$. Accordingly, an equivalent formulation of condition (iii) of Theorem 6.9 is commutativity of the diagram:

$$\begin{tikzcd}
X(\mu) \arrow[rr, "J_T^{(p)}"] \arrow[dr, "\mathrm{M}_{g^{1/q}}"'] & & L^p(m_T)\,. \\
& L^q(\mu) \arrow[ur, "\mathrm{M}_{g^{-1/q}}"'] &
\end{tikzcd}$$

(III) In condition (vi) (respectively, (vii)) the assumption $T \in \mathcal{F}_{[p]}(X(\mu), E)$ implies that the linear map $J_T^{(p)}$ (respectively, $\beta_{[p]}$) is defined because Theorem 5.7 ensures that $X(\mu) \subseteq L^p(m_T)$ (respectively, $X(\mu)_{[p]} \subseteq L^1(m_T)$).

(IV) The equivalences of (i) with (vi)–(viii) indicate essentially that we can separate p-th power factorability and q-concavity type properties of a μ-determined operator when characterizing its bidual (p, q)-power-concavity. $\square$

We need some preparatory results in order to establish Theorem 6.9.

Definition 6.11. A family Ψ of $\mathbb{R}$-valued functions defined on a non-empty set W is called *concave* if, for every finite set $\{\psi_1, \dots, \psi_n\} \subseteq \Psi$ with $n \in \mathbb{N}$ and non-negative scalars $c_1, \dots, c_n$ satisfying $\sum_{j=1}^n c_j = 1$, there exists $\psi \in \Psi$ such that

$$\sum_{j=1}^n c_j \psi_j \le \psi,$$

pointwise on W.

Lemma 6.12. *Let W be a compact convex subset of a Hausdorff topological vector space and let Ψ be a concave family of lower semi-continuous, convex, $\mathbb{R}$-valued functions on W. Let $c \in \mathbb{R}$. Suppose, for every $\psi \in \Psi$, that there exists $x_\psi \in W$ such that $\psi(x_\psi) \le c$. Then there exists $x \in W$ such that $\psi(x) \le c$ for all $\psi \in \Psi$.*

The previous result is known as Ky Fan's Lemma; see, for example, [127, E.4].

Lemma 6.13. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ) and E be a Banach space. For any $1 \le p < \infty$ and $0 < q < \infty$, the following assertions for an operator $T \in \mathcal{L}(X(\mu), E)$ are equivalent.*

(i) *The operator T is bidual (p, q)-power-concave, that is, there exists a constant $C_1 > 0$ satisfying*

$$\sum_{j=1}^n \big\|T(f_j)\big\|_E^{q/p} \le C_1 \Big\| \sum_{j=1}^n |f_j|^{q/p} \Big\|_{\mathrm{b}, X(\mu)_{[q]}}, \qquad f_1, \dots, f_n \in X(\mu), \quad n \in \mathbb{N}.$$

(ii) *There exists a non-negative function $g \in \big(X(\mu)_{[q]}\big)'$ satisfying* (6.17).

Proof. (i) $\Rightarrow$ (ii). Endow the convex subset

$$\mathbf{B}^+\big[(X(\mu)_{[q]})^*\big] := \mathbf{B}\big[(X(\mu)_{[q]})^*\big] \cap \big((X(\mu)_{[q]})^*\big)^+$$

of $(X(\mu)_{[q]})^*$ with the relative weak* topology, in which case $\mathbf{B}^+\big[(X(\mu)_{[q]})^*\big]$ is compact; see the proof of Proposition 2.13(vii) with $X(\mu)_{[q]}$ in place of $X(\mu)$.

Fix $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X(\mu)$, in which case $\sum_{j=1}^n \big|f_j\big|^{q/p} \in X(\mu)_{[q]}$; see (6.1). Consequently, the function $\psi_{f_1,\dots,f_n} : \mathbf{B}^+[(X(\mu)_{[q]})^*] \to \mathbb{R}$ defined by

$$\psi_{f_1,\dots,f_n}(\xi) := \sum_{j=1}^n \big\|T(f_j)\big\|_E^{q/p} - C_1 \Big\langle \sum_{j=1}^n |f_j|^{q/p}, \ \xi \Big\rangle, \qquad \xi \in \mathbf{B}^+\big[(X(\mu)_{[q]})^*\big],$$

is continuous. Moreover, given $0 \le a \le 1$, we have

$$\psi_{f_1,\dots,f_n}\big(a\xi + (1-a)\eta\big) = a \cdot \psi_{f_1,\dots,f_n}(\xi) + (1-a) \cdot \psi_{f_1,\dots,f_n}(\eta),$$

for $\xi, \eta \in \mathbf{B}^+\big[(X(\mu)_{[q]})^*\big]$, and so the function $\psi_{f_1,\dots,f_n}$ is convex. Note that $X(\mu)_{[q]}$ is σ-o.c.; see Lemma 2.21(iii). So, by Proposition 2.13(vii) with $X(\mu)_{[q]}$ in place of $X(\mu)$ and $\sum_{j=1}^{n} |f_j|^{q/p}$ in place of f, there exists $\xi_1 \in \mathbf{B}^+\big[(X(\mu)_{[q]})^*\big]$, which depends on $f_1, \ldots, f_n$, such that

$$\Big\langle \sum_{j=1}^{n} |f_j|^{q/p},\ \xi_1 \Big\rangle \;=\; \Big\| \sum_{j=1}^{n} |f_j|^{q/p} \Big\|_{\mathrm{b}, X(\mu)_{[q]}}.$$

This and (6.2) yield that

$$\psi_{f_1 \ldots, f_n}(\xi_1) \;\le\; 0. \tag{6.18}$$

Next we claim that the collection

$$\Psi := \big\{ \psi_{f_1, \ldots, f_n} : f_1 \ldots, f_n \in X(\mu), n \in \mathbb{N} \big\}$$

of $\mathbb{R}$-valued continuous, convex functions on the compact set $\mathbf{B}^+[(X(\mu)_{[q]})^*]$ is concave in the sense of Definition 6.11. In fact, given numbers $c_j \in [0,1]$, for $j = 1, \ldots, n$, with $n \in \mathbb{N}$ and $\sum_{j=1}^{n} c_j = 1$ and finite collections of functions $\big\{f_1^{(j)}, \ldots, f_{k(j)}^{(j)}\big\} \subseteq X(\mu)$ with $k(j) \in \mathbb{N}$, for $j = 1, \ldots, n$, direct calculation shows that

$$\sum_{j=1}^{n} c_j \, \psi_{f_1^{(j)}, \ldots, f_{k(j)}^{(j)}} \;=\; \psi_{c_1^{p/q} f_1^{(1)}, \ldots, c_1^{p/q} f_{k(1)}^{(1)}, \ldots, c_n^{p/q} f_1^{(n)}, \ldots, c_n^{p/q} f_{k(n)}^{(n)}},$$

from which it follows that Ψ is a concave family.

Now Lemma 6.12, with $W := \mathbf{B}^+[(X(\mu)_{[q]})^*]$ and $c := 0$ and in combination with (6.18), implies that there exists an element $\xi_0 \in \mathbf{B}^+[(X(\mu)_{[q]})^*]$ satisfying

$$\psi_{f_1, \ldots, f_n}(\xi_0) \;\le\; 0 \qquad \text{whenever} \quad n \in \mathbb{N} \quad \text{and} \quad f_1, \ldots, f_n \in X(\mu).$$

Apply this to each single function $f \in X(\mu)$ to obtain that $\psi_f(\xi_0) \le 0$, that is,

$$\big\|T(f)\big\|_E^{q/p} \;\le\; C_1 \big\langle |f|^{q/p},\ \xi_0 \big\rangle \;<\; \infty. \tag{6.19}$$

Since $X(\mu)_{[q]}$ is σ-o.c. (see Lemma 2.21(iii)), we can identify $\xi_0 \in \mathbf{B}^+[(X(\mu)_{[q]})^*]$ with a non-negative function $h \in \big(X(\mu)_{[q]}\big)'$ via (2.37) (with h in the role of g): see Proposition 2.16(ii). So, we can rewrite the inequality (6.19) as

$$\big\|T(f)\big\|_E^{q/p} \le C_1 \int_\Omega |f|^{q/p} h \, d\mu < \infty. \tag{6.20}$$

This is precisely (ii) with $g := C_1 h \ge 0$.

(ii) $\Rightarrow$ (i). The function $g \in \big(X(\mu)_{[q]}\big)'$ corresponds to the element $\xi_g \in \big(X(\mu)_{[q]}\big)^*$ via (2.37) with $X(\mu)_{[q]}$ in place of $X(\mu)$. Let C_1 denote the norm of ξ_g

in $\big(X(\mu)_{[q]}\big)^*$. Given $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in X(\mu)$, it follows from (6.17) that

$$\begin{aligned}
&\sum_{j=1}^{n} \big\|T(f_j)\big\|_E^{q/p} \le \sum_{j=1}^{n} \int_\Omega |f_j|^{q/p} g\, d\mu = \int_\Omega \Big(\sum_{j=1}^{n} |f_j|^{q/p}\Big) g\, d\mu \\
&= C_1 \Big\langle \sum_{j=1}^{n} |f_j|^{q/p},\ C_1^{-1}\xi_g \Big\rangle \le C_1 \sup\Big\{ \Big| \Big\langle \sum_{j=1}^{n} |f_j|^{q/p},\ \xi \Big\rangle \Big| \,:\, \xi \in \mathbf{B}\big[\big(X(\mu)_{[q]}\big)^*\big] \Big\} \\
&= C_1 \Big\| \sum_{j=1}^{n} |f_j|^{q/p} \Big\|_{\mathrm{b}, X(\mu)_{[q]}}.
\end{aligned}$$

So, T is bidual (p,q)-power-concave. $\square$

Before proving Theorem 6.9, let us assume condition (i) of Lemma 6.13 and derive a useful fact from the proof of (i) ⇒ (ii) above. We can identify the non-negative function $h \in (X(\mu)_{[q]})'$ with the functional $\xi_0 \in \mathbf{B}^+[(X(\mu)_{[q]})^*]$. Then, of course, $C_1\xi_0 \in (X(\mu)_{[q]})^*$ is identified with the function $g := C_1 h \in (X(\mu)_{[q]})'$. Now, let us verify that

$$\Big(\int_\Omega |f|^{q/p} g\, d\mu\Big)^{p/q} \le C_1^{p/q} \|f\|_{X(\mu)_{[p]}} \le C_1^{p/q} \big\|i_{[p]}\big\| \cdot \|f\|_{X(\mu)}, \tag{6.21}$$

for $f \in X(\mu) \subseteq X(\mu)_{[p]}$, with $\big\|i_{[p]}\big\|$ denoting the operator norm (see (2.1)) of the canonical injection $i_{[p]} : X(\mu) \to X(\mu)_{[p]}$. Fix $f \in X(\mu)$, in which case $|f|^{q/p} \in X(\mu)_{[q]}$ (see (6.1)). Since $\xi_0 \in \mathbf{B}^+[(X(\mu)_{[q]})^*]$, it follows, from the definition of the dual norm $\|\cdot\|_{(X(\mu)_{[q]})^*}$ (see (2.16) with $X(\mu)_{[q]}$ in place of $X(\mu)$), that

$$\begin{aligned}
&\Big(\int_\Omega |f|^{q/p} g\, d\mu\Big)^{p/q} = \big\langle |f|^{q/p}, C_1\xi_0 \big\rangle^{p/q} \\
&\le \big\| |f|^{q/p} \big\|_{X(\mu)_{[q]}}^{p/q} \cdot \big\|C_1\xi_0\big\|_{(X(\mu)_{[q]})^*}^{p/q} = C_1^{p/q} \big\| |f|^{1/p} \big\|_{X(\mu)}^{p} \cdot \big\|\xi_0\big\|_{(X(\mu)_{[q]})^*}^{p/q} \\
&\le C_1^{p/q} \|f\|_{X(\mu)_{[p]}} = C_1^{p/q} \big\|i_{[p]}(f)\big\|_{X(\mu)_{[p]}} \le C_1^{p/q} \big\|i_{[p]}\big\| \cdot \|f\|_{X(\mu)},
\end{aligned}$$

that is, (6.21) is established. It is important to note that the function g satisfying (6.21) depends on the constant C_1 in condition (i) of Lemma 6.13.

Proof of Theorem 6.9. (i) ⇒ (ii). Observe that condition (i) of Theorem 6.9 is equivalent to condition (i) of Lemma 6.13. So, by (ii) of the same lemma we can find a non-negative function $g \in \big(X(\mu)_{[q]}\big)'$ satisfying (6.17). Then, with $A := g^{-1}(\{0\})$, it follows that

$$\begin{aligned}
\big\|m_T(B)\big\|_E = \big\|T(\chi_B)\big\|_E &\le \Big(\int_\Omega \big|\chi_B\big|^{q/p} g\, d\mu\Big)^{p/q} \\
&= \Big(\int_B g\, d\mu\Big)^{p/q} = 0, \qquad B \in \Sigma \cap A,
\end{aligned}$$

which means precisely that A is m_T-null. Hence, A is also μ-null because T is μ-determined. Thus, $g > 0$ (μ-a.e.). So, (ii) holds.

(ii) $\Rightarrow$ (i). This has already been established in Lemma 6.13.

(ii) $\Rightarrow$ (iv). As noted in Remark 6.10(I) we can conclude that $g \in L^1(\mu)$. Since $g \in (X(\mu)_{[q]})'$, Proposition 2.29 (with q in place of p) implies that $X(\mu) \subseteq L^q(g\,d\mu)$ continuously. Note, with $Y(\mu) := L^{q/p}(g\,d\mu)$, that we always have

$$X(\mu) \subseteq L^q(g\,d\mu) \iff X(\mu)_{[p]} \subseteq Y(\mu), \tag{6.22}$$

with continuous inclusions. Moreover, $1 \le p < \infty$ implies that $X(\mu) \subseteq X(\mu)_{[p]}$ continuously (see Lemma 2.21(iv)) and so $X(\mu) \subseteq Y(\mu)$ continuously. Since (6.17) is then precisely the inequality (4.14), it follows from Corollary 4.16 that $Y(\mu) \subseteq L^1(m_T)$ continuously. So, we have $X(\mu)_{[p]} \subseteq Y(\mu) \subseteq L^1(m_T)$ with continuous inclusions which is precisely condition (iv).

(iv) $\Rightarrow$ (ii). It follows from $\chi_\Omega \in X(\mu)_{[p]} \subseteq L^{q/p}(g\,d\mu)$ that $g \in L^1(\mu)$. Moreover, with $Y(\mu) := L^{q/p}(g\,d\mu)$, the assumed inclusions of condition (iv) together with $X(\mu) \subseteq X(\mu)_{[p]}$ imply that $X(\mu) \subseteq Y(\mu) \subseteq L^1(m_T)$ continuously. Then Corollary 4.16 implies that (6.17) holds. Moreover, via (6.22) and the inclusion $X(\mu)_{[p]} \subseteq L^{q/p}(g\,d\mu)$ we can conclude that $X(\mu) \subseteq L^q(g\,d\mu)$ continuously. Then Proposition 2.29 implies that $g \in (X(\mu)_{[q]})'$. Hence, condition (ii) holds.

(iii) $\Leftrightarrow$ (iv). These statements are clearly equivalent. Indeed, as already observed in (6.22), it is enough to compute the p-th power (or the $(1/p)$-th power) of the spaces to obtain one chain of inclusions from the other one; see Lemma 2.20(ii).

(iv) $\Rightarrow$ (v). First, from Theorem 5.7 we have that $T \in \mathcal{F}_{[p]}(X(\mu), E)$ because (iv) gives $X(\mu)_{[p]} \subseteq L^1(m_T)$. The inclusion $X(\mu)_{[p]} \subseteq L^{q/p}(g\,d\mu)$ allows us to define the multiplication operator $M_{g^{p/q}} : X(\mu)_{[p]} \to L^{q/p}(\mu)$. Now, for every $f \in L^{q/p}(\mu)$, we have $g^{-p/q} f \in L^{q/p}(g\,d\mu) \subseteq L^1(m_T)$. Accordingly, we can define $S(f) := I_{m_T}\big(g^{-p/q} f\big)$. The so-defined linear operator $S : L^{q/p}(\mu) \to E$ is continuous and satisfies $T_{[p]} = S \circ M_{g^{p/q}}$ because S can be viewed as the composition of three continuous linear operators: $M_{g^{-p/q}} : L^{q/p}(\mu) \to L^{q/p}(g\,d\mu)$, the canonical inclusion map from $L^{q/p}(g\,d\mu)$ into $L^1(m_T)$, and the integration operator $I_{m_T} : L^1(m_T) \to E$. To show that S is μ-determined, let $A \in \Sigma$ be an m_S-null set. As noted above, $g^{-p/q} f \in L^1(m_T)$ whenever $f \in L^{q/p}(\mu)$ and hence, with $f := \chi_\Omega$, we see that $g^{-p/q} \in L^1(m_T)$. Then

$$\int_B g^{-p/q} dm_T = S(\chi_B) = m_S(B) = 0, \qquad B \in \Sigma \cap A.$$

This means that $\big(g\chi_A\big)(\omega) = 0$ for m_T-a.e. $\omega \in \Omega$ and hence, for μ-a.e. $\omega \in \Omega$ because T is μ-determined. Since $g > 0$ (μ-a.e.) by assumption, it follows that A is μ-null. Consequently, the m_S-null and μ-null sets coincide (because the μ-null sets are always m_S-null). So, S is μ-determined. Finally, the continuous inclusion $X(\mu)_{[p]} \subseteq L^{q/p}(g\,d\mu)$ and the fact that $\|g^{p/q}f\|_{L^{q/p}(\mu)} = \|f\|_{L^{q/p}(g\,d\mu)}$ for all $f \in L^{q/p}(g\,d\mu)$ imply that $g^{p/q} \in \mathcal{M}\big(X(\mu)_{[p]},\ L^{q/p}(\mu)\big)$. So, condition (v) holds.

(v) $\Rightarrow$ (ii). Set $g_1 := g$ with g as in (v). The calculation

$$\begin{aligned}\big\|T(f)\big\|_E = \big\|T_{[p]}(f)\big\|_E &\le \|S\| \cdot \big\|M_{g_1^{p/q}}(f)\big\|_{L^{q/p}(\mu)} \\ &= \|S\| \left(\int_\Omega |f|^{q/p}\, g_1\, d\mu\right)^{p/q} < \infty,\end{aligned}$$

is valid for all $f \in X(\mu) \subseteq X(\mu)_{[p]}$. So, (ii) holds with g there defined as $\|S\|^{q/p} g_1$.

(iii) $\Rightarrow$ (vi). First, the inclusion $X(\mu) \subseteq L^p(m_T)$ implies that $T \in \mathcal{F}_{[p]}(X(\mu), E)$ via Theorem 5.7. Let $\gamma_1 : X(\mu) \to L^q(g\,d\mu)$ and $\gamma_2 : L^q(g\,d\mu) \to L^p(m_T)$ denote the respective inclusion map. Then, for each $f \in X(\mu)$, we have

$$\begin{aligned}\big\|J_T^{(p)}(f)\big\|_{L^p(m_T)} &= \big\|(\gamma_2 \circ \gamma_1)(f)\big\|_{L^p(m_T)} \\ &\le \|\gamma_2\| \cdot \big\|\gamma_1(f)\big\|_{L^q(g\,d\mu)} = \|\gamma_2\| \Big(\int_\Omega |f|^q g\, d\mu\Big)^{1/q} < \infty.\end{aligned}$$

It follows from $\chi_\Omega \in X(\mu) \subseteq L^q(g\,d\mu)$ that $g \in L^1(\mu)$. Furthermore, because of the inclusion $X(\mu) \subseteq L^q(g\,d\mu)$, Proposition 2.29 gives that $g \in \big(X(\mu)_{[q]}\big)'$. So, Lemma 6.13 (with $E := L^p(m_T)$, the operator $J_T^{(p)}$ in place of T and 1 in place of p) yields that $J_T^{(p)}$ is bidual $(1,q)$-power-concave, that is, bidual q-concave.

(vi) $\Rightarrow$ (vii). Since $J_T^{(p)} \in \mathcal{A}_{1,q}\big(X(\mu),\ L^p(m_T)\big)$, we can choose a constant $C_3 > 0$ satisfying

$$\sum_{j=1}^n \big\|J_T^{(p)}(g_j)\big\|_{L^p(m_T)}^q \le C_3 \Big\| \sum_{j=1}^n |g_j|^q \Big\|_{\mathrm{b},X(\mu)_{[q]}}, \qquad n \in \mathbb{N}, \quad g_1, \dots, g_n \in X(\mu); \tag{6.23}$$

see (6.3) with $J_T^{(p)}$ in place of T and $L^p(m_T)$ in place of E. To prove that $\beta_{[p]} \in \mathcal{A}_{1,q/p}\big(X(\mu)_{[p]}, L^1(m_T)\big)$, fix $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X(\mu)_{[p]}$. By (6.23) with the functions $g_j := |f_j|^{1/p} \in X(\mu)$ for $j = 1, \dots, n$, we have that

$$\begin{aligned}\sum_{j=1}^n \big\|\beta_{[p]}(f_j)\big\|_{L^1(m_T)}^{q/p} &= \sum_{j=1}^n \big\|f_j\big\|_{L^1(m_T)}^{q/p} = \sum_{j=1}^n \big\|\,|f_j|^{1/p}\big\|_{L^p(m_T)}^q \\ &= \sum_{j=1}^n \big\|J_T^{(p)}(|f_j|^{1/p})\big\|_{L^p(m_T)}^q \le C_3 \Big\| \sum_{j=1}^n |f_j|^{q/p} \Big\|_{\mathrm{b},X(\mu)_{[q]}} \\ &= C_3 \Big\| \sum_{j=1}^n |f_j|^{q/p} \Big\|_{\mathrm{b},(X(\mu)_{[p]})_{[q/p]}},\end{aligned}$$

where $X(\mu)_{[q]} = (X(\mu)_{[p]})_{[q/p]}$ follows from Lemma 2.20(i). Thus, the operator $\beta_{[p]}$ is bidual (q/p)-concave, that is, (vii) holds.

(vii) $\Rightarrow$ (viii). Setting $Z := E$ and $E := L^1(m_T)$ in Proposition 6.2(vi), with $\beta_{[p]}$ in place of T, the operator $T_{[p]}$ in place of T, the space $X(\mu)_{[p]}$ in place of $X(\mu)$ and $S := I_{m_T}$, we have from Proposition 6.2(vi) (also with 1 in place of p and (q/p) in place of q) that (viii) holds because $T_{[p]} = I_{m_T} \circ \beta_{[p]}$ and I_{m_T} is continuous.

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[d, "i_{[p]}"'] & & E \\ X(\mu)_{[p]} \arrow[urr, "T_{[p]}"] \arrow[rr, "\beta_{[p]}"] & & L^1(m_T). \arrow[u, "I_{m_T}"'] \end{tikzcd}$$

(viii) $\Rightarrow$ (i). Since $T_{[p]} \in \mathcal{A}_{1,q/p}\big(X(\mu)_{[p]},\, E\big)$, we can choose a constant $C_4 > 0$ (see (6.3)) such that

$$\sum_{j=1}^{n} \big\|T_{[p]}(g_j)\big\|_E^{q/p} \le C_4 \Big\| \sum_{j=1}^{n} |g_j|^{q/p} \Big\|_{\mathrm{b},(X(\mu)_{[p]})_{[q/p]}} = C_4 \Big\| \sum_{j=1}^{n} |g_j|^{q/p} \Big\|_{\mathrm{b},X(\mu)_{[q]}} \tag{6.24}$$

for all $n \in \mathbb{N}$ and $g_1, \dots, g_n \in X(\mu)_{[p]}$. Fix $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X(\mu) \subseteq X(\mu)_{[p]}$. Then, (6.24) with $g_j := f_j$ for $j = 1, \dots, n$ yields that

$$\sum_{j=1}^{n} \big\|T(f_j)\big\|_E^{q/p} = \sum_{j=1}^{n} \big\|T_{[p]}(f_j)\big\|_E^{q/p} \le C_4 \Big\| \sum_{j=1}^{n} |f_j|^{q/p} \Big\|_{\mathrm{b},X(\mu)_{[q]}}$$

because $T_{[p]}$ is an extension of T to $X(\mu)_{[p]}$. Thus, part (i) holds for the constant $C := C_4^{1/q}$. $\square$

The arguments in the proof of Theorem 6.9 have further implications.

Remark 6.14. Let the assumptions be as in Theorem 6.9. Suppose that we can find a function g satisfying condition (ii) there. In view of the arguments establishing the implications (ii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v), let $S \in \mathcal{L}(L^{q/p}(\mu),\, E)$ be the operator defined there by $h \mapsto I_{m_T}(g^{-p/q}h)$ for $h \in L^{q/p}(\mu)$. Then $T_{[p]} = S \circ M_{g^{p/q}}$ with the multiplication operator $M_{g^{p/q}} : X(\mu)_{[p]} \to L^{q/p}(\mu)$ as in condition (v). We shall show that

$$\|S\| \le 1; \tag{6.25}$$

note that S is defined according to the *particular* g satisfying condition (ii). To establish (6.25), first recall the inequality $\|T(f)\|_E \le \Big(\int_\Omega |f|^{q/p} g \, d\mu \Big)^{p/q}$ for each

$f \in X(\mu)$ as given in (6.17). We claim that

$$\|I_{m_T}(\psi)\|_E \leq \Big(\int_\Omega |\psi|^{q/p} g\, d\mu\Big)^{p/q}, \qquad \psi \in L^{q/p}(g\, d\mu). \tag{6.26}$$

Note that the continuous inclusion $L^{q/p}(g\ d\mu) \subseteq L^1(m_T)$ is contained in part (iv), which is equivalent to (ii). To verify our claim, fix $\psi \in L^{q/p}(g\, d\mu)$ and take a sequence $\{s_n\}_{n=1}^\infty$ from the dense subspace $\operatorname{sim}\Sigma$ of the σ-order continuous q-B.f.s. $L^{q/p}(g\ d\mu)$ such that $|s_n| \uparrow |\psi|$ and $\lim_{n\to\infty} \|s_n - \psi\|_{L^{p/q}(g\, d\mu)} = 0$. Now, the continuous inclusion $L^{q/p}(g\ d\mu) \subseteq L^1(m_T)$ ensures that $s_n \to \psi$ in $L^1(m_T)$ and hence, that $T(s_n) = I_{m_T}(s_n) \to I_{m_T}(\psi)$ in the Banach space E as $n \to \infty$. This, the Lebesgue Dominated Convergence Theorem for the scalar measure μ, and (6.17) with $f := s_n \in \operatorname{sim}\Sigma \subseteq X(\mu)$ for $n \in \mathbb{N}$ jointly yield (6.26).

Now, let $h \in L^{q/p}(\mu)$. Since $hg^{-p/q} \in L^{q/p}(g\, d\mu)$ and $\|hg^{-p/q}\|_{L^{q/p}(g\, d\mu)} = \|h\|_{L^{q/p}(\mu)}$, it follows from (6.26), with $\psi := hg^{-p/q}$, that

$$\|S(h)\|_E = \big\|I_{m_T}(hg^{-p/q})\big\|_E \leq \Big(\int_\Omega \big|hg^{-p/q}\big|^{q/p} g\, d\mu\Big)^{p/q} = \|h\|_{L^{q/p}(\mu)},$$

which establishes (6.25). □

The bidual (p, q)-power-concavity of $T : X(\mu) \to E$ and of the natural injection $J_T : X(\mu) \to L^1(m_T)$ are directly connected.

Corollary 6.15. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ). Let E be a Banach space. Given values $1 \leq p < \infty$ and $0 < q < \infty$, a μ-determined operator $T \in \mathcal{L}(X(\mu), E)$ is bidual (p, q)-power-concave if and only if the natural inclusion map $J_T : X(\mu) \to L^1(m_T)$ is bidual (p, q)-power-concave.*

Proof. Suppose that T is bidual (p, q)-power-concave. It follows from condition (vii) of Theorem 6.9 that $T \in \mathcal{F}_{[p]}\big(X(\mu), E\big)$ and the inclusion map $\beta_{[p]} : X(\mu)_{[p]} \to L^1(m_T)$ is bidual (q/p)-concave. Now, we know from Theorem 5.7 that $J_T \in \mathcal{F}_{[p]}\big(X(\mu), L^1(m_T)\big)$ and hence, J_T admits the unique continuous linear extension $(J_T)_{[p]} : X(\mu)_{[p]} \to L^1(m_T)$. Since the continuous inclusion $X(\mu) \subseteq X(\mu)_{[p]}$ is the natural one, it is clear that $(J_T)_{[p]} = \beta_{[p]}$. In short, $J_T \in \mathcal{F}_{[p]}\big(X(\mu), L^1(m_T)\big)$ and its canonical extension $(J_T)_{[p]}$ is bidual (q/p)-concave. Therefore, via the equivalence (i) ⇔ (viii) in Theorem 6.9 (with J_T in place of T), we deduce that J_T is bidual (p, q)-power-concave.

Conversely, suppose that J_T is bidual (p, q)-power-concave. Then Proposition 6.2(vi) yields that T is bidual (p, q)-power-concave because $T = I_{m_T} \circ J_T$. □

The following result is another consequence of Theorem 6.9.

Corollary 6.16. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ) and E be a Banach space. Suppose that $1 \leq p_1 \leq p_2 < \infty$*

and $0 < q < \infty$. *Then*

$$\mathcal{A}_{p_2,q}\big(X(\mu),\, E\big) \,\subseteq\, \mathcal{A}_{p_1,q}\big(X(\mu),\, E\big).$$

Proof. Let $T \in \mathcal{A}_{p_2,q}\big(X(\mu),\, E\big)$. Let Ω_1, μ_1 and $X(\mu_1)$ be as in Remark 6.8. Then the μ_1-determined operator $T_1 : X(\mu_1) \to E$ is bidual (p_2, q)-power-concave via Remark 6.8. So, there exists $g \in L^0(\mu_1)$ with $g > 0$ (μ_1-a.e.) such that, with continuous inclusions,

$$X(\mu_1) \,\subseteq\, L^q(g\, d\mu_1) \,\subseteq\, L^{p_2}(m_{T_1}) \tag{6.27}$$

(see condition (iii) of Theorem 6.9 with $p := p_2$). On the other hand, we have $L^{p_2}(m_{T_1}) \subseteq L^{p_1}(m_{T_1})$ because $p_1 \le p_2$; see (3.49) and Lemma 2.21(iv). This and (6.27) imply that

$$X(\mu_1) \,\subseteq\, L^q(g\, d\mu_1) \,\subseteq\, L^{p_1}(m_{T_1}),$$

that is, condition (iii) of Theorem 6.9 is satisfied with $p := p_1$. So,

$$T_1 \in \mathcal{A}_{p_1,q}(X(\mu_1), E).$$

Now we conclude, again via Remark 6.8, that $T \in \mathcal{A}_{p_1,q}(X(\mu), E)$, which completes the proof. □

In the special case when $p := 1$, Theorem 6.9 reduces to the following corollary in which we shall provide equivalent conditions for an operator $T \in \mathcal{L}(X(\mu), E)$ to be bidual q-concave (with $0 < q < \infty$) when $X(\mu)$ is a σ-order continuous q-B.f.s. and E is a Banach space. Our result is motivated by the Maurey–Rosenthal type factorization theorem presented in [30, Corollary 5] with a slightly different setting. In that version the domain space is assumed to be q-convex and the codomain space is a quasi-Banach space. Moreover, the quasi-Banach function spaces in [30] are assumed to satisfy additional conditions (see (II) and (III) on p. 155 in [30]).

Corollary 6.17. *Let* $0 < q < \infty$*, let* $X(\mu)$ *be a* σ*-order continuous q-B.f.s. over a positive, finite measure space* (Ω, Σ, μ) *and let* E *be a Banach space. The following assertions are equivalent for a* μ*-determined operator* $T \in \mathcal{L}(X(\mu), E)$.

(i) *There is a constant* $C > 0$ *such that*

$$\Big(\sum_{j=1}^{n} \big\|T(f_j)\big\|_E^q\Big)^{1/q} \le C \Big\|\sum_{j=1}^{n} |f_j|^q\Big\|_{\mathrm{b},X(\mu)_{[q]}}^{1/q}, \qquad n \in \mathbb{N}, \quad f_1, \dots, f_n \in X(\mu),$$

that is, T *is bidual* q*-concave.*

(ii) *There is a function* $g \in \big(X(\mu)_{[q]}\big)'$ *satisfying* $g > 0$ (μ*-a.e.*) *such that*

$$\big\|T(f)\big\|_E \le \Big(\int_\Omega |f|^q g\, d\mu\Big)^{1/q}, \qquad f \in X(\mu).$$

(iii) *There exists $g \in L^0(\mu)$ with $g > 0$ (μ-a.e.) such that the inclusions*

$$X(\mu) \subseteq L^q(g\,d\mu) \subseteq L^1(m_T)$$

hold and are continuous.

(iv) *There exist a function $g \in \big(X(\mu)_{[q]}\big)'$ with $g > 0$ (μ-a.e.) and an operator $S \in \mathcal{L}(L^q(\mu), E)$ satisfying $T = S \circ M_{g^{1/q}}$. That is, the following diagram commutes:*

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, "M_{g^{1/q}}"'] & & E \\ & L^q(\mu) \arrow[ur, "S"'] & \end{tikzcd}.$$

(v) *The natural inclusion map $J_T : X(\mu) \to L^1(m_T)$ is bidual q-concave.*

If, in addition, $X(\mu)$ happens to be q-convex, then each one of (i)–(v) *is equivalent to*

(vi) *T is q-concave.*

In this case, there exists $g_0 \in \big(X(\mu)_{[q]}\big)'$ with $g_0 > 0$ (μ-a.e.) such that

$$\big\|T(f)\big\|_E \leq \Big(\int_\Omega |f|^q g_0\,d\mu\Big)^{1/q} \leq \Big(\mathbf{M}_{(q)}[T] \cdot \mathbf{M}^{(q)}[X(\mu)]\Big)\,\|f\|_{X(\mu)}, \qquad (6.28)$$

for every $f \in X(\mu)$. Consequently, for the operator $M_{g_0^{1/q}} : X(\mu) \to L^q(\mu)$ of multiplication by $g_0^{1/q}$ (see part (iv) *), we have*

$$\big\|M_{g_0^{1/q}}\big\| \leq \mathbf{M}_{(q)}[T] \cdot \mathbf{M}^{(q)}[X(\mu)]. \qquad (6.29)$$

Proof. Note that parts (i)–(iii) of Corollary 6.17 correspond precisely to (i)–(iii) of Theorem 6.9 (with $p := 1$). Part (iv) of Corollary 6.17 corresponds to (v) of Theorem 6.9 (also with $p := 1$) because T is always 1-th power factorable and because the condition $g^{1/q} \in \mathcal{M}\big(X(\mu), L^q(\mu)\big)$ (coming from (v) of Theorem 6.9 when $p = 1$) is equivalent to $g \in (X(\mu)_{[q]})'$, which occurs in (iv) of Corollary 6.17; see Proposition 2.29. Note that part (v) of Corollary 6.17 corresponds to (vi) of Theorem 6.9 (with $p := 1$).

Suppose now, in addition, that $X(\mu)$ is q-convex. Then $\mathcal{A}_{1,q}(X(\mu), E) = \mathcal{B}_{1,q}(X(\mu), E) = \mathcal{K}_{(q)}(X(\mu), E)$; for the first equality see Proposition 6.2(iv) with $p := 1$, whereas the second equality is exactly (6.6). This gives the equivalence (i) $\Leftrightarrow$ (vi) in Corollary 6.17.

Still under the assumption that $X(\mu)$ is q-convex, suppose that the operator $T : X(\mu) \to E$ is q-concave, that is, (vi) holds. In order to show that (6.28) holds, we first claim that

$$\Big(\sum_{j=1}^n \big\|T(f_j)\big\|_E^q\Big)^{1/q} \leq \Big(\mathbf{M}_{(q)}[T] \cdot \mathbf{M}^{(q)}[X(\mu)]\Big)\,\Big\|\sum_{j=1}^n |f_j|^q\Big\|_{\mathrm{b},X(\mu)_{[q]}}^{1/q} \qquad (6.30)$$

whenever $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X(\mu)$. Indeed, since $X(\mu)$ is q-convex, apply Proposition 2.23(ii), with q in place of p, to find an equivalent lattice norm $\eta_{[q]}$ on $X(\mu)_{[q]}$ satisfying

$$\eta_{[q]}(f) \le \|f\|_{X(\mu)_{[q]}} \le \Big(\mathbf{M}^{(q)}[X(\mu)]\Big)^q \eta_{[q]}(f), \qquad f \in X(\mu)_{[q]}. \tag{6.31}$$

Recalling the definition of the seminorm $\|\cdot\|_{\mathrm{b},X(\mu)_{[q]}}$ we can, via (6.31), derive that

$$\eta_{[q]}(f) \le \|f\|_{\mathrm{b},X(\mu)_{[q]}} \le \Big(\mathbf{M}^{(q)}[X(\mu)]\Big)^q \eta_{[q]}(f), \qquad f \in X(\mu)_{[q]}. \tag{6.32}$$

Indeed, in the proof of (c) $\Rightarrow$ (a) in Proposition 2.13(vi) we have shown that (2.28) implies (2.29). Here, (6.31) implying (6.32) is a special case of this. It follows from (6.31) and (6.32) that

$$\|f\|^{1/q}_{X(\mu)_{[q]}} \le \Big(\mathbf{M}^{(q)}[X(\mu)]\Big)\,(\eta_{[q]}(f))^{1/q} \le \Big(\mathbf{M}^{(q)}[X(\mu)]\Big)\,\|f\|^{1/q}_{\mathrm{b},X(\mu)_{[q]}}, \tag{6.33}$$

for $f \in X(\mu)_{[q]}$. Now, given $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X(\mu)$ we have $|f_j|^q \in X(\mu)_{[q]}$ for $j = 1, \dots, n$ and so $\sum_{j=1}^n |f_j|^q \in X(\mu)_{[q]}$, that is, $\big(\sum_{j=1}^n |f_j|^q\big)^{1/q} \in X(\mu)$. So, the q-concavity of T as well as (6.33), with $f := \sum_{j=1}^n |f_j|^q$, imply (6.30) because

$$\begin{aligned}
\Big(\sum_{j=1}^n \|T(f_j)\|_E^q\Big)^{1/q} &\le \Big(\mathbf{M}_{(q)}[T]\Big) \Big\|\Big(\sum_{j=1}^n |f_j|^q\Big)^{1/q}\Big\|_{X(\mu)} \\
&= \Big(\mathbf{M}_{(q)}[T]\Big) \Big\|\sum_{j=1}^n |f_j|^q\Big\|^{1/q}_{X(\mu)_{[q]}} \\
&\le \Big(\mathbf{M}_{(q)}[T]\Big)\Big(\mathbf{M}^{(q)}[X(\mu)]\Big) \Big\|\Big(\sum_{j=1}^n |f_j|^q\Big)\Big\|^{1/q}_{\mathrm{b},X(\mu)_{[q]}}.
\end{aligned}$$

In short, with $C_1 := \big(\mathbf{M}_{(q)}[T] \cdot \mathbf{M}^{(q)}[X(\mu)]\big)^q$, we have

$$\sum_{j=1}^n \|T(f_j)\|_E^q \le C_1 \Big\|\sum_{j=1}^n |f_j|^q\Big\|_{\mathrm{b},X(\mu)_{[q]}}, \qquad f_1, \dots, f_n \in X(\mu), \quad n \in \mathbb{N}.$$

This is exactly condition (i) of Lemma 6.13 with $p := 1$. Now, for this *particular constant* C_1, the arguments immediately after the proof of Lemma 6.13 deliver a non-negative function $g_0 \in \big(X(\mu)_{[q]}\big)'$ satisfying (6.21) with $p := 1$ and $g := g_0$. In other words,

$$\Big(\int_\Omega |f|^q g_0 \, d\mu\Big)^{1/q} \le C_1^{1/q} \|i_{[1]}\| \cdot \|f\|_{X(\mu)} = \Big(\mathbf{M}_{(q)}[T] \cdot \mathbf{M}^{(q)}[X(\mu)]\Big) \cdot \|f\|_{X(\mu)}$$

whenever $f \in X(\mu)$; note that $i_{[1]}$ is the identity operator on $X(\mu)$. This and part (ii), which is valid with $g := g_0$, establish (6.28).

Finally, since $\big\|M_{g_0^{1/q}}(f)\big\|_{L^q(\mu)} = \big(\int_\Omega |f|^q g_0\,d\mu\big)^{1/q}$ for $f \in X(\mu)$, the inequality (6.29) follows immediately from (6.28). $\square$

Remark 6.18. Given $0 < q < \infty$, let $X(\mu)$ be a q-convex, σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ). Suppose that E is a Banach space and $T \in \mathcal{L}(X(\mu), E)$ is a μ-determined, q-concave operator. In other words, condition (vi) of Corollary 6.17 is satisfied. Then, we claim that it is possible to select a particular function $g_0 \in \big(X(\mu)_{[q]}\big)'$ with $g_0 > 0$ (μ-a.e.) and a particular operator $S_0 \in \mathcal{L}\big(L^q(\mu), E\big)$ such that $T = S_0 \circ M_{g_0^{1/q}}$ and

$$\|T\| \le \|S_0\| \cdot \big\|M_{g_0^{1/q}}\big\| \;\le\; \mathbf{M}_{(q)}[T] \cdot \mathbf{M}^{(q)}[X(\mu)]; \tag{6.34}$$

in short, we can include an extra condition (6.34) in part (iv) (with $g := g_0$ and $S := S_0$) of Corollary 6.17. The inequality (6.34) was originally given in [30, Corollary 5]. Now let us verify our claim. Take a function $g_0 \in \big(X(\mu)_{[q]}\big)'$ with $g_0 > 0$ (μ-a.e.) satisfying (6.28) for every $f \in X(\mu)$. Then, $g_0^{1/q} \in \mathcal{M}\big(X(\mu),\, L^q(\mu)\big)$ and $T = S_0 \circ M_{g_0^{1/q}}$ with $S_0 \in \mathcal{L}(L^q(\mu), E)$ denoting the operator $f \mapsto I_{m_T}\big(g_0^{-1/q} f\big)$; see the arguments used to prove the implications (ii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) in Theorem 6.9 with $p := 1$. Moreover, $\|S_0\| \le 1$ via Remark 6.14) (with $S := S_0$). Consequently, it follows from (6.29) that

$$\|T\| = \big\|S_0 \circ M_{g_0^{1/q}}\big\| \le \|S_0\| \cdot \big\|M_{g_0^{1/q}}\big\| \le \big\|M_{g_0^{1/q}}\big\| \le \mathbf{M}_{(q)}[T] \cdot \mathbf{M}^{(q)}[X(\mu)]. \quad \square$$

The σ-order continuity of a q-B.f.s. $X(\mu)$ in Theorem 6.9 ensures, amongst other things, that the set function $m_T : A \mapsto T(\chi_A)$ on Σ associated with a Banach-space valued, continuous linear operator T on $X(\mu)$ is actually σ-additive. Such a vector measure m_T played a crucial role in the proof of Theorem 6.9. Let us now demonstrate, by example, that if $X(\mu)$ is not σ-o.c., then there may exist continuous linear operators on $X(\mu)$ which do *not* admit a factorization by means of a multiplication operator as in part (iv) of Corollary 6.17 (which is exactly part (v) of Theorem 6.9 with $p := 1$).

In the following example, we shall use the fact that $L^\infty(\mu)$ over a positive, finite measure space (Ω, Σ, μ) is q-convex whenever $1 \le q < \infty$. To see this, fix $1 \le q < \infty$. Given $n \in \mathbb{N}$ and functions $f_1, \ldots, f_n \in L^\infty(\mu)$, it is clear that $\big(\sum_{j=1}^n |f_j(\omega)|^q\big)^{1/q} \le \big(\sum_{j=1}^n \|f_j\|_{L^\infty(\mu)}^q\big)^{1/q}$ for μ-a.e. $\omega \in \Omega$ and hence, that

$$\Big\|\Big(\sum_{j=1}^n |f_j|^q\Big)^{1/q}\Big\|_{L^\infty(\mu)} \;\le\; \Big(\sum_{j=1}^n \|f_j\|_{L^\infty(\mu)}^q\Big)^{1/q}.$$

So, according to Definition 2.46, the B.f.s. $L^\infty(\mu)$ is q-convex with $\mathbf{M}^{(q)}\big[L^\infty(\mu)\big] \le 1$. On the other hand, $\mathbf{M}^{(q)}\big[L^\infty(\mu)\big] \ge 1$ by (2.107) and consequently, we have $\mathbf{M}^{(q)}\big[L^\infty(\mu)\big] = 1$.

Example 6.19. Suppose that (Ω, Σ, μ) is a positive, finite measure space such that Σ is σ-decomposable relative to μ. Let $1 \leq q < \infty$. The B.f.s. $X(\mu) := L^\infty(\mu)$ is *not* σ-o.c. So, $L^\infty(\mu)^* \neq L^\infty(\mu)' = L^1(\mu)$. Since Σ is σ-decomposable relative to μ, the $L^\infty(\mu)$-valued set function $A \mapsto \chi_A$ is not norm σ-additive on Σ. Hence, there exists $x^* \in L^\infty(\mu)^*$ such that the finitely additive, scalar-valued set function $\lambda : A \mapsto \langle \chi_A, x^* \rangle$ is not σ-additive on Σ. Clearly x^* can be chosen from $(L^\infty(\mu)^*)^+$, in which case $\lambda \geq 0$. Let $E := \mathbb{C}$ and $T \in \mathcal{L}(X(\mu), E)$ denote the functional x^*, that is, $T(f) = \langle f, x^* \rangle$ for all $f \in L^\infty(\mu)$. Note that E is q-concave, and so the positive operator T is also q-concave via Corollary 2.70. Moreover, $L^\infty(\mu)$ is q-convex; see the discussion just prior to this example.

Now, assume that we can factorize T as in part (iv) of Corollary 6.17, i.e., $T = S \circ M_{g^{1/q}}$ for some function $g \in \big(X(\mu)_{[q]}\big)'$ with $g > 0$ (μ-a.e.) and some operator $S \in \mathcal{L}\big(L^q(\mu), E\big)$. Whenever $A(n) \downarrow \emptyset$ with $A(n) \in \Sigma$ for $n \in \mathbb{N}$, it follows that $\big\langle \chi_{A(n)}, x^* \big\rangle = T(\chi_{A(n)}) = S \circ M_{g^{1/q}}(\chi_{A(n)}) \to 0$ as $n \to \infty$ because $\big\|S \circ M_{g^{1/q}}(\chi_{A(n)})\big\| \leq \|S\| \cdot \|M_{g^{1/q}}(\chi_{A(n)})\|$ and

$$\big\|M_{g^{1/q}}(\chi_{A(n)})\big\|_{L^q(\mu)} = \Big(\int_{A(n)} g \, d\mu\Big)^{1/q} \to 0 \qquad \text{as} \quad n \to \infty.$$

We have used here the fact that $g^{1/q} = M_{g^{1/q}}(\chi_\Omega) \in L^q(\mu)$, that is, $g \in L^1(\mu)$. So, $T(\chi_{A(n)}) \to 0$ as $n \to \infty$ whenever $\Sigma \ni A(n) \downarrow \emptyset$. In other words, λ is σ-additive. This is a contradiction and hence, T does not admit a factorization as in Corollary 6.17(iv). $\square$

The following example shows that q-convexity of $X(\mu)$ is necessary if, in part (i) of Corollary 6.17, we wish to replace the bidual q-concavity condition on T assumed there with q-concavity of T.

Example 6.20. Let (Ω, Σ, μ) be a positive, finite measure space for which Σ is σ-decomposable relative to μ. Fix any $1 \leq r < \infty$ and let $X(\mu) := L^r(\mu)$ and $E := L^r(\mu)$, in which case $X(\mu)$ is both r-convex and r-concave. We denote the identity operator on $L^r(\mu)$ by T and interpret $T : X(\mu) \to E$. Given $r < q < \infty$, it follows that $X(\mu) = L^r(\mu)$ is q-concave but not q-convex (see (i-b) and (ii-a) of Example 2.73).

To show that T is not bidual q-concave, assume the contrary. It then follows from Corollary 6.17 that there exists $g \in L^0(\mu)$ with $g > 0$ (μ-a.e.) for which

$$L^r(\mu) \;=\; X(\mu) \;\subseteq\; L^q(g \, d\mu) \;\subseteq\; L^1(m_T). \tag{6.35}$$

Since $m_T(A) = T(\chi_A) = \chi_A \in X(\mu)$ for $A \in \Sigma$, we have that $L^1(m_T) = L^r(\mu)$, with their given norms being equal; see Corollary 3.66(ii)(a). Hence, (6.35) implies that $L^r(\mu) = X(\mu) = L^q(g \, d\mu)$ as isomorphic B.f.s.'. Since $L^q(g \, d\mu)$ is q-convex (see Example 2.73(i)), this contradicts the fact that $L^r(\mu)$ is not q-convex. Therefore, $T \notin \mathcal{A}_{1,q}\big(X(\mu), E\big)$. On the other hand, $T \in \mathcal{K}_{(q)}(X(\mu), E)$ because its codomain $E = L^r(\mu)$ is q-concave and T is a positive operator; see Corollary 2.70.

Noting that $\chi_\Omega \in X(\mu) \subseteq L^q(g\,d\mu)$, we can conclude that $g \in L^1(\mu)$. Then Corollary 6.3(iii) can be applied to deduce that $T \in \mathcal{A}_{1,r}\big(X(\mu), E\big)$.

Finally, suppose that $0 < q < r$. Then T, being the identity operator on $L^r(\mu)$, is not q-concave because $L^r(\mu)$ is not q-concave (see Example 2.73(ii-b)). So, again T fails to be bidual q-concave; see Proposition 6.2(i) with $p := 1$. □

Given $1 \le p < \infty$ and $0 < q < \infty$, we now exhibit a (p,q)-power-concave operator defined on a non-q-convex q-B.f.s. The choice $p := 1$ will then show that Corollary 6.17 allows us to consider a Maurey–Rosenthal type factorization on a non-q-convex space (whereas that due to A. Defant [31, Corollary 5] requires the operator to act in a *q-convex space* $X(\mu)$).

Example 6.21. Let $\varphi \in \ell^1$ satisfy $\varphi(n) > 0$ for every $n \in \mathbb{N}$. Define a finite measure $\mu : 2^{\mathbb{N}} \to [0,\infty)$ by $\mu(\{n\}) := \varphi(n)$ for every $n \in \mathbb{N}$. Fix $1 \le p < \infty$ and $0 < q < \infty$.

(i) Given $0 < r < q < \infty$, the q-B.f.s. $X(\mu) := \ell^r(\mu)$ based on the finite measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ is not q-convex; see Example 2.73(ii-a). By applying condition (v) of Theorem 6.9 we shall exhibit a bidual (p,q)-power-concave multiplication operator defined on $X(\mu)$ and taking values in the Banach space $E := \ell^u(\mu)$; here $1 \le u < \infty$ is arbitrary (but fixed from now on). To this end, apply Lemma 2.80(ii) (with (r/p) in place of q and (q/p) in place of u) to deduce that

$$\mathcal{M}\big(\ell^{r/p}(\mu),\, \ell^{q/p}(\mu)\big) \;=\; \varphi^{(p/r)-(p/q)} \cdot \ell^\infty. \tag{6.36}$$

Next we shall show that

(i-a) if $p < (q/u)$, that is, $u < (q/p)$, then

$$\mathcal{M}\big(\ell^{q/p}(\mu),\, \ell^{u}(\mu)\big) \;=\; \varphi^{(p/q)-(1/u)} \cdot \ell^{(qu)/(q-pu)},$$

and

(i-b) if $p \ge (q/u)$, that is, $u \ge (q/p)$, then

$$\mathcal{M}\big(\ell^{q/p}(\mu),\, \ell^{u}(\mu)\big) = \varphi^{(p/q)-(1/u)} \cdot \ell^\infty.$$

For (i-a), since $u < (q/p)$, take $w > 0$ satisfying $(p/q) + (1/w) = (1/u)$. Then Example 2.30(i) gives that

$$\mathcal{M}\big(\ell^{q/p}(\mu),\, \ell^{u}(\mu)\big) = \ell^w(\mu) = \varphi^{-(1/w)} \cdot \ell^w \;=\; \varphi^{(p/q)-(1/u)} \cdot \ell^{(qu)/(q-pu)}.$$

So, (i-a) holds. Next, applying Lemma 2.80(ii) with (q/p) in place of q establishes (i-b). Note that both (i-a) and (i-b) are *independent* of the condition $0 < r < q$ assumed on r.

Consider now the function $\psi := \sum_{n=1}^\infty 2^{-n}\chi_{\{n\}}$ defined pointwise on $\mathbb{N}$. Then $\psi(n) > 0$ for every $n \in \mathbb{N}$ and

$$\psi \in \bigcap_{0<s\le\infty} \ell^s. \tag{6.37}$$

The function $h := \varphi^{(p/q)-(1/u)}\psi$ satisfies

$$M_h \in \mathcal{L}\big(\ell^{q/p}(\mu),\, \ell^u(\mu)\big). \tag{6.38}$$

In fact, if $1 \le p < (q/u)$, then $(qu)/(q-pu) > 0$ and so (6.37) gives $\psi \in \ell^{(qu)/(q-pu)}$. Hence, (6.38) holds via (i-a). On the other hand, if $p \ge (q/u)$, then (i-b) and the fact that $\psi \in \ell^\infty$ again imply that (6.38) holds. Moreover, the operator M_h is μ-determined because $h(n) > 0$ for every $n \in \mathbb{N}$; see Example 4.7(ii).

Recall that $0 < r < q$. With $g := \big(\varphi^{(p/r)-(p/q)}\big)^{q/p}$, we have $(p/r)-(p/q) > 0$ and $g(n) > 0$ for every $n \in \mathbb{N}$. Moreover, $g^{p/q} = \varphi^{(p/r)-(p/q)}\chi_{\mathbb{N}}$ and so the function $g^{p/q} \in \mathcal{M}\big(\ell^{r/p}(\mu),\, \ell^{q/p}(\mu)\big) = \mathcal{M}\big(X(\mu)_{[p]},\, \ell^{q/p}(\mu)\big)$ via (6.36).

Therefore, with $i_{[p]} : X(\mu) = \ell^r(\mu) \to X(\mu)_{[p]} = \ell^{r/p}(\mu)$ denoting the natural inclusion map, define the operator $T \in \mathcal{L}(X(\mu), E)$ as the composition

$$T := M_h \circ M_{g^{p/q}} \circ i_{[p]}. \tag{6.39}$$

Being the multiplication operator by the function $h\,g^{p/q}$, which satisfies $h\,g^{p/q} > 0$ pointwise on $\mathbb{N}$, the operator T is μ-determined via Example 4.7(ii). Moreover, the operator $T_{[p]} := M_h \circ M_{g^{p/q}}$ belongs to $\mathcal{L}(X(\mu)_{[p]}, E)$ and satisfies $T = T_{[p]} \circ i_{[p]}$. According to Definition 5.1 the operator T is p-th power factorable and hence $T \in \mathcal{F}_{[p]}(X(\mu), E)$. It follows from (6.39) and condition (v) of Theorem 6.9 (with $S := M_h$) that T is bidual (p,q)-power-concave.

(ii) With $X(\mu) := \ell^1(\mu)$ and $E := \ell^1(\mu)$ we shall show that

$$M_z \in \mathcal{A}_{1,2}\big(X(\mu), E\big)$$

whenever $z \in \ell^2$ and $z(n) > 0$ for all $n \in \mathbb{N}$. We can write $z = \big(\varphi^{-(1/2)}z\big)\cdot\varphi^{1/2}$. Moreover, part (ii) of Lemma 2.80 (with $q := 1$ and $u := 2$) gives

$$\varphi^{1/2} \;=\; \varphi^{1-(1/2)}\cdot\chi_{\mathbb{N}} \,\in\, \varphi^{1-(1/2)}\cdot\ell^\infty \;=\; \mathcal{M}\big(\ell^1(\mu),\, \ell^2(\mu)\big),$$

whereas part (i) of the same lemma (with $r := 2$ and $q := 1$, in which case $w = 2$) gives

$$\varphi^{-(1/2)}z \,\in\, \varphi^{-(1/2)}\cdot\ell^2 \;=\; \ell^2(\mu) \;=\; \mathcal{M}\big(\ell^2(\mu),\, \ell^1(\mu)\big).$$

Since $z > 0$ pointwise on $\mathbb{N}$, the multiplication operator $M_z : X(\mu) \to E$ is μ-determined; see Example 4.7(ii). Moreover, the discussion immediately after the identity (5.3) implies that $M_z \in \mathcal{F}_{[1]}(X(\mu), E)$. Accordingly, M_z is bidual 2-concave because the composition

$$M_z \;=\; M_{\varphi^{-1/2}z} \,\circ\, M_{\varphi^{1/2}}$$

ensures that condition (v) of Theorem 6.9 with $p := 1$, $q := 2$, $g := \varphi$, $T := M_z$ and $S := M_{\varphi^{-1/2}z}$ holds, that is, we have:

$$\begin{tikzcd} X(\mu) = \ell^1(\mu) \arrow[rr, "T=M_z"] \arrow[dr, "M_{\varphi^{1/2}}"'] & & E = \ell^1(\mu)\,. \\ & \ell^2(\mu) \arrow[ur, "S=M_{\varphi^{-1/2}z}"'] & \end{tikzcd}$$

□

In order to exhibit an example of a bidual (p, q)-power concave operator on a q-B.f.s. over a non-atomic, positive, finite measure space, we first require the following fact.

Remark 6.22. Let $X(\mu)$ be a q-B.f.s. based on a non-atomic, positive, finite measure space (Ω, Σ, μ) and E be a Banach space. If $1 \leq p < \infty$ and $0 < q < p$, then

$$\mathcal{A}_{p,q}(X(\mu), E) = \{0\}. \tag{6.40}$$

If $E = \{0\}$, then (6.40) is clear. So, suppose that $E \neq \{0\}$ and assume, on the contrary, that there does exist a non-zero operator $T \in \mathcal{A}_{p,q}(X(\mu), E)$. Then we may as well assume that T is μ-determined. Otherwise we simply consider the restriction T_1 of T to its essential carrier Ω_1 because the restriction μ_1 of μ to Ω_1 is also non-atomic and because T_1 is μ_1-determined; see Remark 6.8. Now apply condition (iv) of Theorem 6.9 to find $g \in L^0(\mu)$, with $g > 0$ (μ-a.e.), such that $X(\mu)_{[p]} \subseteq L^{q/p}(g\,d\mu) \subseteq L^1(m_T)$. However, since $(q/p) < 1$ and since the indefinite integral $\mu_g = g\,d\mu : \Sigma \to [0, \infty)$ is also non-atomic and finite (as $g \in L^1(\mu)$, which can be seen from the proof of (iv)⇒(ii) in Theorem 6.9), Example 2.10 yields that $\big(L^{q/p}(g\,d\mu)\big)^* = \{0\}$. So, recalling that $L^1(m_T)$ is a Banach space (and non-trivial as μ is non-atomic with $\mathcal{N}_0(\mu) = \mathcal{N}_0(m_T)$) enables us to obtain from Lemma 2.9 that

$$\mathcal{L}\big(L^{q/p}(g\,d\mu),\, L^1(m_T)\big) = \{0\}.$$

But, the linear map corresponding to the inclusion $L^{q/p}(g\,d\mu) \subseteq L^1(m_T)$ is non-trivial and belongs to $\mathcal{L}\big(L^{q/p}(g\;d\mu),\, L^1(m_T)$; a contradiction. Therefore, (6.40) must hold. □

We now present a bidual (p, q)-power-concave operator defined on a *non-q-convex* q-B.f.s. (over a non-atomic, finite measure space).

Example 6.23. Let $1 < q < \infty$ and $0 < r < q$. Consider the Lorentz space $L^{q,r}(\mu)$ over a non-atomic, positive, finite measure space (Ω, Σ, μ). Then

$$L^{q,r}(\mu) \subseteq L^q(\mu) \subseteq L^1(\mu),$$

via Example 2.76(ii-c), and $L^{q,r}(\mu)$ is not q-convex; see Example 2.76(vii). Let $T \neq 0$ denote the natural injection from $X(\mu) := L^{q,r}(\mu)$ into the Banach space $E := L^1(\mu)$. Given $1 \leq p < \infty$, we claim that $T \in \mathcal{A}_{p,q}(X(\mu), E)$ if and only if

$p \le q$. Indeed, if $T \in \mathcal{A}_{p,q}(X(\mu), E)$, then we necessarily have $p \le q$ via Remark 6.22. Conversely, assume that $p \le q$. Again, by Example 2.76(ii-c), we have (since $(r/p) < (q/p)$ and $(q/p) \ge 1$) that

$$L^{(q/p),\,(r/p)}(\mu) \;\subseteq\; L^{q/p}(\mu) \;\subseteq\; L^1(\mu). \tag{6.41}$$

Observe that $X(\mu)_{[p]} = L^{q,r}(\mu)_{[p]} = L^{(q/p),\,(r/p)}(\mu)$; see Example 2.76(iv). Since

$$m_T(A) = T(\chi_A) = \chi_A, \qquad A \in \Sigma, \tag{6.42}$$

Corollary 3.66(ii)(a) (with $X(\mu) := L^1(\mu)$) implies that $L^1(m_T) = L^1(\mu)$ with equal norms. Therefore, we can rewrite (6.41) as

$$X(\mu)_{[p]} \;\subseteq\; L^{q/p}(\mu) \;\subseteq\; L^1(m_T).$$

Moreover, (6.42) implies that $\mathcal{N}_0(m_T) = \mathcal{N}_0(\mu)$, that is, T is μ-determined. So, condition (iv) of Theorem 6.9 is satisfied and hence, $T \in \mathcal{A}_{p,q}(X(\mu), E)$. □

The following result will be applied to the Volterra operators and convolution operators considered in Examples 6.25 and 6.26 below, respectively.

Lemma 6.24. *Let (Ω, Σ, μ) be a non-atomic, positive, finite measure space. Given $1 < r < \infty$, consider the B.f.s. $L^r(\mu)$ and a Banach space $E \neq \{0\}$. Let $T \in \mathcal{L}(L^r(\mu), E)$ be any μ-determined operator (necessarily non-zero). Fix $1 \le p < \infty$ and $0 < q < \infty$.*

- (i) *If $T \in \mathcal{A}_{p,q}\big(L^r(\mu), E\big)$, then we necessarily have $p \le q \le r$.*
- (ii) *Suppose that $p \le q \le r$. If there exists a number u satisfying $1 \le u \le (q/p)$ such that $L^u(\mu) \subseteq L^1(m_T)$, then $T \in \mathcal{A}_{p,q}\big(L^r(\mu), E\big)$.*
- (iii) *Assume that $L^1(\mu) \subseteq L^1(m_T)$. Then $T \in \mathcal{A}_{p,q}\big(L^r(\mu), E\big)$ if and only if we have $p \le q \le r$.*

Proof. (i) It was noted in Example 6.5 that there *exist* μ-determined operators in $\mathcal{L}(L^r(\mu), E)$. Example 6.4(ii) and Remark 6.22 yield that $q \le r$ and $p \le q$, respectively. So, (i) holds.

(ii) Note that the assumed conditions on p, q, r and u imply that the inequalities $1 \le u \le (q/p) \le q \le r$ hold and hence, we have

$$L^r(\mu)_{[p]} = L^{r/p}(\mu) \;\subseteq\; L^{q/p}(\mu) \;\subseteq\; L^u(\mu) \;\subseteq\; L^1(m_T).$$

In other words, condition (iv) of Theorem 6.9 is satisfied with $g := \chi_\Omega$ and $X(\mu) := L^r(\mu)$. Hence, $T \in \mathcal{A}_{p,q}\big(L^r(\mu), E\big)$.

(iii) This follows from (i) and (ii) (with $u := 1$). □

Example 6.25. Let $1 < r < \infty$ and μ be Lebesgue measure. Given $1 \le p < \infty$ and $0 < q < \infty$, the Volterra integral operator $V_r : L^r([0,1]) = L^r(\mu) \to L^r([0,1])$ (see (3.27)) is bidual (p,q)-power-concave if and only if $1 \le p \le q \le r$. This follows from

Lemma 6.24(iii) because it follows from (3.45) that the vector measure $m_{V_r} = \nu_r$ associated with V_r (see Example 3.10) satisfies

$$L^1(\mu) \subseteq L^1\big((1-t)^{1/r}dt\big) \subseteq L^1(\nu_r) = L^1(m_{V_r}). \qquad \square$$

Example 6.26. Let μ be normalized Haar measure on the circle group $\mathbb{T}$. Given $1 < r < \infty$ and $g \in L^1(\mathbb{T})\setminus\{0\}$ consider the convolution operator $C_g^{(r)} : L^r(\mathbb{T}) \to L^r(\mathbb{T})$ as given in (4.10). Let $m_g^{(r)} : \mathcal{B}(\mathbb{T}) \to L^r(\mathbb{T})$ denote the vector measure associated with $C_g^{(r)}$ (see Example 4.11), that is, $m_g^{(r)}$ denotes $m_{C_g^{(r)}}$. Fix $1 \le p < \infty$ and $0 < q < \infty$.

(i) Assume that $g \in L^r(\mathbb{T})$. Then $L^1(m_g^{(r)}) = L^1(\mu) = L^1(\mathbb{T})$ (see Theorem 7.50 or [123, Theorem 1.2]). So, it follows from Lemma 6.24(iii) that $C_g^{(r)} \in \mathcal{A}_{p,q}\big(L^r(\mathbb{T}),\, L^r(\mathbb{T})\big)$ if and only if $p \le q \le r$. The situation for a general compact abelian group G in place of $\mathbb{T}$ is analogous; see Example 7.51 in Chapter 7.

(ii) Let $1 \le s < r$ and choose $u \ge 1$ according to $(1/s) + (1/u) = (1/r) + 1$. Then, for any function $g \in L^s(\mathbb{T}) \setminus L^r(\mathbb{T})$, we have $L^u(\mu) = L^u(\mathbb{T}) \subseteq L^1(m_g^{(r)})$; see [123, Remark 3.3(ii)]. So, we can apply Lemma 6.24(ii) to obtain that

$$C_g^{(r)} \in \mathcal{A}_{p,q}\big(L^r(\mathbb{T}),\, L^r(\mathbb{T})\big) \qquad \text{whenever} \qquad 1 \le u \le pu \le q \le r.$$

Again for a general group G (of the type in part (i)) in place of $\mathbb{T}$, the situation is analogous; see Remark 7.45(iv) in Chapter 7. $\square$

6.3 Bidual (p,q)-power-concave operators

Given an operator $T \in \mathcal{L}(X(\mu), E)$, not necessarily μ-determined, we shall present equivalent conditions for the bidual (p,q)-power-concavity of T; see Proposition 6.27 below. These conditions do not explicitly contain the associated vector measure m_T. With T_1 denoting the restriction of T to its essential carrier we could, of course, make use of our previous Theorem 6.9 by applying it to T_1. However, we prefer to prove the result directly, without recourse to vector measures. We require some preparation before providing a formal statement.

Given are a positive, finite measure space (Ω, Σ, μ) and $0 < q < \infty$. Let $X(\mu)$ be a q-B.f.s. over (Ω, Σ, μ) and consider any non-negative function $g \in \big(X(\mu)_{[q]}\big)'$. Then $\int_\Omega g|h|\, d\mu < \infty$ for all $h \in X(\mu)_{[q]}$ and hence, with $h := \chi_\Omega \in X(\mu)_{[q]}$, we see that $g \in L^1(\mu)^+$. According to Proposition 2.29, it follows that $\int_\Omega |f|^q g\, d\mu < \infty$ for every $f \in X(\mu)$. If, in addition, $g > 0$ (μ-a.e.), then the weighted L^q-space $L^q(\mu_g) = L^q(g\, d\mu)$ is also a q-B.f.s. over (Ω, Σ, μ) and we have the inclusion $X(\mu) \subseteq L^q(g\, d\mu)$, in which case the corresponding natural inclusion map is exactly the multiplication operator $M_{\mathbf{1}} : X(\mu) \to L^q(g\, d\mu)$ determined by the constant function $\mathbf{1} := \chi_\Omega$. Even for the general case when $g \ge 0$ (i.e., $M_{\mathbf{1}}$ is not necessarily injective), let us still use the same notation $M_{\mathbf{1}} : X(\mu) \to L^q(g\, d\mu)$ to denote the corresponding multiplication operator. That is, for each $f \in X(\mu)$, the function $M_{\mathbf{1}}(f)$ is considered as an element of $L^q(g\,d\mu)$. Recall that the simple functions are dense in any q-B.f.s. which is σ-o.c.; see Remark 2.6.

According to Corollary 6.3(i), for each $1 \le p < \infty$, the p-th power factorable operators and the bidual (p,q)-power-concave operators defined on $L^q(g\,d\mu)$ are the same.

Proposition 6.27. *Let $1 \le p < \infty$ and $0 < q < \infty$. Suppose that $X(\mu)$ is a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ) and E is a Banach space. Then the following conditions for an operator $T \in \mathcal{L}(X(\mu), E)$ are equivalent.*

(i) *T is bidual (p,q)-power-concave.*

(ii) *There exists a non-negative function $g \in \big(X(\mu)_{[q]}\big)'$ such that*

$$\big\|T(f)\big\|_E \le \Big(\int_\Omega |f|^{q/p} g\,d\mu\Big)^{p/q} < \infty, \qquad f \in X(\mu).$$

(iii) *There exist a non-negative function $g \in \big(X(\mu)_{[q]}\big)'$ and a bidual (p,q)-power-concave operator $R : L^q(gd\mu) \to E$ such that $T = R \circ M_{\mathbf{1}}$, with the operator $M_{\mathbf{1}} : X(\mu) \to L^q(g\,d\mu)$ denoting multiplication by $\mathbf{1} := \chi_\Omega$, that is, we have*

$$\begin{tikzcd}
X(\mu) \arrow[r, "T"] \arrow[d, "M_{\mathbf{1}}"'] & E \\
L^q(g\,d\mu) \arrow[ur, "R"] \arrow[r, "\gamma_{[p]}"] & L^{q/p}(g\,d\mu) \arrow[u, "R_{[p]}"']
\end{tikzcd}$$

where $\gamma_{[p]}$ denotes the canonical inclusion map.

(iv) *T is p-th power factorable and there exist $g \in L^1(\mu)^+$ and $S \in \mathcal{L}\big(L^{q/p}(\mu), E\big)$ such that $g^{p/q} \in \mathcal{M}\big(X(\mu)_{[p]}, L^{q/p}(\mu)\big)$ and $T_{[p]} = S \circ M_{g^{p/q}}$, that is, we have*

$$\begin{tikzcd}
X(\mu) \arrow[r, "T"] \arrow[d, "i_{[p]}"'] & E \\
X(\mu)_{[p]} \arrow[ur, "T_{[p]}"] \arrow[r, "M_{g^{p/q}}"] & L^{q/p}(\mu). \arrow[u, "S"']
\end{tikzcd}$$

(v) *T is p-th power factorable and there exist a function $h \in \mathcal{M}\big(X(\mu)_{[p]}, L^{q/p}(\mu)\big)$ and $U \in \mathcal{L}\big(L^{q/p}(\mu), E\big)$ such that $T_{[p]} = U \circ M_h$, that is, we have*

$$\begin{tikzcd}
X(\mu) \arrow[r, "T"] \arrow[d, "i_{[p]}"'] & E \\
X(\mu)_{[p]} \arrow[ur, "T_{[p]}"] \arrow[r, "M_h"] & L^{q/p}(\mu). \arrow[u, "U"']
\end{tikzcd}$$

(vi) *T is p-th power factorable and $T_{[p]} : X(\mu)_{[p]} \to E$ is bidual (q/p)-power-concave.*

Proof. (i) $\Leftrightarrow$ (ii). This is exactly Lemma 6.13.

(ii) $\Rightarrow$ (iii). Since $f := \chi_\Omega \in X(\mu)$, it follows from condition (ii) that $g \in L^1(\mu)$. So, we can define a linear operator R_1 on the subspace $\operatorname{sim}\Sigma$ of $L^{q/p}(g\,d\mu)$ by

$$R_1(s) := T(s), \qquad s \in \operatorname{sim}\Sigma.$$

Here note that s in the left-hand side is considered as an element of $L^{q/p}(g\,d\mu)$ whereas s in the right-hand side is considered as an element of $X(\mu)$. Then condition (ii) gives that

$$\big\|R_1(s)\big\|_E \le \Big(\int_\Omega |s|^{q/p} g\,d\mu\Big)^{p/q} < \infty, \qquad s \in \operatorname{sim}\Sigma,$$

that is, $\|R_1(s)\|_E \le \|s\|_{L^{q/p}(g\,d\mu)}$ for all $s \in \operatorname{sim}\Sigma \subseteq L^{q/p}(g\,d\mu)$. So, R_1 is continuous on the dense subspace $\operatorname{sim}\Sigma$ of the σ-order continuous q-B.f.s. $L^{q/p}(g\,d\mu)$ and hence, admits a unique E-valued, continuous linear extension to $L^{q/p}(g\,d\mu)$. This extension is denoted also by R_1. Let

$$\gamma_{[p]} : L^q(g\,d\mu) \to L^q(g\,d\mu)_{[p]} = L^{q/p}(g\,d\mu)$$

denote the canonical embedding; see Lemma 2.21(iv) with $L^q(g\,d\mu)$ in place of $X(\mu)$ there. The composition $R := R_1 \circ \gamma_{[p]} \in \mathcal{L}\big(L^q(g\,d\mu), E\big)$ is clearly p-th power factorable with $R_{[p]} = R_1$ and hence, $R \in \mathcal{A}_{p,q}(L^q(g\,d\mu), E)$ via Corollary 6.3(ii).

Now, consider the multiplication operator $M_{\mathbf{1}} : X(\mu) \to L^q(g\,d\mu)$ as discussed prior to the proposition. Then the E-valued continuous linear operators $R \circ M_{\mathbf{1}}$ and T coincide on the dense subspace $\operatorname{sim}\Sigma$ of $X(\mu)$ and hence, $R \circ M_{\mathbf{1}} = T$ on $X(\mu)$.

(iii) $\Rightarrow$ (iv). The bidual (p,q)-operator R is p-th power factorable (see parts (i) and (ii) of Proposition 6.2) and hence, admits a continuous linear extension

$$R_{[p]} : L^{q/p}(g\,d\mu) = L^q(g\,d\mu)_{[p]} \to E;$$

see Definition 5.1. We can then define a linear operator $S : L^{q/p}(\mu) \to E$ by

$$S(h) := R_{[p]}(g^{-p/q}h), \qquad h \in L^{q/p}(\mu),$$

because $g^{-p/q}h \in L^{q/p}(g\,d\mu)$ for every $h \in L^{q/p}(\mu)$. Then

$$\|S(h)\|_E \;\le\; \|R_{[p]}\| \cdot \|g^{-p/q}h\|_{L^{q/p}(g\,d\mu)} \;=\; \|R_{[p]}\| \cdot \|h\|_{L^{q/p}(\mu)}$$

for every $h \in L^{q/p}(\mu)$.

In the discussion prior to Proposition 6.27 we saw that the non-negative function $g \in \big(X(\mu)_{[q]}\big)'$ belongs to $L^1(\mu)^+$ and hence, via Proposition 2.29, we have

$$g^{p/q} \in \mathcal{M}\big(X(\mu)_{[p]},\, L^{q/p}(\mu)\big).$$

So, given $f \in X(\mu) \subseteq X(\mu)_{[p]}$, we have (because $g^{p/q}f \in L^{q/p}(\mu)$) that

$$\big(S \circ M_{g^{p/q}}\big)(f) \;=\; S(g^{p/q}f) \;=\; R_{[p]}(f) \;=\; \big(R \circ M_{\mathbf{1}}\big)(f) = T(f),$$

that is, $S \circ M_{g^{p/q}} : X(\mu)_{[p]} \to E$ is a continuous linear extension of T, which establishes (iv).

(iv) $\Rightarrow$ (v). This is immediate by setting $h := g^{p/q}$ and $U := S$.

(v) $\Rightarrow$ (vi). For every $f \in X(\mu)_{[p]}$ we have

$$\begin{aligned}\|T_{[p]}(f)\|_E = \big\|\big(U \circ M_h\big)(f)\big\|_E \;&\le\; \|U\| \cdot \|hf\|_{L^{q/p}(\mu)}\\ &= \|U\| \Big(\int_\Omega |f|^{q/p} |h|^{q/p}\, d\mu\Big)^{p/q} \;<\; \infty.\end{aligned}$$

In particular, $\int_\Omega |f|^{q/p}|h|^{q/p}\,d\mu \;<\; \infty$ for every $f \in X(\mu)_{[p]}$. Since every $\psi \in (X(\mu)_{[p]})_{[q/p]} = X(\mu)_{[q]}$ satisfies $|\psi| = (|\psi|^{p/q})^{q/p}$, with $f := |\psi|^{p/q} \in X(\mu)_{[p]}$, it follows that $|h|^{q/p}$ belongs to $\big((X(\mu)_{[p]})_{[q/p]}\big)'$. So, by setting $g := \|U\|^{q/p} \cdot |h|^{q/p}$ we see that (6.17) holds (with 1 in place of p and (q/p) in place of q and with $X(\mu)_{[p]}$ in place of $X(\mu)$ and $T_{[p]}$ in place of T). According to Lemma 6.13 we can conclude that $T_{[p]} \in \mathcal{A}_{1,q/p}\big(X(\mu)_{[p]}, E\big)$.

(vi) $\Rightarrow$ (i). The proof of (viii) $\Rightarrow$ (i) in Theorem 6.9 works for this case without any change (and makes no use of the μ-determinedness of T or the vector measure m_T). □

Remark 6.28. Let $1 \le p < \infty$ and $0 < q < \infty$.

(i) Condition (iii) in Proposition 6.27 illustrates that a prototype of bidual (p, q)-power-concave operators are those operators induced by p-th power factorable operators defined on the L^q-space of some finite scalar measure.

(ii) Condition (v) in Proposition 6.27 allows us to describe the class of all bidual (p, q)-power-concave operators from a σ-order continuous q-B.f.s. $X(\mu)$ into a Banach space E as follows:

$$\mathcal{A}_{p,q}\big(X(\mu), E\big) = \Big\{U \circ M_h \circ i_{[p]} : h \in \mathcal{M}\big(X(\mu)_{[p]},\, L^{q/p}(\mu)\big),\; U \in \mathcal{L}\big(L^{q/p}(\mu), E\big)\Big\}.$$

□

Let us immediately apply this remark.

Example 6.29. Let $1 \le q < \infty$ and $0 < r < q$. Consider the Lorentz space $X(\mu) := L^{q,r}(\mu)$ over a non-atomic, positive, finite measure space (Ω, Σ, μ); see Example 2.76. Given a Banach space E, our aim is to verify that

$$\mathcal{A}_{p,q}(X(\mu), E) = \Big\{ U \circ M_h \circ i_{[p]} : h \in L^\infty(\mu),\ U \in \mathcal{L}(L^{q/p}(\mu),\ E) \Big\} \tag{6.43}$$

whenever $1 \le p \le q$.

(i) We shall first show that

$$\mathcal{M}\big(L^{1,s}(\mu),\ L^1(\mu)\big) = L^\infty(\mu), \qquad 0 < s < 1. \tag{6.44}$$

It follows from Theorem 1.4.17(ii) and its proof in [69] that

$$L^{1,s}(\mu)^* = L^{1,s}(\mu)' = L^\infty(\mu); \tag{6.45}$$

see also Proposition 2.16 and Example 2.76(iii) for the identification $L^{1,s}(\mu)^* = L^{1,s}(\mu)'$. Now let $h \in \mathcal{M}\big(L^{1,s}(\mu),\ L^1(\mu)\big)$. Since $M_h \in \mathcal{L}\big(L^{1,s}(\mu),\ L^1(\mu)\big)$ and $L^1(\mu)^* = L^1(\mu)' = L^\infty(\mu)$, the linear functional

$$f \mapsto \int_\Omega hf\, d\mu = \int_\Omega M_h(f)\, d\mu, \qquad f \in L^{1,s}(\mu),$$

is continuous. This and (6.45) give that $h \in L^\infty(\mu)$.

Conversely, the inclusion $L^{1,s}(\mu) \subseteq L^1(\mu)$ (see Example 2.76(ii-c)), together with the inequality $\|fg\|_{L^1(\mu)} \le \|g\|_{L^\infty(\mu)} \|f\|_{L^1(\mu)}$, valid for all $f \in L^1(\mu)$ and $g \in L^\infty(\mu)$, yield that

$$\mathcal{M}\big(L^{1,s}(\mu),\ L^1(\mu)\big) \supseteq L^\infty(\mu).$$

So, (6.44) does indeed hold.

(ii) Next we show that if $1 \le u < \infty$ and $0 < w < u$, then

$$\mathcal{M}\big(L^{u,w}(\mu),\ L^u(\mu)\big) = L^\infty(\mu).$$

In fact, Example 2.76(iv) yields that $L^{1,(w/u)}(\mu) = L^{u,w}(\mu)_{[u]}$. From (2.77) (with $X(\mu) := L^{u,w}(\mu)$, $Y(\mu) := L^u(\mu)$, and $p := u$) and part (i) (with $s := w/u$), it follows that

$$\begin{aligned}
\mathcal{M}\big(L^{u,w}(\mu),\ L^u(\mu)\big) &= \Big(\mathcal{M}\big(L^{u,w}(\mu),\ L^u(\mu)\big)_{[u]}\Big)_{[1/u]} \\
&= \mathcal{M}\big(L^{u,w}(\mu)_{[u]},\ L^u(\mu)_{[u]}\big)_{[1/u]} \\
&= \mathcal{M}\big(L^{1,(w/u)}(\mu),\ L^1(\mu)\big)_{[1/u]} = L^\infty(\mu)_{[1/u]} = L^\infty(\mu).
\end{aligned}$$

(iii) Let $1 \le p < \infty$. It follows from part (ii) (with $u := q/p$ and $w := r/p$) that

$$\mathcal{M}\big(L^{(q/p),(r/p)}(\mu),\ L^{q/p}(\mu)\big) \ = \ L^\infty(\mu),$$

that is, $\mathcal{M}\big(L^{q,r}(\mu)_{[p]},\ L^{q/p}(\mu)\big) = L^\infty(\mu)$; see (2.167). This and Remark 6.28(ii) yield (6.43). $\square$

In the previous Example 6.29 we have assumed that $r < q$. If $r = q$, then $L^{q,q}(\mu) = L^q(\mu)$ via Example 2.76(ii-a) and hence, Corollary 6.3(ii) (with $g := \chi_\Omega$) gives a description of the class of all bidual (p,q)-power concave operators defined on $L^q(\mu)$. So, we now consider the remaining case when $r > q$.

Example 6.30. Suppose that $1 \leq q < r < \infty$ and that (Ω, Σ, μ) is a non-atomic, positive, finite measure space. Given any Banach space E, we claim that

$$\mathcal{A}_{p,q}\big(L^{q,r}(\mu), E\big) \;=\; \{0\}, \qquad 1 \leq p < \infty. \tag{6.46}$$

To prove this, it suffices to assume that $1 \leq p \leq q$ because (6.46) has already been verified when $p > q$ (see Remark 6.22).

First we shall show that

$$\mathcal{M}\big(L^{(q/p),\,(r/p)}(\mu),\; L^{q/p}(\mu)\big) \;=\; \{0\}. \tag{6.47}$$

Since $1 \leq (q/p)$, we have $L^{1,\,(r/q)}(\mu) = L^{(q/p),\,(r/p)}(\mu)_{[q/p]}$ via Example 2.76(iv) and since $L^1(\mu) = L^{q/p}(\mu)_{[q/p]}$, it follows from (2.77), with $X(\mu) := L^{(q/p),\,(r/p)}(\mu)$ and $Y(\mu) := L^{q/p}(\mu)$ and with (q/p) in place of p, that

$$\begin{aligned}\mathcal{M}\big(L^{1,\,(r/q)}(\mu),\; L^1(\mu)\big) &= \mathcal{M}\big(L^{(q/p),\,(r/p)}(\mu)_{[q/p]},\; L^{q/p}(\mu)_{[q/p]}\big)\\ &= \mathcal{M}\big(L^{(q/p),\,(r/p)}(\mu),\; L^{q/p}(\mu)\big)_{[q/p]}.\end{aligned} \tag{6.48}$$

However, since $1 < (r/q) < \infty$, the q-B.f.s. $L^{1,\,(r/q)}(\mu)$ has trivial dual (see [69, Theorem 1.4.17(iii)]) and hence, Lemma 2.9 with $E := L^1(\mu)$ implies that $\mathcal{L}\big(L^{1,\,(r/q)}(\mu),\, L^1(\mu)\big) = \{0\}$. In particular, it follows that

$$\mathcal{M}\big(L^{1,\,(r/q)}(\mu),\, L^1(\mu)\big) \;=\; \{0\} \tag{6.49}$$

because each $h \in \mathcal{M}\big(L^{1,\,(r/q)}(\mu),\, L^1(\mu)\big)$ corresponds to the multiplication operator $M_h \in \mathcal{L}\big(L^{1,\,(r/q)}(\mu),\, L^1(\mu)\big)$. Now, (6.48) and (6.49) yield (6.47).

With $i_{[p]} : L^{q,r}(\mu) \to L^{q,r}(\mu)_{[p]} = L^{(q/p),\,(r/p)}(\mu)$ denoting the natural embedding, Remark 6.28(ii) gives

$$\begin{aligned}&\mathcal{A}_{p,q}\big(L^{q,r}(\mu),\, E\big)\\ &= \Big\{U \circ M_h \circ i_{[p]} \,:\, U \in \mathcal{L}\big(L^{q/p}(\mu), E\big),\; h \in \mathcal{M}\big(L^{(q/p),\,(r/p)}(\mu),\; L^{q/p}(\mu)\big)\Big\}.\end{aligned}$$

So, (6.46) follows from this and (6.47). □

The situation for purely atomic measures is somewhat different.

Example 6.31. Let $\varphi \in \ell^1$ with $\varphi(n) > 0$ for all $n \in \mathbb{N}$ and $\mu : 2^{\mathbb{N}} \to [0, \infty)$ be the finite, positive measure defined by $\mu(\{n\}) := \varphi(n)$ for $n \in \mathbb{N}$. Fix positive numbers q, r, u such that $q < r$ and $u \geq 1$. Given any $1 \leq p < \infty$, our aim is to identify those functions $\psi \in \mathcal{M}\big(\ell^r(\mu), \ell^u(\mu)\big)$ for which $M_\psi \in \mathcal{A}_{p,q}\big(\ell^r(\mu), \ell^u(\mu)\big)$.

Recall from Example 2.73(i-a) that the q-B.f.s. $\ell^r(\mu)$ over the finite measure space $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ is q-convex because $0 < q < r$. Therefore it follows from Proposition 6.2(iv), with $X(\mu) := \ell^r(\mu)$ and $E := \ell^u(\mu)$, that

$$\mathcal{A}_{p,q}\big(\ell^r(\mu),\ \ell^u(\mu)\big) \;=\; \mathcal{B}_{p,q}\big(\ell^r(\mu),\ \ell^u(\mu)\big). \tag{6.50}$$

This is to be compared with Example 6.20 in which the assumption $q > r$ imposes non-q-convexity on $\ell^r(\mu)$.

(i) Observing that $0 < (p/r) < (p/q)$, let $w > 0$ be the number satisfying

$$\frac{p}{r} + \frac{1}{w} = \frac{p}{q}, \qquad \text{i.e.,} \qquad w = \frac{qr}{p(r-q)}.$$

Apply Lemma 2.80(i) (with (r/p) in place of r and (q/p) in place of q and noting that $(q/p) < (r/p)$) to deduce that

$$\mathcal{M}\big(\ell^{r/p}(\mu), \ell^{q/p}(\mu)\big) \;=\; \ell^w(\mu) \;=\; \varphi^{-(1/w)} \cdot \ell^w \;=\; \varphi^{(p/r)-(p/q)} \cdot \ell^w. \tag{6.51}$$

(ii) There are two cases: $(q/p) \le u$ and $(q/p) > u$, which will be treated separately.

(ii-a) Assume first that $(q/p) \le u$. It then follows from Lemma 2.80(ii) (also from Example 6.21(i-b)) that

$$\mathcal{M}\big(\ell^{q/p}(\mu), \ell^u(\mu)\big) \;=\; \varphi^{(p/q)-(1/u)} \cdot \ell^\infty.$$

This, the identity $\ell^\infty \cdot \ell^w = \ell^w$, and (6.51) imply that

$$\begin{aligned}&\mathcal{M}\big(\ell^{q/p}(\mu), \ell^u(\mu)\big) \cdot \mathcal{M}\big(\ell^{r/p}(\mu), \ell^{q/p}(\mu)\big)\\ &= \big(\varphi^{(p/q)-(1/u)} \cdot \ell^\infty\big) \cdot \big(\varphi^{(p/r)-(p/q)} \cdot \ell^w\big) \;=\; \varphi^{(p/r)-(1/u)} \cdot \ell^w.\end{aligned} \tag{6.52}$$

Given $\psi \in \mathcal{M}\big(\ell^r(\mu), \ell^u(\mu)\big)$, we shall show that

$$M_\psi \in \mathcal{A}_{p,q}\big(\ell^r(\mu), \ell^u(\mu)\big) \iff \psi \in \varphi^{(p/r)-(1/u)} \cdot \ell^w = \varphi^{(p/r)-(1/u)} \cdot \ell^{(qr)/(p(r-q))}. \tag{6.53}$$

To this end, assume first that $M_\psi \in \mathcal{A}_{p,q}\big(\ell^r(\mu), \ell^u(\mu)\big)$. Via Proposition 6.27(v), there exist a function $h \in \mathcal{M}\big(\ell^r(\mu)_{[p]}, \ell^{q/p}(\mu)\big) = \mathcal{M}\big(\ell^{r/p}(\mu), \ell^{q/p}(\mu)\big)$ and an operator $U \in \mathcal{L}\big(\ell^{q/p}(\mu), \ell^u(\mu)\big)$ such that

$$\big(M_\psi\big)_{[p]} \;=\; U \circ M_h. \tag{6.54}$$

Here $\big(M_\psi\big)_{[p]} : \ell^{r/p} = \ell^r(\mu)_{[p]} \to \ell^u(\mu)$ denotes the continuous linear extension of the operator $M_\psi : \ell^r(\mu) \to \ell^u(\mu)$ (necessarily p-th power factorable by Proposition 6.27(v)) to the p-th power $\ell^r(\mu)_{[p]}$ of $\ell^r(\mu)$. Of course, $\big(M_\psi\big)_{[p]}$ is also the operator of multiplication by ψ. Let $A := h^{-1}\big(\mathbb{C} \setminus \{0\}\big)$. Then the continuous linear operator $U_1 : \ell^{q/p}(\mu) \to \ell^u(\mu)$ defined by $U_1(f) := U(f\chi_A)$ for every $f \in \ell^{q/p}(\mu)$ is continuous and satisfies (because $hg\chi_A = hg$ for every $g \in \ell^{r/p}(\mu)$)

$$\big(M_\psi\big)_{[p]} \;=\; U_1 \circ M_h. \tag{6.55}$$

Now we have

$$U_1(\chi_{\{n\}}) \;=\; U(\chi_{\{n\}}\chi_A) \;=\; U(0) \;=\; 0, \qquad n \in \mathbb{N} \setminus A. \tag{6.56}$$

On the other hand, (6.55) yields that

$$\psi(n)\chi_{\{n\}} \;=\; \big(M_\psi\big)_{[p]}\big(\chi_{\{n\}}\big) \;=\; U_1\big(h\chi_{\{n\}}\big) \;=\; h(n)U_1\big(\chi_{\{n\}}\big), \qquad n \in A. \tag{6.57}$$

We claim that $U_1 \in \mathcal{L}\big(\ell^{q/p}(\mu),\, \ell^u(\mu)\big)$ is the multiplication operator by $\psi_1 := (\psi/h)\chi_A$. Indeed, given $f \in \ell^{q/p}(\mu)$, we have $f = \sum_{n=1}^\infty f(n)\chi_{\{n\}}$ in the topology of the q-B.f.s. $\ell^{q/p}(\mu)$. For the case $f \geq 0$ this follows from $\ell^{q/p}(\mu)$ being σ-o.c. and the fact that $\sum_{n=1}^k f(n)\chi_{\{n\}} \uparrow f$ in the order of $\ell^{q/p}(\mu)$. For $\mathbb{R}$-valued f, the conclusion then follows by considering f^+ and f^- and the case of $\mathbb{C}$-valued f by then considering $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$. The previous fact, together with (6.56) and (6.57), imply that

$$\begin{aligned}
U_1(f) = U_1\Big(\sum_{n=1}^\infty f(n)\chi_{\{n\}}\Big) \;&=\; \sum_{n=1}^\infty f(n)U_1(\chi_{\{n\}}) \\
= \sum_{n=1}^\infty \big(\psi(n)/h(n)\big)\,\chi_A(n)\,f(n)\,\chi_{\{n\}} \;&=\; (\psi/h)\chi_A f \\
= \psi_1 f.&
\end{aligned}$$

Therefore $\psi_1 \in \mathcal{M}\big(\ell^{q/p}(\mu),\, \ell^u(\mu)\big)$ and $U_1 = M_{\psi_1}$, which establishes our claim. Now, this together with (6.55), give that

$$\psi g \;=\; \big(M_\psi\big)_{[p]}(g) \;=\; \big(M_{\psi_1} \circ M_h\big)(g), \qquad g \in \ell^{r/p}(\mu)$$

and hence, via (6.52), we have that

$$\psi = \psi_1 h \in \mathcal{M}\big(\ell^{q/p}(\mu),\, \ell^u(\mu)\big) \cdot \mathcal{M}\big(\ell^{r/p}(\mu),\, \ell^{q/p}(\mu)\big) = \varphi^{(p/r)-(1/u)} \cdot \ell^w.$$

To prove the reverse implication in (6.53), assume that $\psi \in \ell^{(p/r)-(1/u)} \cdot \ell^w$. Then (6.52) implies that the multiplication operator $\widetilde{M_\psi} : f \mapsto \psi f$ from $\ell^{r/p}(\mu)$ into $\ell^u(\mu)$ factorizes through $\ell^{q/p}(\mu)$ by multiplications; see the discussion prior to Lemma 2.80. So, $\widetilde{M_\psi}$ is the continuous linear extension of $M_\psi : \ell^r(\mu) \to \ell^u(\mu)$ to $\ell^r(\mu)_{[p]} = \ell^{r/p}(\mu)$, that is, M_ψ is p-th power factorable with $\widetilde{M_\psi} = (M_\psi)_{[p]}$ and consequently,

$$M_\psi = M_{h_2} \circ M_{h_1} \circ i_{[p]}$$

for some $h_1 \in \mathcal{M}\big(\ell^{r/p}(\mu),\, \ell^{q/p}(\mu)\big)$ and $h_2 \in \mathcal{M}\big(\ell^{q/p}(\mu),\, \ell^{u}(\mu)\big)$. Therefore, $M_\psi \in \mathcal{A}_{p,q}\big(\ell^{r}(\mu),\, \ell^{u}(\mu)\big)$ via condition (v) of Proposition 6.27 or Remark 6.28(ii). So we have established (6.53).

(ii-b) Now let us consider the case when $(q/p) > u$. Let the number $v > 0$ satisfy

$$\frac{p}{q} + \frac{1}{v} = \frac{1}{u}, \qquad \text{i.e.,} \quad v = \frac{qu}{q - pu}.$$

Then Lemma 2.80(i) (with (q/p) in place of r and u in place of q) implies that

$$\mathcal{M}\big(\ell^{q/p}(\mu),\, \ell^{u}(\mu)\big) \;=\; \ell^{v}(\mu) \;=\; \ell^{1}(\mu)_{[1/v]}.$$

Therefore, via (6.51) and Lemma 2.21(i) we have

$$\begin{aligned}
&\mathcal{M}\big(\ell^{q/p}(\mu),\; \ell^{u}(\mu)\big) \cdot \mathcal{M}\big(\ell^{r/p}(\mu),\; \ell^{q/p}(\mu)\big) \\
&= \ell^{v}(\mu) \cdot \ell^{w}(\mu) = \; \ell^{1}(\mu)_{[1/v]} \cdot \ell^{1}(\mu)_{[1/w]} \;=\; \ell^{1}(\mu)_{[(1/v)+(1/w)]} \\
&= \ell^{1}(\mu)_{[(1/u)-(p/r)]} \;=\; \ell^{(ru)/(r-pu)}(\mu).
\end{aligned} \tag{6.58}$$

Given a function $\psi \in \mathcal{M}\big(\ell^{r}(\mu),\, \ell^{u}(\mu)\big)$, we can conclude that

$$M_\psi \in \mathcal{A}_{p,q}(\ell^{r}(\mu), \ell^{u}(\mu)) \iff \psi \in \ell^{(ru)/(r-pu)}(\mu).$$

We omit the proof of this because it is similar to that of (6.53), which requires Proposition 6.27 and the discussion prior to Lemma 2.80.

(iii) Let us assume further that $p = 1$ and $q < u$. Then the following conditions for a function $\psi \in \mathcal{M}\big(\ell^{r}(\mu),\, \ell^{u}(\mu)\big)$ are equivalent.

(a) $M_\psi : \ell^{r}(\mu) \to \ell^{u}(\mu)$ is bidual q-concave.

(b) M_ψ is q-concave.

(c) $\psi \in \varphi^{(1/r)-(1/u)} \cdot \ell^{(qr)/(r-q)}$.

(d) M_ψ factorizes through $\ell^{q}(\mu)$ via multiplications.

In fact, the equivalence (a) $\Leftrightarrow$ (b) is from (6.50) with $p := 1$ and (6.6). The above (ii-a) (see (6.50)) with $p := 1$ verifies the equivalence (a) $\Leftrightarrow$ (c). Finally, in view of (6.52) with $p := 1$, we can see that condition (c) is equivalent to

$$\psi \in \mathcal{M}\big(\ell^{q}(\mu),\, \ell^{u}(\mu)\big) \cdot \mathcal{M}\big(\ell^{r}(\mu),\, \ell^{q}(\mu)\big),$$

which in turn is equivalent to (d) in view of the discussion prior to Lemma 2.80. Hence, we have established the equivalence of (a) to (d).

Note that the equivalence (b) $\Leftrightarrow$ (d) answers the question posed after Lemma 2.80 with the assumption that $u \geq 1$. $\square$

As an application of Proposition 6.27, we now provide a characterization of q-convex B.f.s.' for $1 \leq q < \infty$. For real B.f.s.', such a characterization has already been given in [32, Theorem 2.4], which is recorded as Lemma 6.32 below; it is needed for our characterization of the complex case as given in Proposition 6.33. As before, $\ell^q_{\mathbb{R}}$ denotes the real part of the complex sequence space ℓ^q for $1 \leq q < \infty$. Given a function $\psi : \mathbb{N} \to \mathbb{R}$ satisfying $\psi \cdot \ell^q_{\mathbb{R}} \subseteq \ell^p_{\mathbb{R}}$ with $1 \leq p, q < \infty$, let $M^{(\mathbb{R})}_\psi : \ell^q_{\mathbb{R}} \to \ell^p_{\mathbb{R}}$ denote the corresponding multiplication operator. Of course, if we consider complex spaces ℓ^p and ℓ^q and $\psi \in \mathcal{M}(\ell^q,\, \ell^p)$, then the corresponding multiplication operator will be denoted by $M_\psi : \ell^q \to \ell^p$ as usual.

Lemma 6.32. *Let $1 < q < \infty$. The following assertions are equivalent for a real B.f.s. $X_{\mathbb{R}}(\mu)$ over a positive, finite measure space (Ω, Σ, μ).*

(i) *$X_{\mathbb{R}}(\mu)$ is q-convex.*

(ii) *There is a number $p \in [1, q)$ such that every positive operator $S : X_{\mathbb{R}}(\mu) \to \ell^p_{\mathbb{R}}$ admits a factorization $S = M^{(\mathbb{R})}_\psi \circ R$ through $\ell^q_{\mathbb{R}}$ for some $\mathbb{R}$-linear operator $R \in \mathcal{L}\big(X_{\mathbb{R}}(\mu),\, \ell^q_{\mathbb{R}}\big)$ and some $\mathbb{R}$-valued function ψ on $\mathbb{N}$ satisfying $\psi \cdot \ell^q_{\mathbb{R}} \subseteq \ell^p_{\mathbb{R}}$.*

$$\begin{tikzcd}
X_{\mathbb{R}}(\mu) \arrow[rr, "S \geq 0"] \arrow[dr, "R"'] & & \ell^p_{\mathbb{R}} \\
& \ell^q_{\mathbb{R}} \arrow[ur, "M^{(\mathbb{R})}_\psi"'] &
\end{tikzcd}.$$

We shall now settle the complex case in Proposition 6.33 below. Our proof requires an application of Corollary 2 in [30] to a continuous $\mathbb{R}$-linear operator with values in the real Banach lattice $\ell^1_{\mathbb{R}}$ (which is the real part of ℓ^1). For this, we need to check that $\ell^1_{\mathbb{R}}$ is a B.f.s. in the sense of [30, p. 155], where the definition is different from ours. First, observe that $\ell^1_{\mathbb{R}}$ is an order ideal of the real vector lattice $\mathbb{R}^{\mathbb{N}}$ (in the pointwise order) which is, of course, the L^0-space corresponding to counting measure on $(\mathbb{N}, 2^{\mathbb{N}})$. So, condition (I) in [30, p. 155] holds. Moreover, $\ell^1_{\mathbb{R}}$ being a real Banach lattice, condition (II) in [30, p. 155] holds with $t := 1$ there. Finally, for $\ell^1_{\mathbb{R}}$, condition (III) in [30, p. 155] requires that $\|\psi_n\|_{\ell^1_{\mathbb{R}}} \to \|\psi\|_{\ell^1_{\mathbb{R}}}$ whenever ψ_n $(n \in \mathbb{N})$ and ψ are elements of $\ell^1_{\mathbb{R}}$ satisfying $\psi_n \uparrow \psi$ pointwise on $\mathbb{N}$; this surely holds because $\ell^1_{\mathbb{R}}$ is σ-o.c. Therefore, $\ell^1_{\mathbb{R}}$ is indeed a B.f.s. in the sense of [30].

Proposition 6.33. *Let $1 \leq q < \infty$ and $X(\mu)$ be a σ-order continuous B.f.s. over a positive, finite measure space (Ω, Σ, μ). The following assertions are equivalent.*

(i) *$X(\mu)$ is q-convex.*

(ii) *For each q-concave Banach lattice E and each positive operator $T : X(\mu) \to E$, there exist a non-negative function $g \in \mathcal{M}\big(X(\mu), L^q(\mu)\big)$ and a linear operator*

$U \in \mathcal{L}\big(L^q(\mu), E\big)$ such that T factorizes as $T = U \circ M_g$.

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, "M_g"'] & & E \\ & L^q(\mu) \arrow[ur, "U"'] & \end{tikzcd} .$$

(iii) *For every positive operator $T : X(\mu) \to \ell^1$ there exist a non-negative function $g \in \mathcal{M}\big(X(\mu), L^q(\mu)\big)$ and an operator $U \in \mathcal{L}\big(L^q(\mu),\, \ell^1\big)$ such that T factorizes as $T = U \circ M_g$.*

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, "M_g"'] & & \ell^1 \\ & L^q(\mu) \arrow[ur, "U"'] & \end{tikzcd} .$$

(iv) *There is a number $p \in [1, q]$ such that every positive operator $T : X(\mu) \to \ell^p$ admits a factorization $T = M_\psi \circ V$ through ℓ^q for some $V \in \mathcal{L}\big(X(\mu),\, \ell^q\big)$ with $V\big(X_{\mathbb{R}}(\mu)\big) \subseteq \ell^q_{\mathbb{R}}$ and some $\mathbb{R}$-valued function ψ on $\mathbb{N}$ satisfying $\psi \cdot \ell^q_{\mathbb{R}} \subseteq \ell^p_{\mathbb{R}}$.*

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, "V"'] & & \ell^p \\ & \ell^q \arrow[ur, "M_\psi"'] & \end{tikzcd} .$$

Proof. (i) $\Rightarrow$ (ii). Let E and T be as in the statement of (ii). The positive operator T is q-concave because E is q-concave (see Corollary 2.70). This and (i) imply part (ii) via Corollary 6.17, whenever T is μ-determined. Otherwise, argue for the restriction T_1 of T to an essential carrier Ω_1 (see Remark 6.8) and note that T_1 is also positive and q-concave and that $X(\mu_1)$ is again q-concave.

(ii) $\Rightarrow$ (iii). Part (iii) is a special case of (ii) because ℓ^1 is 1-concave and hence, also q-concave (see Example 2.73(i-b)).

(iii) $\Rightarrow$ (iv). Fix a positive operator $T : X(\mu) \to \ell^q$. According to (iii), we can choose a non-negative function $g \in \mathcal{M}\big(X(\mu),\, L^q(\mu)\big)$ and an operator $U \in \mathcal{L}\big(L^q(\mu),\, \ell^1\big)$ such that $T = U \circ M_g$. Set $A := \{\omega \in \Omega : g(\omega) \neq 0\}$ and define an operator $U_1 \in \mathcal{L}\big(L^q(\mu),\, \ell^1\big)$ by

$$U_1(h) := U(h\chi_A), \qquad h \in L^q(\mu).$$

Then

$$U \circ M_g = U_1 \circ M_g \qquad \text{on} \quad X(\mu).$$

We claim that U_1 is positive. In fact, let $h \in L^q(\mu)^+$. It follows from Proposition 2.27 that the range $\mathcal{R}(M_g)$ of M_g is dense in the closed subspace $\chi_A \cdot L^q(\mu)$ of $L^q(\mu)$. Since $g \geq 0$, we can select a sequence $\{f_n\}_{n=1}^\infty \subseteq X(\mu)^+$ such that $M_g(f_n) \to h\chi_A$ in $L^q(\mu)$ as $n \to \infty$. So,

$$U_1(h) = U(h\chi_A) \;=\; \lim_{n\to\infty} U\big(M_g(f_n)\big) \;=\; \lim_{n\to\infty} T(f_n) \;\geq\; 0,$$

which implies that U_1 is positive.

The positive operator U_1 on the q-convex B.f.s. $L^q(\mu)$ is q-convex (see Corollary 2.65). In view of the inclusion $U_1\big(L^q_{\mathbb{R}}(\mu)\big) \subseteq \ell^1_{\mathbb{R}}$, due to the positivity of $U_1 \in \mathcal{L}(L^q(\mu), \ell^1)$, let $U_1^{(\mathbb{R})} : L^q_{\mathbb{R}}(\mu) \to \ell^1_{\mathbb{R}}$ denote the restriction of U_1 to $L^q_{\mathbb{R}}(\mu)$, with codomain $\ell^1_{\mathbb{R}}$. Then $U_1^{(\mathbb{R})}$ is also q-convex, which is an immediate consequence of Definition 2.46 and Remark 2.48. Moreover, being 1-concave, $\ell^1_{\mathbb{R}}$ is also q-concave; see Lemma 2.49(ii) and Example 2.73(i-b). Consequently, $U_1^{(\mathbb{R})}$ is a q-convex $\mathbb{R}$-linear operator from the real Banach space $L^q_{\mathbb{R}}(\mu)$ into the q-concave space $\ell^1_{\mathbb{R}}$. Recall that $\ell^1_{\mathbb{R}}$ is a B.f.s. over $(\mathbb{N}, 2^{\mathbb{N}})$, relative to counting measure, in the sense of [30, p. 155]; see the discussion immediately prior to this proposition. So, it follows from [30, Corollary 2] that $U_1^{(\mathbb{R})} = M_\psi^{(\mathbb{R})} \circ W^{(\mathbb{R})}$ for some $\mathbb{R}$-linear operator $W^{(\mathbb{R})} \in \mathcal{L}\big(L^q_{\mathbb{R}}(\mu), \ell^q_{\mathbb{R}}\big)$ and non-negative function ψ on $\mathbb{N}$ with $\psi \cdot \ell^q_{\mathbb{R}} \subseteq \ell^1_{\mathbb{R}}$. Here, $M_\psi^{(\mathbb{R})} : \ell^q_{\mathbb{R}} \to \ell^1_{\mathbb{R}}$ is, of course, the multiplication operator corresponding to ψ.

Recalling that $g \geq 0$, let $M_g^{(\mathbb{R})}$ denote the corresponding multiplication operator from the real part $X_{\mathbb{R}}(\mu)$ of $X(\mu)$ into the real part $L^q_{\mathbb{R}}(\mu)$ of $L^q(\mu)$. Now, given $f \in X_{\mathbb{R}}(\mu) \subseteq X(\mu)$, it follows that

$$T(f) \;=\; \big(U \circ M_g\big)(f) \;=\; \big(U_1 \circ M_g\big)(f) \;=\; \big(M_\psi^{(\mathbb{R})} \circ (W^{(\mathbb{R})} \circ M_g^{(\mathbb{R})})\big)(f). \tag{6.59}$$

Let $V \in \mathcal{L}(X(\mu), \ell^q)$ denote the natural $\mathbb{C}$-linear extension of the $\mathbb{R}$-linear operator $(W^{(\mathbb{R})} \circ M_g^{(\mathbb{R})}) : X_{\mathbb{R}}(\mu) \to \ell^q_{\mathbb{R}}$, that is,

$$V\big(f_1 + if_2\big) := \big(W^{(\mathbb{R})} \circ M_g^{(\mathbb{R})}\big)(f_1) \;+\; i\big(W^{(\mathbb{R})} \circ M_g^{(\mathbb{R})}\big)(f_2)$$

for all $f = f_1 + if_2 \in X_{\mathbb{R}}(\mu) + iX_{\mathbb{R}}(\mu) = X(\mu)$; see [149, p. 135], for example. Then we have from (6.59) that $T = M_\psi \circ V$ because, given $f_1, f_2 \in X_{\mathbb{R}}(\mu)$, it follows that

$$\begin{aligned}
T\big(f_1 + if_2\big) &= T(f_1) + iT(f_2) \\
&= \big(M_\psi^{(\mathbb{R})} \circ (W^{(\mathbb{R})} \circ M_g^{(\mathbb{R})})\big)(f_1) + i\big(M_\psi^{(\mathbb{R})} \circ (W^{(\mathbb{R})} \circ M_g^{(\mathbb{R})})\big)(f_2) \\
&= M_\psi\Big((W^{(\mathbb{R})} \circ M_g^{(\mathbb{R})})(f_1) + i(W^{(\mathbb{R})} \circ M_g^{(\mathbb{R})})(f_2)\Big) \\
&= \big(M_\psi \circ V\big)(f_1 + if_2).
\end{aligned}$$

In other words, part (iv) holds with $p := 1$.

(iv) $\Rightarrow$ (i). If $q = 1$ in the assumptions of Proposition 6.33, then part (i) *always* holds (without any recourse to parts (ii)–(iv)) because every B.f.s. is always 1-convex (see Proposition 2.77).

Now, assume that $1 < q < \infty$ and let p be as in part (iv). In order to establish condition (ii) of Lemma 6.32, let $S : X_{\mathbb{R}}(\mu) \to \ell^p_{\mathbb{R}}$ be any positive operator. The natural continuous $\mathbb{C}$-linear extension $T : X(\mu) \to \ell^p$ of S is defined by

$$T(f_1 + if_2) := \; S(f_1) + iS(f_2), \qquad f = f_1 + if_2 \in X_{\mathbb{R}}(\mu) + iX_{\mathbb{R}}(\mu) = X(\mu);$$

again see [149, p. 135]. Clearly T is positive because S is. Since $V\big(X_{\mathbb{R}}(\mu)\big) \subseteq \ell^q_{\mathbb{R}}$, let $V^{(\mathbb{R})} : X_{\mathbb{R}}(\mu) \to \ell^q_{\mathbb{R}}$ denote the restriction of V to $X_{\mathbb{R}}(\mu)$, with codomain $\ell^q_{\mathbb{R}}$. Then the factorization $T = M_\psi \circ V$, as given by the assumptions in part (iv), yields that $S = M_\psi^{(\mathbb{R})} \circ V^{(\mathbb{R})}$. That is, condition (ii) of Lemma 6.32 is satisfied with $R := V^{(\mathbb{R})}$ and hence, $X_{\mathbb{R}}(\mu)$ is q-convex via the same lemma. Finally, apply Lemma 2.49(i) to obtain the q-convexity of $X(\mu)$. $\square$

Inspired by the equivalence (i) $\Leftrightarrow$ (vi) in Proposition 6.27, we now present a corresponding equivalence for a (p,q)-power-concave operator.

Proposition 6.34. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ) and E be a Banach space. Suppose that $1 \le p < \infty$ and $0 < q < \infty$. Then a continuous linear operator $T : X(\mu) \to E$ is (p,q)-power-concave if and only if it is p-th power factorable and its continuous linear extension $T_{[p]} : X(\mu)_{[p]} \to E$ is (q/p)-concave.*

Proof. Assume first that T is (p,q)-power-concave. Proposition 6.2(ii) gives that T is p-th power factorable. By the definition of (p,q)-power-concavity, there is $C > 0$ such that

$$\sum_{j=1}^{n} \big\|T(s_j)\big\|_E^{q/p} \le C \Big\| \sum_{j=1}^{n} |s_j|^{q/p} \Big\|_{X(\mu)_{[q]}}, \qquad n \in \mathbb{N}, \quad s_1, \dots, s_n \in \operatorname{sim}\Sigma. \tag{6.60}$$

To prove that $T_{[p]} \in \mathcal{L}(X(\mu)_{[p]}, E)$ is (q/p)-concave, fix $n \in \mathbb{N}$ and $f_1, \dots, f_n \in X(\mu)_{[p]}$. Given $j = 1, \dots, n$, the fact that $X(\mu)_{[p]}$ is σ-o.c. (see Lemma 2.21(iii)) allows us to select a sequence $\big\{s_k^{(j)}\big\}_{n=1}^\infty \subseteq \operatorname{sim}\Sigma$ such that $\lim_{k\to\infty} s_k^{(j)} = f_j$ in the quasi-norm $\|\cdot\|_{X(\mu)_{[p]}}$ and $|s_k^{(j)}| \uparrow |f_j|$ pointwise (relative to k). Then, since $X(\mu)_{[q]}$ is also σ-o.c. and $\sum_{j=1}^n \big|s_k^{(j)}\big|^{q/p} \uparrow \sum_{j=1}^n \big|f_j\big|^{q/p} \in X(\mu)_{[q]}$ pointwise (relative to k), it follows that

$$\lim_{k\to\infty} \sum_{j=1}^{n} \big|s_k^{(j)}\big|^{q/p} = \sum_{j=1}^{n} \big|f_j\big|^{q/p}$$

in the topology of $X(\mu)_{[q]}$. Proposition 2.2(vi) applied to the q-B.f.s.$X(\mu)_{[q]}$ yields that

$$\lim_{k\to\infty} \Big\|\sum_{j=1}^{n} \big|s_k^{(j)}\big|^{q/p}\Big\|_{X(\mu)_{[q]}} \le 4^{1/r} \Big\|\sum_{j=1}^{n} \big|f_j\big|^{q/p}\Big\|_{X(\mu)_{[q]}} \tag{6.61}$$

for some $r > 0$ determined by the quasi-norm $\|\cdot\|_{X(\mu)_{[q]}}$ (see (2.5) with $Z := X(\mu)_{[q]}$). Now, the continuity of $T_{[p]}$ gives that $\lim_{k\to\infty} T_{[p]}\big(s_k^{(j)}\big) = T_{[p]}(f_j)$ in the Banach space E, which implies (since all $s_k^{(j)} \in X(\mu) \subseteq X(\mu)_{[p]}$) that

$$\big\|T_{[p]}(f_j)\big\|_E = \lim_{k\to\infty} \big\|T_{[p]}(s_k^{(j)})\big\|_E = \lim_{k\to\infty} \big\|T(s_k^{(j)})\big\|_E$$

whenever $j = 1,\dots,n$. This, (6.60) and (6.61) yield that

$$\begin{aligned}
\bigg(\sum_{j=1}^n \big\|T_{[p]}(f_j)\big\|_E^{q/p}\bigg)^{p/q} &= \bigg(\sum_{j=1}^n \lim_{k\to\infty} \big\|T\big(s_k^{(j)}\big)\big\|_E^{q/p}\bigg)^{p/q} \\
&= \bigg(\lim_{k\to\infty} \sum_{j=1}^n \big\|T\big(s_k^{(j)}\big)\big\|_E^{q/p}\bigg)^{p/q} \le \bigg(\lim_{k\to\infty} C\Big\|\sum_{j=1}^n |s_k^{(j)}|^{q/p}\Big\|_{X(\mu)_{[q]}}\bigg)^{p/q} \\
&\le \bigg(4^{1/r} C\Big\|\sum_{j=1}^n \big|f_j\big|^{q/p}\Big\|_{X(\mu)_{[q]}}\bigg)^{p/q} = 4^{p/(qr)} C^{p/q} \bigg\|\Big(\sum_{j=1}^n \big|f_j\big|^{q/p}\Big)^{p/q}\bigg\|_{X(\mu)_{[p]}}.
\end{aligned}$$

Here the last equality is a consequence (via (2.47)) of the general fact that

$$\big\|g\big\|_{X(\mu)_{[q]}}^{p/q} = \big\|\,|g|^{p/q}\big\|_{X(\mu)_{[p]}}, \qquad g \in X(\mu)_{[q]}, \tag{6.62}$$

because $\sum_{j=1}^n \big|f_j\big|^{q/p} \in X(\mu)_{[q]}$. According to Definition 2.46(ii), the operator $T_{[p]} : X(\mu)_{[p]} \to E$ is (q/p)-concave.

Conversely, assume that $T_{[p]} : X(\mu)_{[p]} \to E$ is (q/p)-concave in which case there exists a constant $C_1 > 0$ such that

$$\bigg(\sum_{j=1}^n \big\|T_{[p]}(g_j)\big\|_E^{q/p}\bigg)^{p/q} \le C_1 \bigg\|\Big(\sum_{j=1}^n \big|g_j\big|^{q/p}\Big)^{p/q}\bigg\|_{X(\mu)_{[p]}} = C_1 \Big\|\sum_{j=1}^n \big|g_j\big|^{q/p}\Big\|_{X(\mu)_{[q]}}^{p/q} \tag{6.63}$$

for all $n \in \mathbb{N}$ and $g_1,\dots,g_n \in X(\mu)_{[p]}$. Here, for the equality in (6.63), we have again applied (6.62) with $g := \sum_{j=1}^n \big|g_j\big|^{q/p} \in X(\mu)_{[q]}$. Now, fix $n \in \mathbb{N}$ and $h_1,\dots,h_n \in X(\mu) \subseteq X(\mu)_{[p]}$. Then, (6.63) with $g_j := h_j$ for $j = 1,\dots,n$ yields that

$$\sum_{j=1}^n \big\|T(h_j)\big\|_E^{q/p} = \sum_{j=1}^n \big\|T_{[p]}(h_j)\big\|_E^{q/p} \le (C_1)^{q/p} \Big\|\sum_{j=1}^n |h_j|^{q/p}\Big\|_{X(\mu)_{[q]}}$$

because $T_{[p]}$ is an E-valued extension of T from $X(\mu)$ to $X(\mu)_{[p]}$. According to (6.4), we see that $T : X(\mu) \to E$ is (p,q)-power-concave. □

Let us now show that the converse statement of part (ii) of Proposition 6.2 may not hold, in general; see the discussion after Example 6.5.

Example 6.35. Let (Ω,Σ,μ) be σ-decomposable, positive, finite measure space. Take positive numbers p,q,r such that $1 \le p,r < \infty$ and $0 < q < pr$. Fix any continuous linear operator $S : L^r(\mu) \to L^r(\mu)$ which is not (q/p)-concave; for instance, the identity operator on $L^r(\mu)$ will do by applying both the inequality $(q/p) < r$ and Example 2.73(ii-b). Let

$$X(\mu) := L^{pr}(\mu) \qquad \text{and} \qquad E := L^r(\mu)$$

and consider $T := S \circ i_{[p]}$, where $i_{[p]} : X(\mu) = L^{pr}(\mu) \to X(\mu)_{[p]} = L^r(\mu)$ is the canonical injection (note that $(pr) \geq r$). Clearly, $T : X(\mu) \to E$ is p-th power factorable; see (5.3). However, its continuous linear extension $T_{[p]}$ to $X(\mu)_{[p]}$, which equals S, is not (q/p)-concave. Therefore, Proposition 6.34 yields that the p-th power factorable operator T is not (p, q)-power-concave for all $0 < q < (pr)$. $\square$

Corollary 6.36. *Suppose that* $1 \leq p \leq q < \infty$. *Let* $X(\mu)$ *be any* p*-convex q-B.f.s.* (*over a positive, finite measure space* (Ω, Σ, μ)) *for which* $\mathbf{M}^{(p)}[X(\mu)] = 1$. *Then, for every Banach space* E, *we have*

$$\Pi_{q/p}\big(X(\mu)_{[p]}, E\big) \circ i_{[p]} \subseteq \mathcal{B}_{p,q}\big(X(\mu), E\big). \tag{6.64}$$

Proof. The assumption that $\mathbf{M}^{(p)}[X(\mu)] = 1$ guarantees that the given quasi-norm $\|\cdot\|_{X(\mu)_{[p]}}$ is a *norm* on the p-th power $X(\mu)_{[p]}$ (see Proposition 2.23(iii)). That is, $X(\mu)_{[p]}$ is a B.f.s. with $1 \leq (q/p) < \infty$ and hence, we can speak of the E-valued absolutely (q/p)-summing operators on $X(\mu)_{[p]}$; see Example 2.61. Now, fix $S \in \Pi_{q/p}\big(X(\mu)_{[p]}, E\big)$. Then the composition $T := S \circ i_{[p]}$ is a p-th power factorable operator from $X(\mu)$ into E with $T_{[p]} = S$; see (5.3). It follows from Example 2.61 that $T_{[p]} = S : X(\mu)_{[p]} \to E$ is (q/p)-concave. So, T is (p, q)-power-concave via Proposition 6.34. $\square$

The assumption in Corollary 6.36 that $\mathbf{M}^{(p)}[X(\mu)] = 1$ is not essential. In fact, suppose that $X(\mu)$ is any p-convex q-B.f.s. but, $\mathbf{M}^{(p)}[X(\mu)]$ is not necessarily equal to 1. Then the p-th power $X(\mu)_{[p]}$ of $X(\mu)$ admits an equivalent lattice norm $\eta_{[p]}$ (see Proposition 2.23(ii). So, we need to replace (6.64) with

$$\Pi_{q/p}\big((X(\mu)_{[p]}, \eta_{[p]}),\ E\big) \circ i_{[p]} \subseteq \mathcal{B}_{p,q}(X(\mu), E) \tag{6.65}$$

because (in this monograph) absolutely (q/p)-summing operators are only defined on Banach spaces.

Let us give an application of the previous corollary to the construction of (p, q)-power-concave operators arising from a class of classical kernel operators.

Example 6.37. Let $1 \leq p \leq q < r$ and (Ω, Σ, μ) be a positive, finite measure space. Set $X(\mu) := L^r(\mu)$ and $E := L^{q/p}(\mu)$. Then $L^r(\mu)$ is p-convex with $\mathbf{M}^{(p)}\big[L^r(\mu)\big] = 1$ (see Example 2.73(i-a)). We first exhibit an absolutely (q/p)-summing operator from $L^{r/p}(\mu) = X(\mu)_{[p]}$ into $E = L^{q/p}(\mu)$. Let $u := (r/p)'$, that is, $u = r/(r - p)$. Take any strongly μ-measurable function $F : \Omega \to L^u(\mu)$ (see Section 3.2 for the definition) such that the scalar function $\omega \mapsto \big\|F(\omega)\big\|^{q/p}_{L^u(\mu)}$, defined on Ω, belongs to $L^1(\mu)$. This is usually denoted by $F \in L^{q/p}\big(\mu, L^u(\mu)\big)$ in the literature; see

[41, p. 43] and [42, pp. 49–50], for example. Noting that $L^u(\mu)^* = L^{r/p}(\mu)$, it follows from Hölder's inequality that $\big|\langle g,\, F(\omega)\rangle\big|^{q/p} \leq \|g\|^{q/p}_{L^{r/p}(\mu)} \cdot \|F(\omega)\|^{q/p}_{L^u(\mu)}$ for each $\omega \in \Omega$ and each $g \in L^{r/p}(\mu)$. Accordingly, we can define a continuous linear operator $S : L^{r/p}(\mu) \to L^{q/p}(\mu)$ by

$$S(g)(\omega) := \big\langle g,\, F(\omega)\big\rangle, \qquad g \in L^{r/p}(\mu), \quad \omega \in \Omega.$$

Then [41, Example 2.11] can be applied to deduce that S is absolutely (q/p)-summing on $L^{r/p}(\mu) = X(\mu)_{[p]}$. Hence, $S \circ i_{[p]} : X(\mu) = L^r(\mu) \to E = L^{q/p}(\mu)$ is (p,q)-power-concave; see Corollary 6.36. $\square$

Typical examples of operators S of the form given above are kernel operators of Hille-Tamarkin type, [41, p. 43].

The following result was already announced in Remark 6.7(ii).

Corollary 6.38. *Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ) and E be a Banach space. Suppose that $1 \leq p_1 \leq p_2 < \infty$ and $0 < q \leq r < \infty$. Then*

$$\mathcal{B}_{p_2,q}\big(X(\mu), E\big) \subseteq \mathcal{B}_{p_1,r}\big(X(\mu), E\big).$$

Proof. Let $T \in \mathcal{B}_{p_2,q}(X(\mu), E)$. Then it follows from Proposition 6.34 that T is p_2-th power factorable and its continuous linear extension $T_{[p_2]} : X(\mu)_{[p_2]} \to E$ is (q/p_2)-concave. According to Proposition 2.54(iv), the operator $T_{[p_2]}$ is also (r/p_1)-concave because $(r/p_1) \geq (q/p_2)$. Let $M_{\mathbf{1}} : X(\mu)_{[p_1]} \to X(\mu)_{[p_2]}$ denote the multiplication operator by the constant function $\mathbf{1} := \chi_\Omega$ (i.e., it is the natural injection). Since $X(\mu)_{[p_1]} \subseteq X(\mu)_{[p_2]}$ (see Lemma 2.21(iv)), the operator $T : X(\mu) \to E$ is also p_1-th power factorable and we have $T = T_{[p_1]} \circ i_{[p_1]}$ where

$$T_{[p_1]} = T_{[p_2]} \circ M_{\mathbf{1}}.$$

That is, we have

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[d, "i_{[p_1]}"'] \arrow[drr, "i_{[p_2]}"] & & E \\ X(\mu)_{[p_1]} \arrow[rr, "M_{\mathbf{1}}"] & & X(\mu)_{[p_2]}. \arrow[u, "T_{[p_2]}"'] \end{tikzcd}$$

Since $T_{[p_2]}$ is (r/p_1)-concave, the map $T_{[p_1]} = T_{[p_2]} \circ M_{\mathbf{1}}$ is also (r/p_1)-concave via Proposition 2.68(ii) because

$$M_{\mathbf{1}} \in \Lambda_{r/p_1}(X(\mu)_{[p_1]}, X(\mu)_{[p_2]});$$

see Example 2.59. Consequently, it follows from Proposition 6.34 above that $T \in \mathcal{B}_{p_1,r}(X(\mu), E)$. $\square$

6.4 Factorization of the integration operator

Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a finite, positive, measure space (Ω, Σ, μ) and E be a Banach space. The importance of the integration operator $I_{m_T} : L^1(m_T) \to E$ lies in the fact that it is the extension of a μ-determined operator $T : X(\mu) \to E$ to its *optimal domain* $L^1(m_T)$ and hence, plays a crucial role in the detailed understanding of T itself. Hence, it is most useful to be able to find factorizations of T and/or I_{m_T}, whenever they may exist. So, suppose that $T \in \mathcal{F}_{[p]}(X(\mu), E)$ for some $1 \leq p < \infty$. Then Theorem 5.7 guarantees that the natural inclusion map $J_T : X(\mu) \to L^1(m_T)$ factors through $L^p(m_T)$ via the natural inclusions. In other words,

$$X(\mu) \subseteq L^p(m_T) \subseteq L^1(m_T)$$

or equivalently

$$J_T = \alpha_p \circ J_T^{(p)},$$

where $J_T^{(p)} : X(\mu) \to L^p(m_T)$ and $\alpha_p : L^p(m_T) \to L^1(m_T)$ denote the respective natural inclusion map; see Remark 5.8. Given further assumptions on T, Theorem 6.9 gives equivalent conditions for $J_T^{(p)}$ to factorize through $L^q(g\, d\mu)$ for some $g \in L^0(\mu)$ and $0 < q < \infty$. In this section we focus our attention on factorization of the map α_p. However, since the main role of $X(\mu)$ and T in this context is to define the vector measure $m_T : \Sigma \to E$, we shall present our results in the setting of general vector measures. In order to provide further explanations concerning the contents of this section, we begin with a consequence of Corollary 6.17.

Lemma 6.39. *Let $1 \leq p < \infty$. Given a measurable space (Ω, Σ), the following four conditions are equivalent for a Banach-space-valued vector measure $\nu : \Sigma \to E$.*

(i) *The restricted integration operator $I_\nu^{(p)} : L^p(\nu) \to E$ is p-concave.*

(ii) *The inclusion map $\alpha_p : L^p(\nu) \to L^1(\nu)$ is p-concave.*

(iii) *For every Rybakov functional $x^* \in \mathbf{R}_\nu[E^*]$, there exists a function $g \in L^0(\Sigma)$ with $g > 0$ (ν-a.e.) such that*

$$L^p(\nu) \subseteq L^p\big(g\, d|\langle \nu, x^* \rangle|\big) \subseteq L^1(\nu).$$

(iv) *There exists a constant $C > 0$ such that*

$$\Bigg(\sum_{j=1}^{n} \Big\| \int_\Omega s_j \, d\nu \Big\|_E^p \Bigg)^{1/p} \leq C \Bigg\| \Big(\sum_{j=1}^{n} |s_j|^p \Big)^{1/p} \Bigg\|_{L^p(\nu)} \tag{6.66}$$

for all $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in \operatorname{sim} \Sigma$.

If, in addition, we can select a constant $C_1 > 0$ such that

$$\|f\|_{L^1(\nu)} \leq C_1 \Big\| \int_\Omega |f| \, d\nu \Big\|_E, \qquad f \in L^1(\nu), \tag{6.67}$$

then any of (i) *to* (iv) *above is equivalent to the following condition:*

(v) *there exists a constant* $C^* > 0$ *such that*

$$\Big(\sum_{j=1}^{n} \Big\| \int_\Omega s_j \, d\nu \Big\|_E^p \Big)^{1/p} \le C^* \Big\| \int_\Omega \sum_{j=1}^{n} |s_j|^p d\nu \Big\|_E^{1/p} \tag{6.68}$$

for all $n \in \mathbb{N}$ *and* $s_1, \ldots, s_n \in \operatorname{sim} \Sigma$.

Proof. (i) $\Leftrightarrow$ (ii). This is a special case of Corollary 3.73(i).

(i) $\Leftrightarrow$ (iii). For every $x^* \in \mathbf{R}_\nu[E^*]$, the space $L^p(\nu)$ is a B.f.s. over the finite measure space $\big(\Omega, \Sigma, |\langle \nu, x^* \rangle|\big)$; see Proposition 3.28(i) with $\mu := |\langle \nu, x^* \rangle|$. Moreover, recalling that $L^p(\nu)$ is p-convex (see Proposition 3.28(i)), we can apply Corollary 6.17 (with $q := p$, $\mu := |\langle \nu, x^* \rangle|$, $T := I_\nu^{(p)}$ and $X(\mu) := L^p(\nu)$ in which case $m_T = \nu$ and T is μ-determined) to derive the equivalence (i) $\Leftrightarrow$ (iii).

(i) $\Leftrightarrow$ (iv). In view of the fact that $\int_\Omega s \, d\nu = I_\nu^{(p)}(s)$ for all $s \in \operatorname{sim} \Sigma$, the equivalence (i) $\Leftrightarrow$ (iv) is a consequence of Lemma 2.52(ii) (with $q := p$, $Z := L^p(\nu)$, $Z_0 := \operatorname{sim} \Sigma$, $W := E$ and $S := I_\nu^{(p)}$) because $\operatorname{sim} \Sigma$ is dense in the order continuous B.f.s. $L^p(\nu)$ (see Remark 2.6 and Proposition 3.28(i)).

Now we shall prove that part (iv) together with the assumption (6.67) imply part (v). Fix $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in \operatorname{sim} \Sigma$. Then we have, from part (iv) and (3.50), that

$$\begin{aligned} \Big(\sum_{i=1}^{n} \Big\| \int_\Omega s_i \, d\nu \Big\|_E^p \Big)^{1/p} &\le C \Big\| \Big(\sum_{j=1}^{n} |s_j|^p \Big)^{1/p} \Big\|_{L^p(\nu)} \\ = C \Big\| \sum_{j=1}^{n} |s_j|^p \Big\|_{L^1(\nu)}^{1/p} &\le C C_1^{1/p} \Big\| \int_\Omega \sum_{j=1}^{n} |s_j|^p \, d\nu \Big\|_E^{1/p}; \end{aligned}$$

we have used the assumption (6.67) with $f := \sum_{j=1}^n |s_j|^p \in L^1(\nu)$. That is, (v) holds with $C^* := C C_1^{1/p}$.

Finally we shall verify the implication (v) $\Rightarrow$ (iv), for which we do not require (6.67) as will be seen now. Fix $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in \operatorname{sim} \Sigma$. Then part (v) together with (3.50) and (3.99) imply that

$$\begin{aligned} \Big(\sum_{j=1}^{n} \Big\| \int_\Omega s_j \, d\nu \Big\|_E^p \Big)^{1/p} &\le C^* \Big\| \int_\Omega \sum_{j=1}^{n} |s_j|^p \, d\nu \Big\|_E^{1/p} = C^* \Big\| I_\nu \Big(\sum_{j=1}^{n} |s_j|^p \Big) \Big\|_E^{1/p} \\ &\le C^* \|I_\nu\|^{1/p} \Big\| \sum_{j=1}^{n} |s_j|^p \Big\|_{L^1(\nu)}^{1/p} = C^* \Big\| \Big(\sum_{j=1}^{n} |s_j|^p \Big)^{1/p} \Big\|_{L^p(\nu)}. \end{aligned}$$

In other words, part (iv) holds with $C := C^*$. $\square$

Given a general Banach-space-valued vector measure $\nu : \Sigma \to E$, a natural question arising from Lemma 6.39(iii) above is whether or not we can choose

$g := \chi_\Omega$ for an appropriate choice of $x^* \in \mathbf{R}_\nu[E^*]$, that is, whether or not

$$L^p(\nu) \subseteq L^p\big(|\langle \nu, x^*\rangle|\big) \subseteq L^1(\nu)\,? \tag{6.69}$$

A positive answer would be most desirable. Unfortunately, this is not the case in general, as already demonstrated in Example 5.12(ii). Accordingly, our aim in this section is to present two classes of vector measures ν for which (6.69) *does hold*, for a suitable choice of $x^* \in \mathbf{R}_\nu[E^*]$; see Proposition 6.40 and Theorem 6.41 below. Consequently, we shall have available two independent sufficient conditions each of which provides an affirmative answer to Question (B) posed in Section 5.3; see Remark 6.42 below. Moreover, each of Proposition 6.40 and Theorem 6.41 also provides a sufficient condition under which the p-concavity of the restricted integration operator $I_\nu^{(p)}$ is equivalent to condition (v) of Lemma 6.39.

Proposition 6.40. *Let $\nu : \Sigma \to E$ be a Banach-space-valued vector measure on a measurable space (Ω, Σ) such that the integration operator $I_\nu : L^1(\nu) \to E$ is an isomorphism onto its range $\mathcal{R}(I_\nu)$. Given any $1 \le p < \infty$, the following assertions are equivalent.*

(i) *The restricted integration operator $I_\nu^{(p)} : L^p(\nu) \to E$ is p-concave.*

(ii) *There exists $x_0^* \in \mathbf{R}_\nu[E^*]$ such that the scalar measure $\langle \nu, x_0^*\rangle$ is positive and*

$$L^p(\nu) \subseteq L^p(\langle \nu, x_0^*\rangle) \subseteq L^1(\nu).$$

(iii) *There exists a constant $C > 0$ such that*

$$\bigg(\sum_{j=1}^n \Big\| \int_\Omega s_j \, d\nu \Big\|_E^p \bigg)^{1/p} \le C \Big\| \int_\Omega \sum_{j=1}^n |s_j|^p d\nu \Big\|_E^{1/p}$$

for all $n \in \mathbb{N}$ and $s_1, \dots, s_n \in \operatorname{sim} \Sigma$.

Proof. (i) $\Rightarrow$ (ii). Fix any $x_1^* \in \mathbf{R}_\nu[E^*]$. By Lemma 6.39, there exists $g \in L^0(\Sigma)$ with $g > 0$ (ν-a.e.) such that

$$L^p(\nu) \subseteq L^p\big(g\, d|\langle \nu, x_1^*\rangle|\big) \subseteq L^1(\nu).$$

In particular, we have

$$L^1(\nu) = L^p(\nu)_{[p]} \subseteq L^p\big(g\, d|\langle \nu, x_1^*\rangle|\big)_{[p]} = L^1(g\, d|\langle \nu, x_1^*\rangle|);$$

see Lemma 2.20(ii). Since $\chi_\Omega \in L^1(\nu)$, we conclude that the indefinite integral $gd|\langle \nu, x_1^*\rangle|$ is a finite, positive measure and hence, the corresponding linear functional $f \mapsto \int_\Omega f g\, d|\langle \nu, x_1^*\rangle|$ is continuous on $L^1(g\, d|\langle \nu, x_1^*\rangle|)$. It then follows that the linear functional

$$\eta_g : f \mapsto \int_\Omega f g\, d|\langle \nu, x_1^*\rangle|, \qquad f \in L^1(\nu),$$

is continuous on $L^1(\nu)$, that is, $\eta_g \in \big(L^1(\nu)^*\big)^+$. Since I_ν is an isomorphism onto its range, the space $L^1(\nu)$ can be identified with the closed subspace $\mathcal{R}(I_\nu)$ of E. Hence, the dual operator $I_\nu^* : \big(\mathcal{R}(I_\nu)\big)^* \to L^1(\nu)^*$ is a vector space isomorphism. So, via the Hahn-Banach Theorem, there exists $x_0^* \in E^*$ satisfying $\eta_g = I_\nu^*(\tilde{x_0}^*)$, where $\tilde{x_0}^* \in \big(\mathcal{R}(I_\nu)\big)^*$ denotes the restriction of x_0^* to $\mathcal{R}(I_\nu)$. It follows that

$$\langle f, \eta_g\rangle = \big\langle I_\nu(f), x_0^*\big\rangle = \int_\Omega f\, d\langle \nu, x_0^*\rangle, \qquad f \in L^1(\nu).$$

Since $\eta_g \geq 0$, it is easy to deduce that $\langle \nu, x_0^*\rangle$ is a positive scalar measure satisfying $L^p\big(g\, d|\langle \nu, x_1^*\rangle|\big) = L^p\big(\langle \nu, x_0^*\rangle\big)$. Moreover, since $g > 0$ (μ-a.e.), we must have that x_0^* is a Rybakov functional for ν. Accordingly, (ii) holds.

Apply Lemma 6.39 to derive the implication (ii) $\Rightarrow$ (i).

(i) $\Leftrightarrow$ (iii). Since $I_\nu^{-1} : \mathcal{R}(I_\nu) \to L^1(\nu)$ is continuous on $\mathcal{R}(I_\nu)$ equipped with the topology induced by E we have, for any $f \in L^1(\nu)$, that

$$\|f\|_{L^1(\nu)} = \big\|I_\nu^{-1}\big(I_\nu(f)\big)\big\|_{L^1(\nu)} \leq \big\|I_\nu^{-1}\big\| \cdot \Big\| \int_\Omega f\, d\nu \Big\|_E$$

and hence, it follows that

$$\|f\|_{L^1(\nu)} = \big\|\, |f|\, \big\|_{L^1(\nu)} \leq \big\|I_\nu^{-1}\big\| \cdot \Big\| \int_\Omega |f|\, d\nu \Big\|_E.$$

Namely, (6.67) holds with $C_1 := \big\|I_\nu^{-1}\big\|$. Therefore, the equivalence (i) $\Leftrightarrow$ (iii) is a consequence of Lemma 6.39. $\square$

Recall that the class of all vector measures whose associated integration operator is an isomorphism onto its range is exactly that consisting of the evaluations of all spectral measures; see Proposition 3.64.

Let us now consider the case of positive vector measures.

Theorem 6.41. *Suppose that E is a Banach lattice and that $\nu : \Sigma \to E$ is any positive vector measure defined on a measurable space (Ω, Σ). Let $1 \leq p < \infty$. Then the following assertions are equivalent.*

(i) *The restricted integration operator $I_\nu^{(p)} : L^p(\nu) \to E$ is p-concave.*

(ii) *There exist a constant $C > 0$ and a positive element $x_0^* \in \mathbf{B}[E^*]$ such that*

$$\big\|I_\nu^{(p)}(f)\big\|_E \leq C \left(\int_\Omega |f|^p\, d\langle \nu, x_0^*\rangle \right)^{1/p}, \qquad f \in L^p(\nu). \tag{6.70}$$

(iii) *There exists a positive functional $x_0^* \in \mathbf{B}[E^*]$ such that*

$$L^p(\nu) \subseteq L^p\big(\langle \nu, x_0^*\rangle\big) \subseteq L^1(\nu), \tag{6.71}$$

in which case x_0^* is necessarily a Rybakov functional for ν.

(iv) *There exists a constant $C > 0$ such that*

$$\Big(\sum_{j=1}^{n} \Big\| \int_\Omega s_j \, d\nu \Big\|_E^p \Big)^{1/p} \le C \Big\| \int_\Omega \sum_{j=1}^{n} |s_j|^p d\nu \Big\|_E^{1/p}$$

for all $n \in \mathbb{N}$ and $s_1, \dots, s_n \in \operatorname{sim} \Sigma$.

Proof. (i) ⇒ (ii). Equip the convex subset

$$\mathbf{B}^+[E^*] := \mathbf{B}[E^*] \cap (E^*)^+$$

of E^* with the relative weak* topology. Then $\mathbf{B}^+[E^*]$ is compact because $\mathbf{B}[E^*]$ is weak* compact (by the Banach-Alaoglu Theorem) and $(E^*)^+$ is weak* closed. Let $C := \mathbf{M}_{(p)}\big[I_\nu^{(p)}\big]$.

Fix $n \in \mathbb{N}$ and $f_1, \dots, f_n \in L^p(\nu)$. Define a function $\psi_{f_1,\dots,f_n} : \mathbf{B}^+[E^*] \to \mathbb{R}$ by

$$\psi_{f_1,\dots,f_n}(x^*) := \sum_{j=1}^{n} \big\| I_\nu^{(p)}(f_j) \big\|_E^p - C^p \Big\langle \sum_{j=1}^{n} \int_\Omega |f_j|^p \, d\nu, \ x^* \Big\rangle, \qquad x^* \in \mathbf{B}^+[E^*].$$

Then $\psi_{f_1,\dots,f_n}$ is continuous because $\sum_{j=1}^n \int_\Omega |f_j|^p \, d\nu \in E$. Given $x^*, y^* \in \mathbf{B}[E^*]$ and $0 \le a \le 1$, we have

$$\psi_{f_1,\dots,f_n}\big(ax^* + (1-a)y^*\big) = a\psi_{f_1,\dots,f_n}(x^*) + (1-a)\psi_{f_1,\dots,f_n}(y^*).$$

Hence, $\psi_{f_1,\dots,f_n}$ is a convex function on $\mathbf{B}^+[E^*]$. Moreover, since $I_\nu^{(p)}$ is p-concave, it follows that

$$\sum_{j=1}^{n} \big\| I_\nu^{(p)}(f_j) \big\|_E^p \le C^p \Big\| \Big(\sum_{j=1}^{n} |f_j|^p \Big)^{1/p} \Big\|_{L^p(\nu)}^p = C^p \Big\| \sum_{j=1}^{n} |f_j|^p \Big\|_{L^1(\nu)}, \tag{6.72}$$

where we have used (2.104) to obtain the inequality and (3.50) to obtain the equality in (6.72). On the other hand, from Lemma 3.13 we have

$$\Big\| \sum_{j=1}^{n} |f_j|^p \Big\|_{L^1(\nu)} = \Big\| \int_\Omega \sum_{j=1}^{n} |f_j|^p d\nu \Big\|_E = \sup_{x^* \in \mathbf{B}^+[E^*]} \Big\langle \int_\Omega \sum_{j=1}^{n} |f_j|^p d\nu, \ x^* \Big\rangle \tag{6.73}$$

because ν is positive and $\sum_{j=1}^n |f_j|^p \ge 0$. Now, the continuous function

$$x^* \longmapsto \Big\langle \int_\Omega \sum_{j=1}^{n} |f_j|^p d\nu, \ x^* \Big\rangle, \qquad x^* \in \mathbf{B}^+[E^*]$$

on the compact set $\mathbf{B}^+[E^*]$ attains its maximum at some point $y_0^* \in \mathbf{B}^+[E^*]$ (depending on $f_1, \dots, f_n$ and $n \in \mathbb{N}$). This, together with the definition of $\psi_{f_1,\cdots,f_n}$, (6.72) and (6.73), imply that

$$\psi_{f_1,\dots,f_n}\big(y_0^*\big) \le 0. \tag{6.74}$$

Now let $\Psi := \{\psi_{f_1,\ldots,f_n} : f_1,\ldots,f_n \in L^p(\nu),\, n \in \mathbb{N}\}$. Then Ψ is a concave family of $\mathbb{R}$-valued functions, in the sense of Definition 6.11. In fact, given numbers $c_j \in [0,1]$, for $j = 1,\ldots,n$, with $n \in \mathbb{N}$ and $\sum_{j=1}^n c_j = 1$ and finite collections of functions $\{f_1^{(j)},\ldots,f_{k(j)}^{(j)}\} \subseteq X(\mu)$ with $k(j) \in \mathbb{N}$, for $j = 1,\ldots,n$, direct calculation shows that

$$\sum_{j=1}^n c_j \psi_{f_1^{(j)},\ldots,f_{k(j)}^{(j)}} = \psi_{c_1^{1/p} f_1^{(1)},\ldots,c_1^{1/p} f_{k(1)}^{(1)},\cdots,c_n^{1/p} f_1^{(n)},\ldots,c_n^{1/p} f_{k(n)}^{(n)}},$$

from which it follows that Ψ is a concave family.

Apply Lemma 6.12 (with $c := 0$ and $W := \mathbf{B}^+[E^*]$ and in combination with (6.74)) to the concave family Ψ of $\mathbb{R}$-valued continuous, convex functions to find a positive element $x_0^* \in \mathbf{B}[E^*]$ satisfying $\psi_{f_1,\ldots,f_n}(x_0^*) \le 0$, that is,

$$\Big(\sum_{j=1}^n \big\|I_\nu^{(p)}(f_j)\big\|_E^p\Big)^{1/p} \le C\bigg(\int_\Omega \sum_{j=1}^n |f_j|^p \, d\langle\nu, x_0^*\rangle\bigg)^{1/p}$$

whenever $f_1,\ldots,f_n \in L^p(\nu)$ and $n \in \mathbb{N}$. In particular, the case $n := 1$ gives

$$\big\|I_\nu^{(p)}(f)\big\|_E \le C\bigg(\int_\Omega |f|^p \, d\langle\nu, x_0^*\rangle\bigg)^{1/p} = C\,\big\|f\big\|_{L^p(\langle\nu,x_0^*\rangle)}, \qquad f \in L^p(\nu).$$

So, (ii) holds.

(ii) $\Rightarrow$ (i). Let $n \in \mathbb{N}$ and $f_1,\ldots,f_n \in L^p(\nu)$. Then it follows, from (ii), (3.7) and (3.50), that

$$\begin{aligned}\Big(\sum_{j=1}^n \big\|I_\nu^{(p)}(f_j)\big\|_E^p\Big)^{1/p} &\le C\bigg(\int_\Omega \sum_{j=1}^n |f_j|^p \, d\langle\nu, x_0^*\rangle\bigg)^{1/p} \\ &\le C\Big\|\sum_{j=1}^n |f_j|^p\Big\|_{L^1(\nu)}^{1/p} = C\Big\|\Big(\sum_{j=1}^n |f_j|^p\Big)^{1/p}\Big\|_{L^p(\nu)},\end{aligned}$$

which implies that $I_\nu^{(p)}$ is p-concave. Note that in order to be able to apply (3.50) we need to know that $\big(\sum_{j=1}^n |f_j|^p\big)^{1/p} \in L^p(\nu)$, namely, that $\sum_{j=1}^n |f_j|^p \in L^1(\nu)$, which actually is the case because $f_j \in L^p(\nu)$ for all $j = 1,\ldots,n$.

(ii) $\Leftrightarrow$ (iii). If $x_0^* \in \mathbf{B}[E^*] \cap (E^*)^+$ satisfies (6.70), then $x_0^* \in \mathbf{R}_\nu[E^*]$ because (6.70) yields, with $f := \chi_A$, that

$$\|\nu(A)\|_E = \big\|I_\nu^{(p)}(\chi_A)\big\|_E \le C\Big(\langle\nu, x_0^*\rangle(A)\Big)^{1/p}, \qquad A \in \Sigma.$$

Similarly, if x_0^* satisfies (6.69), then the second inclusion in (6.69) ensures the existence of a constant $C_1 > 0$ such that

$$\|\nu(A)\|_E \le \|\chi_A\|_{L^1(\nu)} \le C_1\|\chi_A\|_{L^p(\langle\nu,x_0^*\rangle)} = C_1\Big(\langle\nu, x_0^*\rangle(A)\Big)^{1/p}, \qquad A \in \Sigma,$$

and so again $x_0^* \in \mathbf{R}_\nu[E^*]$. Therefore, we can apply Corollary 4.16, with $\mu := \langle \nu, x_0^* \rangle$, $X(\mu) := L^p(\nu)$ and $Y(\mu) := L^p(\langle \nu, x_0^* \rangle)$ to the μ-determined operator $T := I_\nu^{(p)}$, to establish the equivalence (ii) $\Leftrightarrow$ (iii).

(i) $\Leftrightarrow$ (iv). Since ν is positive, it follows from Lemma 3.13 that $\|f\|_{L^1(\nu)} = \big\| \int_\Omega |f| \, d\nu \big\|_E$ for $f \in L^1(\nu)$. So, (6.67) is satisfied with $C_1 := 1$ and hence, Lemma 6.39 establishes the equivalence (i) $\Leftrightarrow$ (iv). $\square$

By considering vector measures of the form $\nu := m_T$, each of Proposition 6.40 and Theorem 6.41 provides sufficient conditions under which an affirmative answer to Question (B) in Section 5.3 is equivalent to p-concavity of the restricted integration operator $I_{m_T}^{(p)} : L^p(m_T) \to E$. Let us formulate this more precisely. We also include an additional equivalent condition.

Remark 6.42. Let $X(\mu)$ be a σ-order continuous q-B.f.s. over a positive, finite measure space (Ω, Σ, μ) and E be a Banach space. Let $m_T : \Sigma \to E$ denote the vector measure associated with a μ-determined operator $T : X(\mu) \to E$. Suppose that $1 \le p < \infty$. We assume that either:

(a) the associated integration operator $I_{m_T} : L^1(m_T) \to E$ is an isomorphism onto its range, or that

(b) E is a Banach lattice and $T : X(\mu) \to E$ is a positive operator.

Observe that (a) is exactly the assumption of Proposition 6.40 with $\nu := m_T$, whereas (b) is precisely the hypothesis of Theorem 6.41 with $\nu := m_T$ (because m_T is necessarily positive). So, according to Proposition 6.40 and Theorem 6.41, the following assertions are equivalent.

(i) The restricted integration operator $I_{m_T}^{(p)} : L^p(m_T) \to E$ is p-concave.

(ii) The continuous inclusions

$$L^p(m_T) \subseteq L^p\big(\langle m_T, x_0^* \rangle\big) \subseteq L^1(m_T)$$

hold for some $x_0^* \in \mathbf{R}_{m_T}[E^*]$ such that the scalar measure $\langle m_T, x_0^* \rangle \ge 0$.

(iii) There exists a constant $C > 0$ such that

$$\Big(\sum_{j=1}^n \|T(s_j)\|_E^p \Big)^{1/p} \le C \Big\| T\Big(\sum_{j=1}^n |s_j|^p \Big) \Big\|_E^{1/p}$$

for all $n \in \mathbb{N}$ and $s_1, \dots, s_n \in \operatorname{sim} \Sigma$.

Note, under the assumption (b), that there exists a positive functional $x_0^* \in \mathbf{R}_{m_T}[E^*]$ satisfying (ii) (see Theorem 6.41). Regarding (iii) we have, of course, used the fact that $T(s) = \int_\Omega s \, dm_T$ for every $s \in \operatorname{sim} \Sigma$, which is a direct consequence of the definition of the vector measure m_T. $\square$

It is worthwhile to record a special case of Theorem 6.41.

Corollary 6.43. *Let E be a Banach lattice and let ν be any E-valued, positive vector measure. Then the following conditions are equivalent.*

(i) *The associated integration operator $I_\nu : L^1(\nu) \to E$ is* 1*-concave.*

(ii) *There exists a positive $x_0^* \in \mathbf{R}_\nu[E^*]$ satisfying*

$$L^1(\nu) = L^1(\langle \nu, x_0^* \rangle),$$

with the given lattice norms being equivalent.

Proof. This is a consequence of Theorem 6.41 (with $p := 1$) and the fact that we always have $L^1(\nu) \subseteq L^1(|\langle \nu, x^* \rangle|)$ with a continuous inclusion whenever $x^* \in \mathbf{R}_\nu[E^*]$; see (I-1) immediately after Lemma 3.3 and the definition of Rybakov functionals (given just prior to Theorem 3.7). $\square$

Recall that Proposition 3.74 presents several equivalent conditions to I_ν being 1-concave (for a general Banach-space-valued vector measure ν). Corollary 6.43(ii) exhibits a further equivalent condition provided that ν is a *positive* vector measure (with values in a Banach lattice, of course). Can we extend this extra equivalence to a general vector measure, that is, to one which is not necessarily positive? The answer is no as will now be demonstrated.

Example 6.44. Let the notation be as in Example 5.12, where $\nu : \mathcal{B}(\mathbb{R}^2) \to E := \mathbb{C}^2$ as given there is surely *not* positive. Then

$$L^1(\nu) \neq L^1\big(|\langle \nu, x^* \rangle|\big), \qquad x^* \in \mathbf{R}_\nu[E^*]. \tag{6.75}$$

To see this fix $x^* \in \mathbf{R}_\nu[E^*]$.

Given $1 < p < 3/2$, recall from (5.25) that $L^p\big(|\langle \nu, x^* \rangle|\big) \not\subseteq L^1(\nu)$. This, together with the general fact that $L^p\big(|\langle \nu, x^* \rangle|\big) \subseteq L^1\big(|\langle \nu, x^* \rangle|\big)$ for the scalar measure $|\langle \nu, x^* \rangle|$, yield (6.75).

On the other hand, the integration operator $I_\nu : L^1(\nu) \to \mathbb{C}^2$ clearly has finite rank, and hence, is absolutely 1-summing (see [41, Proposition 2.3]). So, apply Example 2.61 with $q := 1$ to conclude that I_ν is 1-concave. $\square$

The following fact, whose proof is obvious, has already been used in Example 6.44.

Lemma 6.45. *Let $\nu : \Sigma \to E$ be a Banach-space-valued measure defined on a measurable space (Ω, Σ). If there exists $x_0^* \in \mathbf{R}_\nu[E^*]$ such that $L^1(\nu) = L^1(|\langle \nu, x_0^* \rangle|)$, then*

$$L^p\big(|\langle \nu, x_0^* \rangle|\big) \subseteq L^1\big(|\langle \nu, x_0^* \rangle|\big) = L^1(\nu), \qquad 1 \le p < \infty.$$

In Corollary 3.19, we have exhibited conditions equivalent to $L^1(\nu) = L^1\big(|\langle \nu, x_0^* \rangle|\big)$ for some Rybakov functional x_0^*. In certain cases, it is possible to identify such a Rybakov functional explicitly; see, for instance, Lemma 3.14(ii)(b) (and its proof) and Example 3.67.

We end this chapter with some examples of positive vector measures ν for which we can find an explicit functional $x_0^* \in \mathbf{B}[E^*] \cap (E^*)^+$ satisfying (6.71). Note that Theorem 6.41 does not provide a general method for constructing such a functional x_0^*.

Example 6.46. Let $1 \leq r \leq \infty$ and consider the Volterra measure $\nu_r : \mathcal{B}([0,1]) \to L^r([0,1])$ of order r; see (3.23) and (3.26). For the case $1 \leq r < \infty$, the constant function $\mathbf{1} := \chi_{[0,1]}$ belongs to $L^{r'}([0,1]) = L^r([0,1])^*$ and satisfies

$$\langle \nu_r, \mathbf{1}\rangle(A) = \int_A (1-t)\,dt, \qquad A \in \mathcal{B}([0,1]), \tag{6.76}$$

which follows easily from the definition of ν_r and Fubini's theorem. In particular, $\mathbf{1}$ is a positive Rybakov functional for ν_r.

(i) Consider $r := 1$. Then, according to Example 3.26(i), we have

$$L^1(\nu_1) = L^1(|\nu_1|) = L^1(\langle \nu_1, \mathbf{1}\rangle) = L^1\big((1-t)\,dt\big).$$

This situation is quite typical for Lemma 3.14(ii)(b). So, we can apply Lemma 6.45 to deduce (6.71) for every $1 \leq p < \infty$.

(ii) Let $1 < r < \infty$. Then, for the function $\mathbf{1} \in L^{r'}([0,1]) = L^r([0,1])^*$, we have $L^p(\langle \nu_r, \mathbf{1}\rangle) = L^p\big((1-t)\,dt\big)$ and so

$$L^p(\nu_r) \subseteq L^p\big((1-t)\,dt\big) \subseteq L^1\big((1-t)^{1/r}dt\big) = L^1(|\nu_r|) \subseteq L^1(\nu_r) \tag{6.77}$$

whenever p satisfies $r \leq p < \infty$; see Example 3.76. In other words, for every $r \leq p < \infty$, (6.71) holds with $\nu := \nu_r$ and $x_0^* := \mathbf{1} \in L^{r'}([0,1]) = L^r([0,1])^*$.

(iii) For $r = \infty$, it follows from (3.27) that the Volterra operator V_∞ satisfies $V_\infty\big(L^\infty([0,1])\big) \subseteq C([0,1])$. So, we have

$$\mathcal{R}(\nu_\infty) \subseteq \mathcal{R}(V_\infty) \subseteq C([0,1]),$$

where $\mathcal{R}(\nu_\infty)$ and $\mathcal{R}(V_\infty)$ denote the ranges of the vector measure ν_∞ and the operator V_∞, respectively. Accordingly, let $\nu_0 : \mathcal{B}([0,1]) \to C([0,1])$ denote the Volterra measure ν_∞ considered as being $C([0,1])$-valued. Then

$$L^1(\nu_\infty) = L^1(\nu_0) = L^1([0,1]); \tag{6.78}$$

see [129, Proposition 3.1]. As usual, the dual space of $C([0,1])$ is identified with the space of all $\mathbb{C}$-valued Borel measures on $[0,1]$. In particular, the Dirac measure δ_1 at the point 1 belongs to the dual space $C([0,1])^*$. With μ denoting Lebesgue measure on $[0,1]$, it follows that

$$\langle \nu_0, \delta_1\rangle(A) = \int_0^1 \Big(\int_0^t \chi_A(u)\,du\Big)\,d\delta_1(t) = \int_0^1 \chi_A(u)\,du = \mu(A), \quad A \in \mathcal{B}([0,1]).$$

This and (6.78) yield that

$$L^1(\nu_0) \;=\; L^1(\langle \nu_0, \delta_1 \rangle) \;=\; L^1([0,1]).$$

(iv) Finally we consider the vector measure $\nu_\infty : \mathcal{B}([0,1]) \to L^\infty([0,1])$. Let $j : C([0,1]) \to L^\infty([0,1])$ denote the natural injection (actually an isometry). That is, $j(f)$ is the equivalence class (modulo μ-a.e.) in $L^\infty([0,1])$ determined by $f \in C([0,1])$. Then $j\big(C([0,1])\big)$ is a closed subspace of $L^\infty([0,1])$ and $\nu_\infty = j \circ \nu_0$ (see Lemma 3.27). Moreover, $\widetilde{\delta_1}$ defined by $\widetilde{\delta_1}\big(j(f)\big) := \langle f, \delta_1 \rangle$, for $f \in C([0,1])$, is a continuous linear functional on $j\big(C([0,1]\big)$ which, by the Hahn-Banach Theorem, has an extension (not unique) to an element $\eta \in L^\infty([0,1])^*$. Since the containment $\mathcal{R}(\nu_\infty) \subseteq j\big(C([0,1])\big)$ holds, it follows that

$$\langle \nu_\infty, \eta \rangle(A) = \langle \nu_\infty(A), \eta \rangle = \big\langle j\big(\nu_0(A)\big), \widetilde{\delta_1} \big\rangle = \big\langle \nu_0(A), \delta_1 \big\rangle = \mu(A), \quad A \in \mathcal{B}([0,1]).$$

This and (6.78) imply that

$$L^1(\nu_\infty) = L^1\big(\langle \nu_\infty, \eta \rangle\big) = L^1([0,1]). \qquad \square$$

Example 6.47. Given a positive, finite measure space (Ω, Σ, μ) which is σ-decomposable and $1 < p < \infty$, select functions $g_j \in L^1(\mu)^+$, for $j \in \mathbb{N}$, such that $\sum_{j=1}^\infty \|g_j\|_{L^1(\mu)} < \infty$ and $g_j > 0$ (μ-a.e.) for each $j \in \mathbb{N}$. Let $\{e_j\}_{j=1}^\infty$ denote the standard unit vector basis for the Banach space ℓ^p. The (norm 1) inclusion $\ell^1 \subseteq \ell^p$ implies that

$$\Big(\sum_{j=1}^\infty \|g_j\|_{L^1(\mu)}^p \Big)^{1/p} \;\le\; \sum_{j=1}^\infty \|g_j\|_{L^1(\mu)} \;<\; \infty,$$

which enables us to define a finitely additive set function $\nu : \Sigma \to \ell^p$ by

$$\nu(A) := \sum_{j=1}^\infty \Big(\int_A g_j \, d\mu \Big) e_j, \qquad A \in \Sigma. \tag{6.79}$$

Note that the separable Banach space ℓ^p contains no copy of ℓ^∞ and that the standard unit vector basis $\{\gamma_n\}_{n=1}^\infty \subseteq (\ell^p)^* = \ell^{p'}$ is a total subset of $(\ell^p)^*$. Let $n \in \mathbb{N}$. Observe that

$$\langle \nu(A), \gamma_n \rangle = \int_A g_n d\mu, \qquad A \in \Sigma,$$

which implies that the finitely additive set function $\langle \nu, \gamma_n \rangle : A \mapsto \langle \nu(A), \gamma_n \rangle$ on Σ is equal to the indefinite integral of $g_n \in L^1(\mu)$ (with respect to μ) and hence, is σ-additive. Consequently, the generalized Orlicz-Pettis Theorem (see Lemma 3.2) implies that ν is σ-additive, that is, ν is a positive, ℓ^p-valued vector measure.

Now let $\psi \in \ell^{p'} = (\ell^p)^*$ be the (non-negative) function defined by the formula $\psi(j) := \|g_j\|_{L^1(\mu)}^{p/p'}$ for $j \in \mathbb{N}$. Then

$$\langle \nu, \psi \rangle(A) = \sum_{j=1}^{n} \psi(j) \int_A g_j \, d\mu, \qquad A \in \Sigma.$$

Our aim is to show, for $x_0^* := \psi$, that

$$L^p(\nu) \subseteq L^p(\langle \nu, x_0^* \rangle) = L^p\big(\langle \nu, \psi \rangle\big) \subseteq L^1(\nu),$$

that is, (6.71) holds. Since $\psi(j) > 0$ for each $j \in \mathbb{N}$, the scalar measure $\langle \nu, \psi \rangle$ and the vector measure ν are mutually absolutely continuous and so, the first inclusion $L^p(\nu) \subseteq L^p\big(\langle \nu, \psi \rangle\big)$ necessarily holds. To prove the second inclusion $L^p\big(\langle \nu, \psi \rangle\big) \subseteq L^1(\nu)$, we shall first verify that

$$\|s\|_{L^1(\nu)} \le \|s\|_{L^p(\langle \nu, \psi \rangle)}, \qquad s \in \operatorname{sim} \Sigma. \tag{6.80}$$

To this end, fix $s \in \operatorname{sim} \Sigma$. It follows from Lemma 3.13 and (6.79) that

$$\begin{aligned} \|s\|_{L^1(\nu)} = \Big\| \int_\Omega |s| \, d\nu \Big\|_{\ell^p} &= \Big\| \sum_{j=1}^{\infty} \Big(\int_\Omega |s| \, g_j \, d\mu \Big) e_j \Big\|_{\ell^p} \\ &= \bigg(\sum_{j=1}^{\infty} \Big(\int_\Omega |s| g_j \, d\mu \Big)^p \bigg)^{1/p} \le \bigg(\sum_{j=1}^{\infty} \psi(j) \int_\Omega |s|^p g_j \, d\mu \bigg)^{1/p}. \end{aligned} \tag{6.81}$$

Here the last inequality in (6.81) is a consequence of the identity $(1/p)+(1/p') = 1$ and Hölder's inequality as follows: for each $j \in \mathbb{N}$ we have

$$\begin{aligned} \Big(\int_\Omega |s| \, g_j \, d\mu \Big)^p &= \Big(\int_\Omega \Big(\psi(j)^{1/p} \, |s| \, g_j^{1/p} \Big) \Big(\psi(j)^{-1/p} \, g_j^{1/p'} \Big) \, d\mu \Big)^p \\ &\le \Big(\int_\Omega \psi(j) \, |s|^p \, g_j \, d\mu \Big)^{p/p} \Big(\int_\Omega \psi(j)^{-p'/p} \, g_j \, d\mu \Big)^{p/p'} \\ &= \Big(\psi(j) \int_\Omega |s|^p \, g_j \, d\mu \Big) \Big(\|g_j\|_{L^1(\mu)}^{-1} \int_\Omega g_j \, d\mu \Big)^{p/p'} = \psi(j) \int_\Omega |s|^p \, g_j \, d\mu. \end{aligned}$$

Since the right-hand side of (6.81) equals $\|s\|_{L^p(\langle \nu, \psi \rangle)}$, the inequality (6.80) does indeed hold.

Now, let $f \in L^p(\langle \nu, \psi \rangle)$. Select a sequence $\{s_n\}_{n=1}^\infty \subseteq \operatorname{sim} \Sigma$ such that $s_n \to f$ both pointwise and in the norm $\| \cdot \|_{L^p(\langle \nu, \psi \rangle)}$ as $n \to \infty$. Then we have from (6.80) that

$$\|s_n - s_k\|_{L^1(\nu)} \le \|s_n - s_k\|_{L^p(\langle \nu, \psi \rangle)}, \qquad n, k \in \mathbb{N},$$

and hence, $\{s_n\}_{n=1}^\infty$ is Cauchy in $L^1(\nu)$. So, Theorem 3.5(ii) yields that $f \in L^1(\nu)$. This establishes the inclusion $L^p(\langle \nu, \psi \rangle) \subseteq L^1(\nu)$. □

Chapter 7

Operators from Classical Harmonic Analysis

Two of the most important operators arising in harmonic analysis are the Fourier transform and convolutions (which include translation operators via convolution with Dirac point measures). The aim of this final chapter is to make a detailed analysis of these two classes of operators, acting in L^p-spaces, from the viewpoint of their optimal domain and properties of the corresponding extended operator. In particular, for the well-known class of L^q-*improving measures*, it turns out that the corresponding convolution operators can be *characterized* as precisely those which are p-th power factorable for a suitable range of p; see Section 7.5. This makes a close and important connection between the results of Chapter 5 and the classical family of convolution operators.

To fix the setting, let G denote a (Hausdorff) *compact abelian group* with *dual group* Γ. It is always assumed that G (hence, also Γ) is infinite. The identity element of G is denoted by 0. Normalised *Haar measure* in G is denoted by μ; it is defined on the Borel σ-algebra $\mathcal{B}(G)$ (i.e., that generated by the open sets). For each $1 \le p < \infty$, let $L^p(G)$ denote the (complex) Banach space of all p-th power μ-integrable functions $f : G \to \mathbb{C}$ with its standard norm $\|f\|_{L^p(G)} := \big(\int_G |f|^p d\mu\big)^{1/p}$, that is, $L^p(G) = L^p(\mu)$. As usual, $L^\infty(G) = L^\infty(\mu)$ is the Banach space of μ-a.e. bounded measurable functions equipped with the essential sup-norm $\|\cdot\|_{L^\infty(G)}$.

The adjoint index to p is denoted by p' and is determined by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Since μ is finite, we have

$$L^\infty(G) \subseteq L^q(G) \subseteq L^p(G) \subseteq L^1(G) \tag{7.1}$$

whenever $1 < p < q < \infty$. Moreover, since $\mu(G) = 1$, we have that all inclusions in (7.1) are continuous with operator norm 1. As G is infinite, these inclusions are all *proper* (adapt the proof of [95, Lemma 4.5.1]). The dual space $L^p(G)^*$ of $L^p(G)$ is identified with $L^{p'}(G)$ via the duality

$$\langle f, \psi \rangle := \int_G f\psi \, d\mu, \qquad f \in L^p(G), \quad \psi \in L^{p'}(G).$$

We will also require the Banach spaces $\ell^p(\Gamma)$, equipped with the norm $\|\xi\|_{\ell^p(\Gamma)} := \big(\sum_{\gamma\in\Gamma} |\xi(\gamma)|^p\big)^{1/p}$ for $1 \le p < \infty$ and $\|\xi\|_{\ell^\infty(\Gamma)} := \sup_{\gamma\in\Gamma} |\xi(\gamma)|$ for $p = \infty$, where elements ξ of $\ell^p(\Gamma)$ are considered as functions $\xi : \Gamma \to \mathbb{C}$ defined on the *discrete space* Γ. The Banach space dual $\ell^p(\Gamma)^*$ of $\ell^p(\Gamma)$ is identified with $\ell^{p'}(\Gamma)$ via the duality

$$\langle \xi, \eta \rangle := \sum_{\gamma\in\Gamma} \xi(\gamma)\eta(\gamma), \qquad \xi \in \ell^p(\Gamma), \quad \eta \in \ell^{p'}(\Gamma).$$

Of course, $c_0(\Gamma)$ denotes the Banach space of all functions $\xi : \Gamma \to \mathbb{C}$ which "vanish at infinity" (i.e., for each $\varepsilon > 0$ the set $\{\gamma \in \Gamma : |\xi(\gamma)| \ge \varepsilon\}$ is finite), equipped with the norm $\|\cdot\|_{c_0(\Gamma)}$ inherited from $\ell^\infty(\Gamma)$. Actually, $c_0(\Gamma)$ is a closed subspace of $\ell^\infty(\Gamma)$. Both $c_0(\Gamma)$ and $\ell^\infty(\Gamma)$ are algebras relative to pointwise multiplication of functions on Γ.

For each $\gamma \in \Gamma$, the value of the *character* γ at a point $x \in G$ is denoted by (x, γ). The identity element of Γ will be denoted by e. Let

$$\mathcal{T}(G) := \operatorname{span}\big\{(\cdot, \gamma) : \gamma \in \Gamma\big\}$$

be the space of all *trigonometric polynomials* on G. Clearly we have $\mathcal{T}(G) \subseteq \bigcap_{1\le r\le\infty} L^r(G)$ and $\big\|(\cdot,\gamma)\big\|_{L^r(G)} = 1$ whenever $\gamma \in \Gamma$ and $1 \le r \le \infty$.

Finally, the space of all $\mathbb{C}$-valued, *regular measures* defined on $\mathcal{B}(G)$ is denoted by $M(G)$; it is a Banach space when equipped with the usual *total variation norm* $\|\cdot\|_{M(G)}$. The *Fourier–Stieltjes transform* $\widehat{\lambda} : \Gamma \to \mathbb{C}$ of a measure $\lambda \in M(G)$ is defined by

$$\widehat{\lambda}(\gamma) := \int_G \overline{(x,\gamma)} \, d\lambda(x), \qquad \gamma \in \Gamma, \tag{7.2}$$

in which case $\widehat{\lambda} \in \ell^\infty(\Gamma)$. If λ is absolutely continuous with respect to μ, denoted by $\lambda \ll \mu$, then it follows from the Radon–Nikodým Theorem that there exists $h \in L^1(G)$ whose indefinite integral

$$\mu_h : A \mapsto \int_E h \, d\mu, \qquad A \in \mathcal{B}(G), \tag{7.3}$$

equals λ. In this case, $\widehat{\lambda}$ is equal to the *Fourier transform* $\widehat{h}$ of h defined by

$$\widehat{h}(\gamma) = \int_G \overline{(x,\gamma)} h(x) \, d\mu(x), \qquad \gamma \in \Gamma. \tag{7.4}$$

Clearly $\|\widehat{f}\|_{\ell^\infty(\Gamma)} \leq \|f\|_{L^1(G)}$ for each $f \in L^1(G)$ and so the Fourier transform map

$$F_1 : L^1(G) \to \ell^\infty(\Gamma)$$

defined by $f \mapsto \widehat{f}$ is continuous. Actually, by the Riemann–Lebesgue Lemma, [140, Theorem 1.2.4], F_1 is $c_0(\Gamma)$-valued. If we wish to consider F_1 as being $c_0(\Gamma)$-valued, then we will denote it by

$$F_{1,0} : L^1(G) \to c_0(\Gamma).$$

Since the inequalities

$$\|\widehat{f}\|_{c_0(\Gamma)} \leq \|f\|_{L^1(G)} \leq \|f\|_{L^p(G)}, \qquad f \in L^p(G), \tag{7.5}$$

hold for each $1 \leq p < \infty$, it follows that the Fourier transform map is also defined on each space $L^p(G)$ and maps it continuously into $c_0(\Gamma)$. We denote this map by

$$F_{p,0} : L^p(G) \to c_0(\Gamma).$$

Actually, more is true: it turns out that $\widehat{f} \in \ell^{p'}(\Gamma)$, whenever $1 \leq p \leq 2$ and $f \in L^p(G)$, with

$$\|\widehat{f}\|_{\ell^{p'}(\Gamma)} \leq \|f\|_{L^p(G)}, \qquad f \in L^p(G). \tag{7.6}$$

This is the *Hausdorff–Young inequality*, [76, p. 227], [95, Theorem F.8.4.]. The continuous linear map $f \mapsto \widehat{f}$ so-defined is denoted by

$$F_p : L^p(G) \to \ell^{p'}(\Gamma),$$

for each $1 \leq p \leq 2$.

For $1 \leq p < \infty$ and each $\lambda \in M(G)$, the *convolution operator* $C_\lambda^{(p)}$ from $L^p(G)$ into itself, defined by

$$C_\lambda^{(p)} : f \mapsto f * \lambda, \quad f \in L^p(G), \tag{7.7}$$

is linear and continuous. Indeed, the function $f * \lambda : G \to \mathbb{C}$ defined by

$$f * \lambda : \ x \mapsto \int_G f(x-y)\, d\lambda(y), \qquad \mu\text{-a.e. } x \in G,$$

belongs to $L^p(G)$ and satisfies

$$\|f * \lambda\|_{L^p(G)} \leq \|\lambda\|_{M(G)} \cdot \|f\|_{L^p(G)}, \tag{7.8}$$

[75, Theorem 20.13]. So, $\big\|C_\lambda^{(p)}\big\| \leq \|\lambda\|_{M(G)}$. For fixed $a \in G$, let δ_a denote the *Dirac measure* at a, that is, $\delta_a(A) = \chi_A(a)$ for $A \in \mathcal{B}(G)$. Then $C_{\delta_a}^{(p)}$ is precisely the *translation operator* $\tau_a \in \mathcal{L}\big(L^p(G)\big)$ given by

$$\tau_a f : \ x \mapsto f(x-a), \qquad x \in G, \tag{7.9}$$

for each $f \in L^p(G)$. It is clear that $\tau_a \circ C_\lambda^{(p)} = C_\lambda^{(p)} \circ \tau_a$ for each $a \in G$.

Since $\mathcal{T}(G) \subseteq L^p(G)$ and

$$C_\lambda^{(p)}\big((\cdot,\gamma)\big) = (\cdot,\gamma) * \lambda = \widehat{\lambda}(\gamma) \cdot (\cdot,\gamma), \qquad \gamma \in \Gamma, \tag{7.10}$$

with equality as elements of $L^p(G)$, we see that the trigonometric polynomials are invariant under each convolution operator $C_\lambda^{(p)}$.

As indicated above, the aim of this final chapter is to make a detailed and systematic study of the convolution operators $C_\lambda^{(p)}$, for $1 \le p < \infty$ and $\lambda \in M(G)$, and the Fourier transform maps $F_{p,0} : L^p(G) \to c_0(\Gamma)$ for $1 \le p < \infty$ and $F_p : L^p(G) \to \ell^{p'}(\Gamma)$ for $1 \le p \le 2$. We begin with the Fourier transform, after noting that each B.f.s. $X(\mu) = L^p(\mu)$, for $1 \le p < \infty$, which appears in this chapter has σ-o.c. norm.

7.1 The Fourier transform

We begin with an analysis of the Fourier transform maps $F_{p,0} : L^p(G) \to c_0(\Gamma)$ for $1 \le p < \infty$. First we require a preliminary result.

Lemma 7.1. *The map $F_{1,0} : L^1(G) \to c_0(\Gamma)$ is not weakly compact.*

Proof. Since Γ is infinite we can choose (and fix) a sequence $\{\gamma_n\}_{n=1}^\infty$ of distinct elements of Γ. Given $h \in L^\infty(G)$ we have

$$\big\langle (\cdot,\gamma_n),\ h \big\rangle = \int_G (x,\gamma_n) h(x)\, d\mu(x) = \widehat{h}(-\gamma_n), \qquad n \in \mathbb{N}.$$

Since $h \in L^1(G)$, we know that $\widehat{h} \in c_0(\Gamma)$ and so $\lim_{n\to\infty} \widehat{h}(-\gamma_n) = 0$. Accordingly, $\big\{(\cdot,\gamma_n)\big\}_{n=1}^\infty$ converges weakly to 0 in $L^1(G)$. Moreover, the *orthogonality relations*

$$\int_G (x,\gamma)\, d\mu(x) = \chi_{\{\gamma\}}(e), \qquad \gamma \in \Gamma,$$

[140, p. 10], imply that

$$(\cdot,\gamma)\widehat{\ } = \chi_{\{\gamma\}}, \qquad \gamma \in \Gamma. \tag{7.11}$$

Suppose that $F_{1,0}$ is weakly compact. Since $L^1(G)$ has the Dunford-Pettis property, it follows that $F_{1,0}$ maps weakly convergent sequences in $L^1(G)$ to norm convergent sequences in $c_0(\Gamma)$, [42, pp. 176–177]. Hence, the sequence $\big\{F_{1,0}(\gamma_n)\big\}_{n=1}^\infty = \big\{(\cdot,\gamma_n)\widehat{\ }\big\}_{n=1}^\infty$ would be norm convergent in $c_0(\Gamma)$; this is surely not the case since (7.11) implies that

$$\big\|(\cdot,\gamma_k)\widehat{\ } - (\cdot,\gamma_n)\widehat{\ }\big\|_{c_0(\Gamma)} = 1, \qquad k \ne n. \tag{7.12}$$

$\square$

Remark 7.2. The proof of Lemma 7.1 also shows that $F_{1,0} : L^1(G) \to c_0(\Gamma)$ is not completely continuous. According to Corollary 2.42, it then follows that $\big\{F_{1,0}(\chi_A) : A \in \mathcal{B}(G)\big\}$ is not a relatively compact subset of $c_0(\Gamma)$. $\square$

Define a set function $m_p : \mathcal{B}(G) \to c_0(\Gamma)$ by

$$m_p : A \mapsto F_{p,0}(\chi_A) = \widehat{\chi}_A, \qquad A \in \mathcal{B}(G). \tag{7.13}$$

According to (7.5) we have

$$\|m_p(A)\|_{c_0(\Gamma)} \le \mu(A), \qquad A \in \mathcal{B}(G),$$

from which it follows that m_p is both σ-additive and has finite variation. Of course, in the notation of Chapter 4, m_p is precisely the vector measure $m_{F_{p,0}}$ induced by the operator $F_{p,0}$. The unit vector $\chi_{\{e\}} \in \ell^1(\Gamma) = c_0(\Gamma)^*$ satisfies

$$\big\langle m_p(A),\ \chi_{\{e\}} \big\rangle \,=\, \widehat{\chi}_A(e) \,=\, \mu(A), \qquad A \in \mathcal{B}(G). \tag{7.14}$$

Accordingly, for each $A \in \mathcal{B}(G)$, we have

$$\mu(A) \,=\, \big|\langle m_p(A),\ \chi_{\{e\}}\rangle\big| \,\le\, \|m_p(A)\|_{c_0(\Gamma)} \,\le\, \mu(A),$$

from which it is clear that actually $|m_p| = \mu$. It is also clear from (7.14) that $F_{p,0} : L^p(G) \to c_0(\Gamma)$ is a μ-determined operator. Moreover, (7.14) implies that

$$L^1(m_p) \subseteq L^1_{\mathrm{w}}(m_p) \subseteq L^1\big(|\langle m_p,\, \chi_{\{e\}}\rangle|\big) \,=\, L^1(G)$$

which, together with $L^1(G) = L^1(|m_p|) \subseteq L^1(m_p)$, yields

$$L^1_{\mathrm{w}}(m_p) \,=\, L^1(m_p) \,=\, L^1(|m_p|) \,=\, L^1(G).$$

Since $\int_G s\, dm_p = \widehat{s}$ for each $s \in \operatorname{sim} \mathcal{B}(G) \subseteq L^p(G)$, we see that I_{m_p}, which is the continuous $c_0(\Gamma)$-valued extension of $F_{p,0} : L^p(G) \to c_0(\Gamma)$ to $L^1(m_p) = L^1(G)$, is precisely the operator $F_{1,0} : L^1(G) \to c_0(\Gamma)$. The following result summarizes the above discussion.

Proposition 7.3. *Let $1 \le p < \infty$ and $m_p : \mathcal{B}(G) \to c_0(\Gamma)$ be the vector measure associated to the (μ-determined) Fourier transform map $F_{p,0} : L^p(G) \to c_0(\Gamma)$ via* (7.13).

(i) *The vector measure m_p has finite variation, satisfies $|m_p| = \mu$, and its range $\mathcal{R}(m_p)$ is not relatively compact in $c_0(\Gamma)$.*

(ii) *We have*

$$L^1_{\mathrm{w}}(m_p) \,=\, L^1(m_p) \,=\, L^1(|m_p|) \,=\, L^1(G) \tag{7.15}$$

and the integration operator $I_{m_p} : L^1(m_p) \to c_0(\Gamma)$ is precisely the Fourier transform map $F_{1,0} : L^1(G) \to c_0(\Gamma)$. In particular, I_{m_p} is injective. Moreover,

$$L^r(m_p) = L^r(G), \qquad 1 \le r < \infty.$$

(iii) *The integration operator $I_{m_p} : L^1(m_p) \to c_0(\Gamma)$ is not weakly compact or completely continuous.*

Proof. Parts (i) and (ii) have already been established above. Part (iii) follows from the identity $I_{m_p} = F_{1,0}$ together with Lemma 7.1 and Remark 7.2. $\square$

Remark 7.4. (i) Since $L^p(G)$ is reflexive, for $1 < p < \infty$, the Fourier transform map $F_{p,0} : L^p(G) \to c_0(\Gamma)$ is necessarily weakly compact. However, its optimal extension $I_{m_p} = F_{1,0}$ is not weakly compact.

(ii) In the notation of the proof of Lemma 7.1, note that the sequence $\{(\cdot,\gamma_n)\}_{n=1}^{\infty}$ belongs to $\mathbf{B}[L^p(G)]$ for every $1 \le p < \infty$. It is clear from (7.12) that $\{(\cdot,\gamma_n)\widehat{\ }\}_{n=1}^{\infty}$ cannot have a convergent subsequence in $c_0(\Gamma)$. Accordingly, the Fourier transform map $F_{p,0} : L^p(G) \to c_0(\Gamma)$ is not a compact operator. $\square$

Remark 7.5. Fix $1 \le p < \infty$. Let $X(\mu) := L^p(G)$ and $E := c_0(\Gamma)$. Then the operator $T := F_{p,0} \in \mathcal{L}(X(\mu), E)$ is μ-determined and $L^1(m_T) = L^1(\mu)$; see Proposition 7.3. Since μ is non-atomic (see Lemma 7.97 below) we can apply Lemma 6.24(iii) (with p and r interchanged) to conclude that

$$F_{p,0} \in \mathcal{A}_{r,q}\big(L^p(G),\, c_0(\Gamma)\big) \quad \text{if and only if} \quad 1 \le r \le q \le p.$$

Therefore, if $1 \le r \le q \le p$, then Theorem 6.9(v) (again with p and r interchanged) guarantees that there exists a Σ-measurable function $g > 0$ (μ-a.e.) and a μ-determined operator $S \in \mathcal{L}(L^{q/r}(\mu), E)$ for which the continuous linear extension $T_{[r]} = \big(F_{p,0}\big)_{[r]} : X(\mu)_{[r]} \to E$ is factorized as $T_{[r]} = \big(F_{p,0}\big)_{[r]} = S \circ M_{g^{r/q}}$. That is, we have

$$\begin{tikzcd}
X(\mu) = L^p(G) \arrow[r, "T=F_{p,0}"] \arrow[d, "i_{[r]}"'] & E = c_0(\Gamma) \\
X(\mu)_{[r]} = L^{p/r}(G) \arrow[r, "M_{g^{r/q}}"] \arrow[ur, "F_{(p/r),0}"] & L^{q/r}(G) \arrow[u, "S"']
\end{tikzcd}$$

because $\big(F_{p,0}\big)_{[r]} = F_{(p/r),\,0}$. However, in this particular case, it is possible to take $g := \mathbf{1} = \chi_G$ and $S := F_{(q/r),0}$ because $F_{(q/r),0} : L^{q/r}(G) \to c_0(\Gamma)$ is a continuous linear extension of both $F_{p,0}$ and $F_{(p/r),0}$. With $M_{\mathbf{1}} : L^{p/r}(G) \to L^{q/r}(G)$ denoting the operator of multiplication by $\mathbf{1}$ (i.e., the natural inclusion of $L^{p/r}(G)$ into $L^{q/r}(G)$), we have a more "concrete" diagram:

$$\begin{tikzcd}
X(\mu) = L^p(G) \arrow[r, "T=F_{p,0}"] \arrow[d, "i_{[r]}"'] & E = c_0(\Gamma) \\
X(\mu)_{[r]} = L^{p/r}(G) \arrow[r, "M_{\mathbf{1}}"] \arrow[ur, "F_{(p/r),0}"] & L^{q/r}(G). \arrow[u, "F_{(q/r),0}"']
\end{tikzcd}$$

$\square$

Proposition 7.3 identifies the main properties of the optimal extension of each Fourier transform map $F_{p,0} : L^p(G) \to c_0(\Gamma)$. If we change the codomain space from $c_0(\Gamma)$ to the space $\ell^{p'}(\Gamma)$, that is, we consider the Fourier transform maps $F_p : L^p(G) \to \ell^{p'}(\Gamma)$ for $1 \le p \le 2$, then the situation *changes dramatically.* Indeed, it turns out that the optimal domain space of F_p (equivalently, of the Hausdorff–Young inequality (7.6)) is no longer an $L^r(G)$-space for any $1 \le r \le \infty$. Moreover, it is possible to give alternative, concrete descriptions of this space (other than saying that it just consists of m_{F_p}-integrable functions); these descriptions show that the optimal domain of F_p is a genuinely new type of B.f.s. Nevertheless, it is still "well behaved" in that it is translation invariant and such classical operators as translations, convolutions and Fourier transforms act continuously in it. Moreover, these optimal domains solve a problem in classical harmonic analysis raised by R.E. Edwards some 40 years ago. It is time to be more precise.

Fix $1 \le p \le 2$ and consider the linear operator $F_p : L^p(G) \to \ell^{p'}(\Gamma)$. According to the Hausdorff–Young inequality (7.6), the finitely additive set function $m_{F_p} : \mathcal{B}(G) \to \ell^{p'}(\Gamma)$ defined by

$$m_{F_p} : A \mapsto F_p(\chi_A) = \widehat{\chi}_A, \qquad A \in \mathcal{B}(G), \tag{7.16}$$

satisfies $\|m_{F_p}(A)\|_{\ell^{p'}(\Gamma)} \le \|\chi_A\|_{L^p(G)} = \mu(A)^{1/p}$. Accordingly, m_{F_p} is σ-additive and $m_{F_p} \ll \mu$. The following result collects together some of the basic properties of m_{F_p}.

Proposition 7.6. *The following statements are valid.*

(i) *For each $1 \le p \le 2$, the vector measure $m_{F_p} : \mathcal{B}(G) \to \ell^{p'}(\Gamma)$ as given by* (7.16) *is mutually absolutely continuous with respect to μ (i.e., m_{F_p} and μ have the same null sets). In particular, the Fourier transform operator $F_p : L^p(G) \to \ell^{p'}(\Gamma)$ is μ-determined.*

(ii) *For each $1 < p \le 2$, the vector measure m_{F_p} has infinite variation.*

(iii) *For each $1 \le p \le 2$, the vector measure m_{F_p} does not have relatively compact range in $\ell^{p'}(\Gamma)$.*

Proof. (i) It is clear from (7.16) and the definition (7.4) of Fourier transforms that if $A \in \mathcal{B}(G)$ is μ-null, then $\widehat{\chi}_B = 0$ for all $B \in \Sigma \cap A$. So, A is m_{F_p}-null.

On the other hand, suppose that $A \in \mathcal{B}(G)$ is m_{F_p}-null. In particular, we have $\widehat{\chi}_A = m_{F_p}(A) = 0$ and hence, by injectivity of the Fourier transform, [76, Theorem 31.5], [140, p. 29], $\chi_A = 0$ in $L^p(G)$. That is, $\mu(A) = 0$.

(ii) Fix $1 < p \le 2$. Suppose that m_{F_p} has finite variation. By part (i) and the fact that the reflexive Banach space $\ell^{p'}(\Gamma)$ has the Radon–Nikodým property, [42, p. 218], there exists a Bochner μ-integrable function $H : G \to \ell^{p'}(\Gamma)$ such that

$$\widehat{\chi}_A = m_{F_p}(A) = \text{(B)-}\int_A H \, d\mu, \qquad A \in \mathcal{B}(G). \tag{7.17}$$

Given $\gamma \in \Gamma$, the element $\chi_{\{\gamma\}} \in \ell^p(\Gamma) = \ell^{p'}(\Gamma)^*$ satisfies

$$\langle m_{F_p}(A),\ \chi_{\{\gamma\}}\rangle \;=\; \widehat{\chi}_A(\gamma) \;=\; \int_A \overline{(x,\gamma)}\, d\mu(x), \qquad A \in \mathcal{B}(G), \tag{7.18}$$

with $\overline{(\cdot,\gamma)} \in L^1(G)$. On the other hand, (7.17) implies that

$$\langle m_{F_p}(A), \chi_{\{\gamma\}}\rangle \;=\; \int_A \langle H(x),\ \chi_{\{\gamma\}}\rangle\, d\mu(x), \qquad A \in \mathcal{B}(G), \tag{7.19}$$

also with $\langle H(\cdot),\ \chi_{\{\gamma\}}\rangle \in L^1(G)$. It follows from (7.18) and (7.19) that

$$\langle H(x),\ \chi_{\{\gamma\}}\rangle \;=\; \overline{(x,\gamma)}, \qquad x \in G \setminus N(\gamma), \tag{7.20}$$

for some μ-null set $N(\gamma) \in \mathcal{B}(G)$. Choose any infinite sequence of distinct elements $\{\gamma_n\}_{n=1}^{\infty} \subseteq \Gamma$, in which case $N := \bigcup_{n=1}^{\infty} N(\gamma_n)$ is μ-null. Since H is $\ell^{p'}(\Gamma)$-valued μ-a.e., we can select $x_0 \in G \setminus N$ such that $H(x_0) \in \ell^{p'}(\Gamma)$. But, by (7.20) it follows that

$$\sum_{n=1}^{\infty} \left|\langle H(x_0),\ \chi_{\{\gamma_n\}}\rangle\right|^{p'} \;=\; \sum_{n=1}^{\infty} \,|\,\overline{(x_0,\gamma_n)}\,|^{p'} \;=\; \infty$$

which is impossible. So, m_{F_p} must have infinite variation.

(iii) For $\gamma \in \Gamma$ fixed, write

$$(\cdot,\gamma) \;=\; [\mathrm{Re}(\cdot,\gamma)]^+ - [\mathrm{Re}(\cdot,\gamma)]^- + i[\mathrm{Im}(\cdot,\gamma)]^+ - i[\mathrm{Im}(\cdot,\gamma)]^-$$

and note that each of the four $[0,1]$-valued functions on the right-hand side, being bounded, is m_{F_p}-integrable. Since the closed convex hull of $\mathcal{R}(m_{F_p})$ is given by

$$C := \overline{\mathrm{co}}\, \mathcal{R}(m_{F_p}) \;=\; \left\{ \int_G f\, dm_{F_p} : 0 \le f \le 1,\ f \in L^\infty(m_{F_p}) \right\},$$

[42, p. 263], we conclude that

$$\chi_{\{\gamma\}} \;=\; (\cdot,\gamma)\widehat{\ } = F_p\big((\cdot,\gamma)\big) \;=\; \int_G (\cdot,\gamma)\, dm_{F_p} \in C + C + iC + iC.$$

So, if $\mathcal{R}(m_{F_p})$ is relatively compact in $\ell^{p'}(\Gamma)$, then so is $C+C+iC+iC$ and hence, also $\{\chi_{\{\gamma\}} : \gamma \in \Gamma\}$. But, this is surely not the case as $\|\chi_{\{\gamma\}} - \chi_{\{\eta\}}\|_{\ell^{p'}(\Gamma)} = 2^{1/p'}$ for all $\gamma \neq \eta$ in Γ. $\square$

Remark 7.7. (i) For $p = 1$, the inequalities

$$\|m_{F_1}(A)\|_{\ell^\infty(\Gamma)} \;=\; \|\widehat{\chi}_A\|_{\ell^\infty(\Gamma)} \;\le\; \|\chi_A\|_{L^1(G)} \;=\; \mu(A), \qquad A \in \mathcal{B}(G),$$

show that the vector measure $m_{F_1} : \mathcal{B}(G) \to \ell^\infty(\Gamma)$ does have finite variation.

(ii) Parts (ii) and (iii) of Proposition 7.6 should be compared with Proposition 7.3(i).

(iii) Let $A \in \mathcal{B}(G)$ satisfy $\mu(A) > 0$. The same proof as for part (ii) of Proposition 7.6 shows that m_{F_p} restricted to $A \cap \mathcal{B}(G)$ has infinite variation. Hence, m_{F_p} has infinite variation on *every* measurable subset of G with positive Haar measure. In other words, the variation measure of m_{F_p} is totally infinite. $\square$

Because the optimal domain spaces $L^1(m_{F_p})$ of the continuous linear operators $F_p : L^p(G) \to \ell^{p'}(\Gamma)$ turn out to be of special interest in harmonic analysis, we will adopt a special notation for them, as introduced in [113] for the case of $G = \mathbb{T}^d$ with $d \in \mathbb{N}$. Namely, we will denote $L^1(m_{F_p})$ by $\mathbf{F}^p(G)$ and the norm $\|\cdot\|_{L^1(m_{F_p})}$ by $\|\cdot\|_{\mathbf{F}^p(G)}$.

Proposition 7.8. *Let $1 \le p \le 2$. For each B.f.s. $\mathbf{F}^p(G)$ over $(G, \mathcal{B}(G), \mu)$, necessarily σ-o.c., we have that $L^p(G)$ is a dense subspace of $\mathbf{F}^p(G)$ with*

$$\|f\|_{\mathbf{F}^p(G)} \le \|f\|_{L^p(G)}, \qquad f \in L^p(G), \tag{7.21}$$

and $\mathbf{F}^p(G) \subseteq L^1(G)$ with

$$\|f\|_{L^1(G)} \le \|f\|_{\mathbf{F}^p(G)}, \qquad f \in \mathbf{F}^p(G). \tag{7.22}$$

Also, $\mathbf{F}^p(G)$ is dense in $L^1(G)$ and the optimal extension $I_{m_{F_p}} : \mathbf{F}^p(G) \to \ell^{p'}(\Gamma)$ of F_p is the Fourier transform map, that is,

$$I_{m_{F_p}}(f) = \int_G f \, dm_{F_p} = \widehat{f}, \qquad f \in \mathbf{F}^p(G). \tag{7.23}$$

In particular, $I_{m_{F_p}}$ is injective.

Proof. As in (7.14), the unit vector $\chi_{\{e\}} \in \ell^p(\Gamma) = \ell^{p'}(\Gamma)^*$ satisfies

$$\langle m_{F_p}(A), \ \chi_{\{e\}} \rangle = \mu(A),$$

that is,

$$\mu(A) = \langle m_{F_p}(A), \ \chi_{\{e\}} \rangle = \big|\langle m_{F_p}, \ \chi_{\{e\}} \rangle\big|(A), \qquad A \in \mathcal{B}(G).$$

So, if $f \in \mathbf{F}^p(G)$, then

$$\int_G |f| \, d\mu = \int_G |f| \, d|\langle m_{F_p}, \ \chi_{\{e\}} \rangle| \le \|f\|_{L^1(m_{F_p})} = \|f\|_{\mathbf{F}^p(G)}.$$

This establishes $\mathbf{F}^p(G) \subseteq L^1(G)$ with (7.22) holding.

According to Proposition 7.6(i), the $\mathcal{B}(G)$-simple functions in $L^1(m_{F_p})$ coincide with those in $L^1(G)$. Since $\operatorname{sim} \mathcal{B}(G)$ is dense in $L^1(G)$, it follows that $L^1(m_{F_p}) = \mathbf{F}^p(G)$ is also dense in $L^1(G)$ because of the continuous inclusions

$$\operatorname{sim} \mathcal{B}(G) \subseteq \mathbf{F}^p(G) \subseteq L^1(G).$$

As $\mathbf{F}^p(G)$ is the optimal domain for $F_p : L^p(G) \to \ell^{p'}(\Gamma)$, we automatically have $L^p(G) \subseteq \mathbf{F}^p(G)$ with a continuous inclusion and

$$\int_G f \, dm_{F_p} = F_p(f) = \widehat{f}, \qquad f \in L^p(G). \tag{7.24}$$

To establish (7.21), observe that the Hausdorff–Young inequality (7.6) means that $\|F_p\| \leq 1$. So, the natural inclusion map $J_{F_p} : L^p(G) \to L^1(m_{F_p}) = \mathbf{F}^p(G)$ satisfies $\|J_{F_p}\| = \|F_p\| \leq 1$ (see Proposition 4.4(ii) with $X(\mu) := L^p(G), E := \ell^{p'}(\Gamma)$ and $T := F_p$) and hence, (7.21) holds because

$$\|f\|_{\mathbf{F}^p(G)} = \big\|J_{F_p}(f)\big\|_{\mathbf{F}^p(G)} \leq \|J_{F_p}\| \cdot \|f\|_{L^p(G)} \leq \|f\|_{L^p(G)}, \qquad f \in L^p(G).$$

Furthermore, recall that $\operatorname{sim}\mathcal{B}(G)$ is dense in $L^1(m_{F_p}) = \mathbf{F}^p(G)$ from Theorem 3.7(ii) with $\nu := m_{F_p}$. So, the continuous inclusions

$$\operatorname{sim}\mathcal{B}(G) \subseteq L^p(G) \subseteq \mathbf{F}^p(G)$$

imply that $L^p(G)$ is dense in $\mathbf{F}^p(G)$.

To establish (7.23) fix $f \in \mathbf{F}^p(G)$. Choose $\{s_n\}_{n=1}^\infty \subseteq \operatorname{sim}\mathcal{B}(G)$ with $s_n \to f$ in $L^1(m_{F_p}) = \mathbf{F}^p(G)$ as $n \to \infty$. By continuity of the associated integration operator $I_{m_{F_p}} : L^1(m_{F_p}) \to \ell^{p'}(\Gamma)$ and (7.24) we have

$$\lim_{n\to\infty} \widehat{s}_n = \lim_{n\to\infty} \int_G s_n \, dm_{F_p} = \int_G f \, dm_{F_p} = I_{m_{F_p}}(f)$$

with convergence in $\ell^{p'}(\Gamma)$. On the other hand, (7.22) implies that $s_n \to f$ in $L^1(G)$ as $n \to \infty$ and so the continuity of $F_1 : L^1(G) \to \ell^\infty(\Gamma)$ yields $\lim_{n\to\infty} \widehat{s}_n = \widehat{f}$ with convergence in $\ell^\infty(\Gamma)$. Since $\ell^{p'}(\Gamma) \subseteq \ell^\infty(\Gamma)$ it follows from uniqueness of Fourier transforms that (7.23) is valid. □

Remark 7.9. (i) For $p = 1$, Proposition 7.8 shows that $\mathbf{F}^1(G) = L^1(G)$ with their given (lattice) norms being equal.

For $p = 2$, the Plancherel Theorem, [140, Theorem 1.6.1], and (7.21) yield, for every $f \in L^2(G)$,

$$\|f\|_{\mathbf{F}^2(G)} \leq \|f\|_{L^2(G)} = \|\widehat{f}\|_{\ell^2(\Gamma)} = \big\|I_{m_{F_2}}(f)\big\|_{\ell^2(\Gamma)} \leq \|f\|_{\mathbf{F}^2(G)},$$

where the last inequality follows from $\|I_{m_{F_2}}\| = 1$; see (3.99) with $\nu := m_{F_2}$. This implies that $\mathbf{F}^2(G) = L^2(G)$ with their given norms being equal.

So, for $p = 1, 2$ both the Fourier transform maps $F_1 : L^1(G) \to \ell^\infty(\Gamma)$ and $F_2 : L^2(G) \to \ell^2(\Gamma)$ are already defined on their optimal domain; no further extension is possible.

(ii) The reflexive spaces $\ell^{p'}(\Gamma)$, for $1 < p \leq 2$, cannot contain an isomorphic copy of c_0, and so we know from the discussion immediately after Remark 3.33

that $L^1_{\mathrm{w}}(m_{F_p}) = L^1(m_{F_p})$. Since m_{F_1} takes its values in the closed subspace $c_0(\Gamma)$ and $m_{F_1} = m_1$ on $\mathcal{B}(G)$, with m_1 defined by (7.13), it follows from (7.15) that $L^1_{\mathrm{w}}(m_{F_1}) = L^1(m_{F_1}) = L^1(G)$. Accordingly,

$$L^1_{\mathrm{w}}(m_{F_p}) \;=\; L^1(m_{F_p}), \qquad 1 \le p \le 2. \tag{7.25}$$

□

Proposition 7.10. *Let G be a compact abelian group.*

(i) *The operator $F_1 : L^1(G) \to \ell^\infty(\Gamma)$ is not weakly compact whereas, for $1 < p \le 2$, each map $F_p : L^p(G) \to \ell^{p'}(\Gamma)$ is weakly compact. For each $1 \le p \le 2$, the same conclusion as for the operator F_p also holds for the extended operator $I_{m_{F_p}} : \mathbf{F}^p(G) \to \ell^{p'}(\Gamma)$.*

(ii) *For each $1 \le p \le 2$, the Fourier transform map $F_p : L^p(G) \to \ell^{p'}(\Gamma)$ is neither compact nor completely continuous. The same conclusion as for F_p is also true of the extended operator $I_{m_{F_p}} : \mathbf{F}^p(G) \to \ell^{p'}(\Gamma)$.*

Proof. (i) Since $\ell^{p'}(\Gamma)$ is reflexive whenever $1 < p \le 2$, it follows that both the operator $F_p : L^p(G) \to \ell^{p'}(\Gamma)$ and its extension $I_{m_{F_p}} : \mathbf{F}^p(G) \to \ell^{p'}(\Gamma)$ are weakly compact. For $p = 1$, we note that $\mathbf{F}^1(G) = L^1(G)$ and that $I_{m_{F_1}} = F_1$ takes its values in the closed subspace $c_0(\Gamma)$ of $\ell^\infty(\Gamma)$, that is, it coincides with $F_{1,0}$. By Lemma 7.1 we conclude that $F_1 = I_{m_{F_1}}$ is not weakly compact.

(ii) Since $I_{m_{F_1}} = F_1$ coincides with $F_{1,0}$ we conclude from Remark 7.2 that $F_1 = I_{m_{F_1}}$ is not completely continuous. Of course, not being weakly compact (by part (i)), it is clear that $F_1 = I_{m_{F_1}}$ also fails to be a compact operator.

Suppose now that $1 < p \le 2$. Fix any sequence of distinct points $\{\gamma_n\}_{n=1}^\infty \subseteq \Gamma$. As in the proof of Lemma 7.1 we have, for each $h \in L^{p'}(G)$, that $\big\langle (\cdot, \gamma_n), h \big\rangle = \widehat{h}(-\gamma_n)$ for each $n \in \mathbb{N}$. Since $h \in L^1(G)$, we have $\widehat{h} \in c_0(\Gamma)$, from which it follows that $\lim_{n\to\infty} \widehat{h}(-\gamma_n) = 0$. That is, $\big\{(\cdot, \gamma_n)\big\}_{n=1}^\infty$ converges weakly to 0 in $L^p(G)$. Again via (7.11) it follows that

$$\big\| (\cdot, \gamma_k)\widehat{\ } - (\cdot, \gamma_n)\widehat{\ } \big\|_{\ell^{p'}(\Gamma)} \;=\; 2^{1/p'}, \qquad k \ne n,$$

and hence, that $\big\{F_p\big((\cdot, \gamma_n)\big)\big\}_{n=1}^\infty$ has no convergent subsequence in $\ell^{p'}(\Gamma)$. Accordingly, the map $F_p : L^p(G) \to \ell^{p'}(\Gamma)$ is neither compact nor completely continuous. The same is then true of its extension $I_{m_{F_p}} : \mathbf{F}^p(G) \to \ell^{p'}(\Gamma)$. □

To describe the space $L^1(m)$, for a general vector measure m, is rather difficult. However, for the vector measures m_{F_p}, with $1 \le p \le 2$, we now show that this is possible. These alternate descriptions of m_{F_p}-integrable functions will allow us to make a more detailed analysis of the optimal domain spaces $\mathbf{F}^p(G)$.

Define a vector subspace $V^p(G)$ of $L^{p'}(G)$, for $1 \le p \le 2$, by

$$V^p(G) := \big\{ h \in L^{p'}(G) : \, h = \overset{\vee}{\varphi} \text{ for some } \varphi \in \ell^p(\Gamma) \big\}.$$

Since $\ell^p(\Gamma) \subseteq \ell^2(\Gamma)$, *the inverse Fourier transform*

$$\overset{\vee}{\varphi}(x) := \sum_{\gamma\in\Gamma} (x,\gamma)\varphi(\gamma), \qquad x \in G, \tag{7.26}$$

is well defined as an element of $L^2(G)$ whenever $\varphi \in \ell^p(\Gamma)$. The Plancherel Theorem implies that $V^2(G) = L^2(G)$. For $p = 1$ we note that $V^1(G) = \{\overset{\vee}{\varphi} : \varphi \in \ell^1(\Gamma)\}$ and hence, $V^1(G) \subseteq C(G)$ where $C(G)$ is the space of all $\mathbb{C}$-valued continuous functions on G (a Banach space for the sup-norm $\|\cdot\|_{C(G)}$). Since G is infinite, it is known that the inclusion $V^1(G) \subseteq C(G)$ is proper, [140, Theorem 4.6.8]. Given $1 \le r \le \infty$, a subspace Y of $L^r(G)$ is called *translation invariant* if $\tau_x(Y) \subseteq Y$ for every $x \in G$. For both $p = 1, 2$, it is clear that $V^p(G)$ is a translation invariant subspace of $L^{p'}(G)$ which is stable under formation of complex conjugates and reflections. By the *reflection* of a function $f : G \to \mathbb{C}$ we mean the function $R_G f : x \mapsto f(-x)$ for $x \in G$. The reflection $R_\Gamma \xi$ of a function $\xi : \Gamma \to \mathbb{C}$ is defined analogously. We now show that the situation for $1 < p < 2$ is similar.

Lemma 7.11. *Let G be a compact abelian group.*

(i) *If $1 \le p < 2$, then*

$$V^p(G) \subseteq L^{p'}(G); \tag{7.27}$$

the containment in (7.27) is always proper and $V^p(G)$ separates points of $\mathbf{F}^p(G)$. Moreover, $V^2(G) = L^2(G)$.

(ii) *For $1 \le p < q \le 2$ it is the case that $V^p(G) \subseteq V^q(G) \subseteq L^2(G)$.*

(iii) *For each $1 \le p \le 2$, $V^p(G)$ is a translation invariant subspace of $L^{p'}(G)$ which contains $\mathcal{T}(G)$ and is stable under formation of complex conjugates and reflections.*

Proof. (i) The cases $p = 1, 2$ were discussed above. That the containment $V^p(G) \subseteq L^{p'}(G)$ is also proper for $1 < p < 2$ is known; see [76, p. 429], for example.

Since $\chi_{\{\gamma\}} \in \ell^p(\Gamma)$, for each $\gamma \in \Gamma$, and $(\cdot,\gamma) = \overset{\vee}{\chi}_{\{\gamma\}}$ (see (7.26)), it follows that $\mathcal{T}(G) \subseteq V^p(G)$. By Lemma 7.15 below, $\langle F, H\rangle := \int_G FH\,d\mu$ exists in $\mathbb{C}$ for each $F \in \mathbf{F}^p(G)$ and $H \in V^p(G)$. Accordingly, if $\langle f, h\rangle = 0$ for some $f \in \mathbf{F}^p(G) \subseteq L^1(G)$ and all $h \in V^p(G)$, then also $\langle f, (\cdot,\gamma)\rangle = \widehat{f}(-\gamma) = 0$ for all $\gamma \in \Gamma$. By injectivity of F_1 it follows that $f = 0$. Hence, $V^p(G)$ separates points of $\mathbf{F}^p(G)$.

(ii) The condition $1 \le p < q \le 2$ implies that $\ell^p(\Gamma) \subseteq \ell^q(\Gamma)$. Moreover, since then $2 \le q' < p'$ we also have $L^{p'}(G) \subseteq L^{q'}(G) \subseteq L^2(G)$. So, if $h \in V^p(G)$, then surely $h \in L^{q'}(G)$. Moreover, $h = \overset{\vee}{\varphi}$ for some $\varphi \in \ell^p(\Gamma) \subseteq \ell^q(\Gamma)$ and so $h \in V^q(G)$.

(iii) Fix $1 \le p \le 2$. It was already noted in the proof of part (i) that $\mathcal{T}(G) \subseteq V^p(G)$.

Fix $a \in G$. Let $h \in V^p(G)$ so that $\widehat{h} = \varphi$ for some (unique) $\varphi \in \ell^p(\Gamma)$. Since $h \in L^{p'}(G)$, also the translate $\tau_a h \in L^{p'}(G)$. Moreover, the identity

$$(\tau_a h)\widehat{\ }(\gamma) = \overline{(a,\gamma)}\,\widehat{h}(\gamma), \qquad \gamma \in \Gamma, \tag{7.28}$$

shows that $(\tau_a h)\widehat{\ } = \overline{(a,\cdot)}\,\varphi \in \ell^p(\Gamma)$ and hence, $\tau_a h \in V^p(G)$. This shows that $V^p(G)$ is translation invariant.

It is routine to check that $\overline{h} \in L^{p'}(G)$ and

$$(\overline{h})\widehat{\ } = \overline{R_\Gamma \varphi} = \overline{R_\Gamma \widehat{h}}. \tag{7.29}$$

Since both R_Γ and the conjugation operator $\xi \mapsto \overline{\xi}$ are isometries on $\ell^p(\Gamma)$, it follows from (7.29) that $\overline{h} \in V^p(G)$. Hence, $V^p(G)$ is also stable under complex conjugation.

Finally, since $R_G h \in L^{p'}(G)$ and $(R_G h)\widehat{\ } = R_\Gamma \varphi$, it follows that the function $R_G h \in V^p(G)$. That is, $V^p(G)$ is stable under formation of reflections. $\square$

For each $f \in L^1(G)$ define a linear map

$$S_f : L^\infty(G) \to c_0(\Gamma)$$

by the formula

$$S_f : h \mapsto (hf)\widehat{\ }, \qquad h \in L^\infty(G). \tag{7.30}$$

Clearly $S_f \in \mathcal{L}\big(L^\infty(G),\ c_0(\Gamma)\big)$ with operator norm $\|S_f\| \le \|f\|_{L^1(G)}$. For each $1 \le p \le 2$ and $T \in \mathcal{L}\big(L^\infty(G),\ \ell^{p'}(\Gamma)\big)$, we denote the operator norm of T by $\|T\|_{\infty,p'}$. We note that if $f \in L^1(G)$ has the property that $S_f\big(L^\infty(G)\big) \subseteq \ell^{p'}(\Gamma)$, then the Closed Graph Theorem implies that $S_f \in \mathcal{L}\big(L^\infty(G),\ \ell^{p'}(\Gamma)\big)$ and, in particular, $\|S_f\|_{\infty,p'} < \infty$. Indeed, let $h_n \to 0$ in $L^\infty(G)$ and $S_f(h_n) \to \varphi$ in $\ell^{p'}(\Gamma)$ as $n \to \infty$. Since $\ell^{p'}(\Gamma) \subseteq c_0(\Gamma)$ continuously, it follows that $S_f(h_n) \to \varphi$ in $c_0(\Gamma)$. On the other hand, $h_n f \to 0$ pointwise μ-a.e. in G and $|h_n f| \le M|f|$ for $n \in \mathbb{N}$ (with $M := \sup_{n\in\mathbb{N}} \|h_n\|_{L^\infty(G)}$). By the Dominated Convergence Theorem $\|h_n f\|_{L^1(G)} \to 0$ as $n \to \infty$ and hence, by continuity of the Fourier transform map $F_{1,0} : L^1(G) \to c_0(\Gamma)$, we can conclude that $S_f(h_n) = (h_n f)\widehat{\ } \to 0$ in $c_0(\Gamma)$ as $n \to \infty$. Accordingly, $\varphi = 0$ and so $S_f : L^\infty(G) \to \ell^{p'}(\Gamma)$ is indeed a closed linear operator. Moreover, the dual operator $S_f^* : \ell^p(\Gamma) \to L^\infty(G)^*$ turns out to be rather "nice".

Proposition 7.12. *Let $1<p<2$ and suppose that $f\in L^1(G)$ satisfies $S_f\big(L^\infty(G)\big)\subseteq \ell^{p'}(\Gamma)$. Then the dual operator $S_f^* \in \mathcal{L}\big(\ell^p(\Gamma),\ L^\infty(G)^*\big)$ actually takes its values in the closed subspace $L^1(G)$ of $L^\infty(G)^*$ and is given by*

$$S_f^* : \xi \mapsto f(\cdot)\sum_{\gamma\in\Gamma} \overline{(\cdot,\gamma)}\,\xi(\gamma), \qquad \xi \in \ell^p(\Gamma). \tag{7.31}$$

Proof. Observe that $h \in L^1(G)$ is identified with the element $F_h \in L^\infty(G)^*$ defined by $\psi \mapsto \int_G h\psi\, d\mu$, for $\psi \in L^\infty(G)$, which then satisfies (with $\|\cdot\|_\infty$ denoting $\|\cdot\|_{L^\infty(G)}$)

$$\|F_h\|_{L^\infty(G)^*} = \sup_{\|\psi\|_\infty\le 1} \big|\langle \psi, F_h\rangle\big| = \sup_{\|\psi\|_\infty\le 1} \Big|\int_G \psi h\, d\mu\Big| = \|h\|_{L^1(G)}.$$

Suppose that $\{h_n\}_{n=1}^{\infty} \subseteq L^1(G)$ satisfies $F_{h_n} \to \rho$ in $L^\infty(G)^*$. Then

$$\|h_n - h_k\|_{L^1(G)} = \|F_{h_n} - F_{h_k}\|_{L^\infty(G)^*}$$

and so $\{h_n\}_{n=1}^{\infty}$ is Cauchy in $L^1(G)$. Accordingly, there exists $h \in L^1(G)$ such that $\|h_n - h\|_{L^1(G)} \to 0$ as $n \to \infty$, that is,

$$\|F_{h_n} - F_h\|_{L^\infty(G)^*} = \|h_n - h\|_{L^1(G)} \to 0, \qquad n \to \infty.$$

It follows that $\rho = F_h$ and hence, that $L^1(G)$ is a closed subspace of $L^\infty(G)^*$.

Let $\psi \in L^\infty(G)$ and $\xi \in \ell^p(\Gamma)$ have finite support. Then

$$\big\langle S_f(\psi),\ \xi\big\rangle = \big\langle \widehat{f\psi},\ \xi\big\rangle = \sum_{\gamma\in\Gamma} \xi(\gamma) \int_G \overline{(x,\gamma)}\, f(x)\psi(x)\, d\mu(x).$$

Since this is a *finite sum*, it follows that

$$\big\langle S_f(\psi),\ \xi\big\rangle = \int_G \psi(x)\Big[f(x) \sum_{\gamma\in\Gamma} \overline{(x,\gamma)}\, \xi(\gamma)\Big]\, d\mu(x) = \big\langle \psi,\ S_f^*(\xi)\big\rangle$$

with $S_f^*(\xi) \in L^1(G)$ as given by (7.31).

For a general element $\xi \in \ell^p(\Gamma)$ there is a countable set $\{\gamma_n : n \in \mathbb{N}\} \subseteq \Gamma$ such that $\xi(\gamma) = 0$ for all $\gamma \notin \{\gamma_n : n \in \mathbb{N}\}$. Define the finitely supported elements $\xi_N := \sum_{n=1}^{N} \xi(\gamma_n)\chi_{\{\gamma_n\}}$ of $\ell^p(\Gamma)$ for each $N \in \mathbb{N}$, in which case $\xi_N \to \xi$ in $\ell^p(\Gamma)$ as $N \to \infty$. By continuity of S_f^* it follows that $S_f^*(\xi_N) \to S_f^*(\xi)$ in $L^\infty(G)^*$ for $N \to \infty$. Since $\big\{S_f^*(\xi_N)\big\}_{N=1}^{\infty} \subseteq L^1(G)$ it follows from the above discussion that $S_f^*(\xi) = F_{\varphi_\xi}$ for some $\varphi_\xi \in L^1(G)$ and that $S_f^*(\xi_N) \to \varphi_\xi$ in $L^1(G)$. By passing to a subsequence, if necessary, we may assume that

$$S_f^*(\xi_N) = f(\cdot) \sum_{\gamma\in\Gamma} \overline{(\cdot,\gamma)}\xi_N(\gamma) \to \varphi_\xi, \qquad N \to \infty, \tag{7.32}$$

pointwise μ-a.e. on G. Since $\ell^p(\Gamma) \subseteq \ell^2(\Gamma)$ continuously and $\xi_N \to \xi$ in $\ell^p(\Gamma)$ as $N \to \infty$, it follows that also $\xi_N \to \xi$ in $\ell^2(\Gamma)$. By Plancherel's Theorem we have that $(\overline{\xi_N})^\vee \to (\overline{\xi})^\vee$ in $L^2(G)$ and hence,

$$\sum_{\gamma\in\Gamma} \overline{(\cdot,\gamma)}\xi_N(\gamma) \to \sum_{\gamma\in\Gamma} \overline{(\cdot,\gamma)}\xi(\gamma), \qquad N \to \infty,$$

in $L^2(G)$. So, for some subsequence

$$\sum_{\gamma\in\Gamma} \overline{(\cdot,\gamma)}\xi_{N(k)}(\gamma) \to \sum_{\gamma\in\Gamma} \overline{(\cdot,\gamma)}\xi(\gamma), \qquad k \to \infty,$$

pointwise μ-a.e. on G. It follows from (7.32) that

$$\varphi_\xi(\cdot) = f(\cdot) \sum_{\gamma\in\Gamma} \overline{(\cdot,\gamma)}\xi(\gamma).$$

In particular, $S_f^*(\xi) = F_{\varphi_\xi}$ belongs to $L^1(G)$ and the formula (7.31) is valid. $\square$

We can now present some alternate descriptions of $L^1(m_{F_p})$, first formulated in [113, Theorem 1.2] for the d-dimensional torus $G = \mathbb{T}^d$.

Proposition 7.13. *Let $1 \le p \le 2$. Each of the spaces*

$$\Delta^p(G) := \big\{ f \in L^1(G) : \int_G |f| \cdot |h| \, d\mu < \infty \text{ for all } h \in V^p(G) \big\},$$

$$\Phi^p(G) := \big\{ f \in L^1(G) : (f\chi_A)\widehat{\ } \in \ell^{p'}(\Gamma) \text{ for all } A \in \mathcal{B}(G) \big\}, \tag{7.33}$$

$$\Lambda^p(G) := \big\{ f \in L^1(G) : S_f\big(L^\infty(G)\big) \subseteq \ell^{p'}(\Gamma) \big\}, \tag{7.34}$$

coincides with the optimal domain $\mathbf{F}^p(G)$ of $F_p : L^p(G) \to \ell^{p'}(\Gamma)$. Moreover, in the case of (7.34) *we have*

$$\|S_f\|_{\infty,p'} = \|f\|_{\mathbf{F}^p(G)}, \qquad f \in \mathbf{F}^p(G). \tag{7.35}$$

Remark 7.14. (i) As noted above, for $G = \mathbb{T}^d$ Proposition 7.13 occurs in [113, Theorem 1.2]; our proof will, to a certain extent, follow the lines of that given there.

(ii) It is not obvious from (7.33) that the space $\Phi^p(G)$ is actually an order ideal in $L^0(\mu)$. Of course, being equal to the B.f.s. $\mathbf{F}^p(G)$, it must have this property. In addition to having σ-o.c. norm it will follow from Proposition 7.13 that the optimal domain spaces $\mathbf{F}^p(G)$, for $1 \le p \le 2$, have other desirable properties.

(iii) For $G = \mathbb{T}$ and $1 \le p \le 2$ the following question was raised some forty years ago by R.E. Edwards, [54, p. 206]:

What can be said about the family of functions $f \in L^1(\mathbb{T})$ having the property that $\widehat{f\chi_A} \in \ell^{p'}(\mathbb{Z})$ for all $A \in \mathcal{B}(\mathbb{T})$?

As noted in [113, Remark 1.3(iii)], Propositions 7.8 and 7.13 provide an exact answer: this family of functions is precisely the optimal domain $\mathbf{F}^p(\mathbb{T})$ of the Fourier transform map $F_p : L^p(\mathbb{T}) \to \ell^{p'}(\mathbb{Z})$. □

The proof of Proposition 7.13 will be via a series of lemmata.

Lemma 7.15. *Let $1 \le p \le 2$. A $\mathcal{B}(G)$-measurable function $f : G \to \mathbb{C}$ belongs to $\mathbf{F}^p(G)$ if and only if*

$$\int_G |f| \cdot |h| \, d\mu < \infty, \qquad h \in V^p(G). \tag{7.36}$$

Proof. Let $1 < p \le 2$. Suppose that $f \in \mathbf{F}^p(G)$, that is, f is m_{F_p}-integrable. Let $h \in V^p(G)$, that is, $h = \overset{\vee}{\varphi}$ for some $\varphi \in \ell^p(\Gamma)$. Since then $\varphi \in \ell^{p'}(\Gamma)^*$, it follows that $\int_G |f| \, d|\langle m_{F_p}, \varphi\rangle| < \infty$. Now, for $A \in \mathcal{B}(G)$, we have

$$\big\langle m_{F_p}(A),\ \varphi \big\rangle = \sum_{\gamma\in\Gamma} \widehat{\chi}_A(\gamma)\widehat{h}(\gamma) = \sum_{\gamma\in\Gamma} \widehat{\chi}_A(\gamma)\overline{(R_G\overline{h})\widehat{\ }(\gamma)}.$$

Since $\chi_A \in L^2(G)$ and $h \in L^{p'}(G) \subseteq L^2(G)$ implies that $R_G\overline{h} \in L^2(G)$, it follows from *Parseval's formula*, [140, p. 27], that

$$\sum_{\gamma\in\Gamma} \widehat{\chi}_A(\gamma)\,\overline{(R_G\overline{h})\widehat{\ }(\gamma)} = \int_G \chi_A(x)\,\overline{R_G\overline{h}(x)}\,d\mu(x) = \int_A R_G h\,d\mu.$$

This establishes the formula (with $h = \overset{\vee}{\varphi}$)

$$\langle m_{F_p}(A),\ \varphi\rangle = \int_A R_G h\,d\mu, \qquad A \in \mathcal{B}(G). \tag{7.37}$$

Accordingly, the variation measure of $\langle m_{F_p}, \varphi\rangle$ is given by

$$\big|\langle m_{F_p},\ \varphi\rangle\big|(A) = \int_A |R_G h|\,d\mu, \qquad A \in \mathcal{B}(G),$$

and so

$$\int_G |f|\cdot|R_G h|\,d\mu = \int_G |f|\,d|\langle m_{F_p},\varphi\rangle| < \infty. \tag{7.38}$$

Since $V^p(G)$ is invariant under formation of reflections (see Lemma 7.11(iii)), it follows that (7.36) holds.

Conversely, suppose that the $\mathcal{B}(G)$-measurable function $f : G \to \mathbb{C}$ satisfies (7.36). Let $\varphi \in \ell^{p'}(\Gamma)^* = \ell^p(\Gamma)$ be arbitrary. According to [76, p. 229] there exists $h \in L^{p'}(G)$ such that $\widehat{h} = \varphi$. Then $h \in V^p(G)$ and so $\int_G |f|\cdot|h|\,d\mu < \infty$. Moreover, the same calculation as above shows that the equality in (7.38) holds and hence, is finite. That is, $f \in L^1_{\mathrm{w}}(m_{F_p})$ and so, by (7.25), we have $f \in L^1(m_{F_p}) = \mathbf{F}^p(G)$.

For $p = 1$, we know that $\mathbf{F}^1(G) = L^1(G)$. Moreover, since the constant function $\mathbf{1} = \overset{\vee}{\chi}_{\{e\}}$ (see (7.26)) belongs to $V^1(G)$ and also $V^1(G) \subseteq C(G)$, it follows easily that the stated assertion holds for $p = 1$. □

For the diligent reader, we point out that the use of Lemma 7.11(iii) in the above proof is logically legitimate as its proof did *not* use Lemma 7.11(i) (whose proof makes use of Lemma 7.15).

Fix $1 < p \le 2$ and let $f \in \Phi^p(G)$; see (7.33). Then the vector-valued set function $\kappa^f : \mathcal{B}(G) \to \ell^{p'}(\Gamma)$ defined by

$$A \mapsto \kappa^f(A) := (\chi_A f)\widehat{\ }, \qquad A \in \mathcal{B}(G), \tag{7.39}$$

is surely finitely additive. Actually, more is true.

Lemma 7.16. *Let $1 < p \le 2$. Then, for each $f \in \Phi^p(G)$, the finitely additive set function κ^f as defined by (7.39) is σ-additive, that is, it is an $\ell^{p'}(\Gamma)$-valued vector measure on $\mathcal{B}(G)$.*

Proof. Let $H := \{\chi_{\{\gamma\}} : \gamma \in \Gamma\}$, considered as a subset of the dual space $\ell^{p'}(\Gamma)^* = \ell^p(\Gamma)$. Fix $\gamma \in \Gamma$. Then the set function $A \mapsto \langle \kappa^f(A),\ \chi_{\{\gamma\}}\rangle$ on $\mathcal{B}(G)$ is σ-additive because it equals the indefinite integral of the function $\overline{(\cdot,\gamma)}f \in L^1(G)$ with respect to μ as seen from the calculation

$$\langle \kappa^f(A),\ \chi_{\{\gamma\}}\rangle = \langle (\chi_A f)\widehat{\ },\ \chi_{\{\gamma\}}\rangle = \int_G \overline{(\cdot,\gamma)} f \chi_A \, d\mu = \int_A \overline{(\cdot,\gamma)} f \, d\mu, \quad A \in \mathcal{B}(G).$$

Since the reflexive space $\ell^{p'}(\Gamma)$ cannot contain an isomorphic copy of the Banach sequence space ℓ^∞ and H is a total set of functionals in $\ell^{p'}(\Gamma)^* = \ell^p(\Gamma)$, it follows from the generalized Orlicz-Pettis Theorem (see Lemma 3.2) that κ^f is σ-additive. □

Lemma 7.17. *Let $1 < p \le 2$. Then $L^1(m_{F_p}) = \Phi^p(G)$.*

Proof. By Proposition 7.8 it is clear that $L^1(m_{F_p}) \subseteq \Phi^p(G)$.

Conversely, suppose that $f \in \Phi^p(G)$. Given $h \in V^p(G)$ there exists $\varphi \in \ell^p(\Gamma)$ such that $\widehat{h} = \varphi$ and (7.37) holds. According to Lemma 7.16,

$$A \mapsto \langle \kappa^f(A),\ \varphi\rangle = \langle (\chi_A f)\widehat{\ },\ \widehat{h}\rangle, \qquad A \in \mathcal{B}(G),$$

is σ-additive. Define $A_n := |f|^{-1}([0,n])$, for $n \in \mathbb{N}$, in which case $(A \cap A_n) \uparrow A$ for $A \in \mathcal{B}(G)$ fixed. By σ-additivity of $\langle \kappa^f, \varphi\rangle$ we have

$$\langle \kappa^f(A),\ \varphi\rangle = \lim_{n\to\infty} \langle (\chi_{A\cap A_n} f)\widehat{\ },\ \widehat{h}\rangle.$$

Since $\chi_{A\cap A_n} \in L^\infty(G) \subseteq L^2(G)$ and $h \in L^{p'}(G) \subseteq L^2(G)$ we can apply Parseval's formula to yield

$$\langle (\chi_{A\cap A_n} f)\widehat{\ },\ \widehat{h}\rangle = \int_G \chi_{A\cap A_n} \cdot f \cdot R_G h \, d\mu, \qquad n \in \mathbb{N}.$$

That is,

$$\langle \kappa^f(A),\ \varphi\rangle = \lim_{n\to\infty} \int_A f_n \, d\eta$$

where the functions $f_n := \chi_{A_n} f \in L^\infty(G)$ converge pointwise on G to f and $d\eta = R_G h \ d\mu$ is a complex measure (as $R_G h \in L^1(G)$). By Lemma 2.17 we conclude that f is η-integrable (i.e., $f \cdot R_G h \in L^1(G)$) and

$$\int_A f \cdot R_G h \, d\mu = \int_A f \, d\eta = \langle \kappa^f(A),\ \varphi\rangle.$$

So, $f \cdot R_G h \in L^1(G)$ for all $h \in V^p(G)$. Then Lemma 7.15 implies that the function $f \in L^1(m_{F_p})$. □

We have an immediate consequence for the spaces $\Lambda^p(G)$ as given by (7.34).

Corollary 7.18. *For each $1 < p \le 2$ we have $L^1(m_{F_p}) = \Lambda^p(G)$.*

Proof. Let $f \in \Lambda^p(G)$. Then the operator S_f (see (7.30)) maps each $h \in L^\infty(G)$ into $\ell^{p'}(\Gamma)$. In particular, for $h = \chi_A$ we have

$$S_f(\chi_A) \;=\; (f\chi_A)\widehat{\ } \in \ell^{p'}(\Gamma), \qquad A \in \mathcal{B}(G),$$

that is, $f \in \Phi^p(G)$. By Lemma 7.17 we have $f \in L^1(m_{F_p})$.

Conversely, suppose that $f \in L^1(m_{F_p})$. Given $h \in L^\infty(G)$, we have that $|h| \le \big(\|h\|_{L^\infty(G)}\big)\,\chi_G$ (μ-a.e.). Since the μ-null sets and m_{F_p}-null sets coincide, it follows that

$$|h| \;\le\; \big(\|h\|_{L^\infty(G)}\big)\,\chi_G, \qquad m_{F_p}\text{-a.e.} \tag{7.40}$$

In particular, $h \in L^\infty(m_{F_p})$ and so $hf \in L^1(m_{F_p})$ by the ideal property of the B.f.s. $L^1(m_{F_p})$. It follows from (7.23) that $S_f(h) = (fh)\widehat{\ } = \int_G fh\, dm_{F_p} \in \ell^{p'}(\Gamma)$ and hence, that

$$\big\|S_f(h)\big\|_{\ell^{p'}(\Gamma)} \;=\; \Big\|\int_G fh\,dm_{F_p}\Big\|_{\ell^{p'}(\Gamma)} \;\le\; \|fh\|_{L^1(m_{F_p})}. \tag{7.41}$$

But, (7.40) yields $|fh| \le \|h\|_{L^\infty(G)}\cdot|f|$ (m_{F_p}-a.e.). Since the norm of $L^1(m_{F_p})$ is a lattice norm we have $\|fh\|_{L^1(m_{F_p})} \le \|h\|_{L^\infty(G)}\cdot\|f\|_{L^1(m_{F_p})}$. Accordingly,

$$\big\|S_f(h)\big\|_{\ell^{p'}(\Gamma)} \;\le\; \|h\|_{L^\infty(G)}\cdot\|f\|_{L^1(m_{F_p})}, \qquad h \in L^\infty(G).$$

This shows that $S_f \in \mathcal{L}\big(L^\infty(G),\, \ell^{p'}(\Gamma)\big)$ with $\|S_f\|_{\infty,p'} \le \|f\|_{L^1(m_{F_p})}$. In particular, $f \in \Lambda^p(G)$. $\square$

Remark 7.19. Let $1 \le p \le 2$. Fix $f \in \Lambda^p(G) = L^1(m_{F_p})$. It was shown in the proof of Corollary 7.18 that $\|S_f\|_{\infty,p'} \le \|f\|_{L^1(m_{F_p})}$ for $1 < p \le 2$; the arguments there also apply to obtain the same inequality for $p = 1$. To prove the reverse inequality, consider the indefinite integral

$$(m_{F_p})_f : A \mapsto \int_A f\,dm_{F_p}, \qquad A \in \mathcal{B}(G).$$

Then (3.2), with $\nu := (m_{F_p})_f$, together with the definition of S_f and (7.23) imply that

$$\begin{aligned}\big\|(m_{F_p})_f\big\|(G) &= \sup\Big\|\sum_{j=1}^{n} a_j\int_{A_j} f\,dm_{F_p}\Big\|_{\ell^{p'}(\Gamma)}\\ &= \sup\Big\|S_f\Big(\sum_{j=1}^{n} a_j\chi_{A_j}\Big)\Big\|_{\ell^{p'}(\Gamma)} \le \|S_f\|_{\infty,p'},\end{aligned}$$

where the supremum is taken over all choices of scalars $a_j \in \mathbb{C}$ with $|a_j| \le 1$ ($j = 1,\dots,n$), Σ-partitions $\{A_j\}_{j=1}^n$ of G, and $n \in \mathbb{N}$. This, together with the fact that $\|f\|_{L^1(m_{F_p})} = \|(m_{F_p})_f\|(G)$ (see (3.8) with $\nu := m_{F_p}$), yield the inequality $\|f\|_{L^1(m_{F_p})} \le \|S_f\|_{\infty,p'}$, which thereby establishes (7.35). $\square$

Proof of Proposition 7.13. Since $\mathbf{F}^p(G) = L^1(m_{F_p})$, it follows from Lemma 7.15 that $\mathbf{F}^p(G) = \Delta^p(G)$ for all $1 \le p \le 2$. For $1 < p \le 2$, Lemma 7.17 states that $\mathbf{F}^p(G) = \Phi^p(G)$. Clearly $L^1(G) = \Phi^1(G)$; see (7.33). Hence, by Remark 7.9(i), also $\mathbf{F}^1(G) = \Phi^1(G)$. For $1 < p \le 2$, Corollary 7.18 states that $\mathbf{F}^p(G) = \Lambda^p(G)$. Clearly $L^1(G) = \Lambda^1(G)$; see (7.34). Again by Remark 7.9(i) it follows that $\mathbf{F}^1(G) = \Lambda^1(G)$. Finally, the equality (7.35) was established in Remark 7.19. □

Remark 7.20. (i) It follows from Lemma 7.11(ii) and the fact that $\mathbf{F}^p(G) = \Delta^p(G)$ that

$$L^2(G) \subseteq \mathbf{F}^q(G) \subseteq \mathbf{F}^p(G) \subseteq L^1(G), \qquad 1 \le p < q \le 2.$$

Moreover, the inclusions are continuous. Indeed, since $\|\xi\|_{\ell^q(\Gamma)} \le \|\xi\|_{\ell^p(\Gamma)}$ and $\langle m_{F_p}, \xi\rangle = \langle m_{F_q}, \xi\rangle$, for all $\xi \in \ell^p(\Gamma) \subseteq \ell^q(\Gamma)$, it follows from the definition

$$\|f\|_{\mathbf{F}^r(G)} := \sup\Big\{ \int_G |f|\, d|\langle m_{F_r}, \xi\rangle| \;:\; \|\xi\|_{\ell^r(\Gamma)} \le 1\Big\}$$

that

$$\|f\|_{\mathbf{F}^p(G)} \le \|f\|_{\mathbf{F}^q(G)}, \qquad f \in \mathbf{F}^q(G).$$

(ii) The optimal domain spaces $\mathbf{F}^p(G)$, for $1 \le p \le 2$, are all translation invariant and stable under formation of reflections and complex conjugates. Indeed, fix $f \in \mathbf{F}^p(G)$ and $a \in G$. For each $h \in V^p(G)$ we have, according to (7.9), that

$$\int_G |\tau_a f| \cdot |h|\, d\mu = \int_G |f| \cdot |\tau_{-a} h|\, d\mu < \infty;$$

note that $\tau_{-a}h \in V^p(G)$ because of Lemma 7.11(iii). It then follows from Lemma 7.15 that $\tau_a f \in \mathbf{F}^p(G)$ and hence, that $\mathbf{F}^p(G)$ is translation invariant. Similarly,

$$\int_G |R_G f| \cdot |h|\, d\mu = \int_G |f| \cdot |R_G h|\, d\mu < \infty$$

with $R_G h \in V^p(G)$; see Lemma 7.11(iii). Again by Lemma 7.15 it follows that $R_G f \in \mathbf{F}^p(G)$, that is, $\mathbf{F}^p(G)$ is reflection invariant. Finally,

$$\int_G |\overline{f}| \cdot |h|\, d\mu = \int_G |f| \cdot |h|\, d\mu < \infty$$

together with Lemma 7.15 show that $\mathbf{F}^p(G)$ is also stable under formation of complex conjugates. □

We now collect together a few Banach space properties of the optimal domain spaces $\mathbf{F}^p(G)$.

Proposition 7.21. *For each $1 \le p \le 2$ the following statements are valid.*

(i) *$\mathbf{F}^p(G)$ is weakly sequentially complete and hence, its associate space satisfies $\mathbf{F}^p(G)' = \mathbf{F}^p(G)^*$.*

(ii) *$\mathbf{F}^p(G)$ has σ-o.c. norm and satisfies the σ-Fatou property.*

(iii) $\mathbf{F}^p(G)$ *is weakly compactly generated.*

(iv) *Whenever* G *is metrizable, the space* $\mathbf{F}^p(G)$ *is separable.*

Proof. By Theorem 3.7 the optimal domain space $\mathbf{F}^p(G) = L^1(m_{F_p})$ has σ-o.c. norm. The equality (7.25), stating that $L^1_{\mathrm{w}}(m_{F_p}) = L^1(m_{F_p})$, is equivalent to $L^1(m_{F_p}) = \mathbf{F}^p(G)$ being weakly sequentially complete (see Proposition 3.38(I) with $p := 1$), which then implies that $L^1(m_{F_p})$ has the σ-Fatou property and that $L^1(m_{F_p}) = L^1(m_{F_p})''$; see (3.84) and (3.85) in Chapter 3. According to Proposition 2.16(ii) we also have $\mathbf{F}^p(G)' = \mathbf{F}^p(G)^*$. This establishes (i) and (ii).

(iii) See Theorem 3.7(ii) with $\nu := m_{F_p}$, after recalling that $\mathbf{F}^p(G) = L^1(m_{F_p})$.

(iv) If G is metrizable (in addition to being compact), then $C(G)$ is separable for the sup-norm $\|\cdot\|_{C(G)}$, [140, Appendix A16]. As a consequence of Lusin's theorem it follows that $L^p(G)$ is separable for every $1 \leq p < \infty$, [140, Appendix E8]. But, $L^p(G)$ is dense and continuously embedded in $\mathbf{F}^p(G)$, for each $1 \leq p \leq 2$, from which it follows that $\mathbf{F}^p(G)$ is also separable. □

Remark 7.22. From the viewpoint of analysis, the weak sequential completeness of $\mathbf{F}^p(G)$ is difficult to use in practice since $\mathbf{F}^p(G)^*$ is not explicitly known. However, there is available a good (partial) substitute in this regard. Indeed, since $\mathbf{F}^p(G) \subseteq L^1(G)$ and $\mathbf{F}^p(G)$ has the σ-Fatou property, we see that $\mathbf{F}^p(G)$ is also a B.f.s in the more restricted sense of [13, p. 2]. Moreover, $\mathbf{F}^p(G) \subseteq L^1(G)$ implies that $L^\infty(G) \subseteq \mathbf{F}^p(G)^* = \mathbf{F}^p(G)'$. Since $L^\infty(G)$ is an order ideal of $\mathbf{F}^p(G)'$ containing $\operatorname{sim}\mathcal{B}(G)$, it follows from [13, Ch. 1, Theorem 5.2] that $\mathbf{F}^p(G)$ is also sequentially $\sigma(\mathbf{F}^p(G), L^\infty(G))$-complete. □

We also record some useful properties of the dual spaces $\mathbf{F}^p(G)^*$.

Proposition 7.23. *For each* $1 < p < 2$, *the dual space* $\mathbf{F}^p(G)^*$ *is a translation invariant B.f.s. satisfying*

$$L^\infty(G) \subseteq \mathbf{F}^p(G)^* \quad \textit{and} \quad V^p(G) \subseteq \mathbf{F}^p(G)^* \quad \textit{and} \quad \mathbf{F}^p(G)^* \subseteq L^{p'}(G). \tag{7.42}$$

The first two inclusions in (7.42) *are always proper, whereas the last inclusion is proper whenever* $L^p(G) \neq \mathbf{F}^p(G)$.

Proof. It follows from $L^p(G) \subseteq \mathbf{F}^p(G) \subseteq L^1(G)$ with continuous inclusions (see Proposition 7.8) that

$$L^\infty(G) \subseteq \mathbf{F}^p(G)^* \subseteq L^{p'}(G). \tag{7.43}$$

Moreover, (7.36) implies that $V^p(G)$ is contained in the associate space $\mathbf{F}^p(G)' = \mathbf{F}^p(G)^*$. This, together with (7.43), establish all three containments in (7.42).

According to [55, p. 151], for each $1 < p < 2$ there exists $h \in C(G) \subseteq L^{p'}(G)$ such that $\widehat{h} \notin \ell^p(\Gamma)$. That is, $h \notin V^p(G)$. Now, if it were the case that $V^p(G) = \mathbf{F}^p(G)'$, then $V^p(G)$ would be a B.f.s. over $(G, \mathcal{B}(G), \mu)$. Since $\chi_G \in V^p(G)$ and $|h| \leq \big(\|h\|_{C(G)}\big)\chi_G$ it would follow from the ideal property that $h \in V^p(G)$. But, this is not the case and so $V^p(G) \neq \mathbf{F}^p(G)'$.

If it were the case that $L^\infty(G) = \mathbf{F}^p(G)'$, then

$$L^1(G) \,=\, L^\infty(G)' = \mathbf{F}^p(G)'' = L^1(m_{F_p})'' = L^1(m_{F_p});$$

see the proof of parts (i) and (ii) of Proposition 7.21 for the last equality. That is, $L^1(G) = \mathbf{F}^p(G)$. But, for $1 < p < 2$, it is known that there always exists $f \in L^1(G)$ such that $\widehat{f} \notin \ell^{p'}(\Gamma)$, [55, p. 160]. Accordingly, $f \notin \mathbf{F}^p(G)$ as the optimal extension of F_p maps $\mathbf{F}^p(G)$ into $\ell^{p'}(\Gamma)$. Hence, $L^\infty(G) \neq \mathbf{F}^p(G)'$. Thus we have established that the first two inclusions in (7.42) are always proper.

If it were the case that $\mathbf{F}^p(G)' = L^{p'}(G)$, then also $\mathbf{F}^p(G)'' = L^p(G)$. But, as already noted above $\mathbf{F}^p(G)'' = \mathbf{F}^p(G)$ and so $\mathbf{F}^p(G) = L^p(G)$ would follow, contrary to our hypothesis that the containment $L^p(G) \subsetneq \mathbf{F}^p(G)$ is proper. So, the last inclusion in (7.42) is proper whenever $L^p(G) \neq \mathbf{F}^p(G)$.

Finally, to see that $\mathbf{F}^p(G)'$ is translation invariant, let $h \in \mathbf{F}^p(G)'$. By the definition of the associate space this means that $\int_G |f| \cdot |h| \, d\mu < \infty$ for all $f \in \mathbf{F}^p(G)$. Fix $a \in G$. Since $\mathbf{F}^p(G)$ is translation invariant (see Remark 7.20(ii)), it follows that

$$\int_G |f| \cdot |\tau_a h| \, d\mu \,=\, \int_G |\tau_{-a} f| \cdot |h| \, d\mu < \infty$$

for all $f \in \mathbf{F}^p(G)$. Accordingly, $\tau_a h$ also belongs to the associate space $\mathbf{F}^p(G)'$. This establishes that $\mathbf{F}^p(G)'$ is translation invariant. $\square$

Remark 7.24. The linear space $V^p(G) \subseteq L^{p'}(G)$ is *not* an ideal in $L^{p'}(G)$. This follows from an argument along the lines of that in proof of Proposition 7.23 where it was established that $V^p(G) \neq L^{p'}(G)$. $\square$

Is was noted in Remark 7.20(ii) that the optimal domain spaces $\mathbf{F}^p(G)$, for $1 \le p \le 2$, are all translation invariant. From the viewpoint of harmonic analysis they are even "nicer".

A *homogeneous Banach space* on G is a linear subspace B of $L^1(G)$ having a norm $\|\cdot\|_B \ge \|\cdot\|_{L^1(G)}$ under which B is a Banach space satisfying the following properties:

(H-1) If $f \in B$ and $a \in G$, then $\tau_a f \in B$ and $\|\tau_a f\|_B = \|f\|_B$.

(H-2) For all $f \in B$ and $a_0 \in G$ we have

$$\lim_{a \to a_0} \|\tau_a f - \tau_{a_0} f\|_B \,=\, 0.$$

This notion, for $G = \mathbb{T}$, was introduced by Y. Katznelson, [84, Ch. I, Definition 2.10]; see also [11] for general G. According to Sections 2 and 7 of Chapter I in [84], formulated for $G = \mathbb{T}$, such spaces are well suited to harmonic analysis.

Proposition 7.25. *Let $1 \le p \le 2$. The following assertions are valid.*

(i) $\mathbf{F}^p(G)$ *is a translation invariant subspace of* $L^1(G)$ *which is stable under formation of reflections and complex conjugation.*

(ii) *For each $a \in G$, the translation operator $\tau_a : \mathbf{F}^p(G) \to \mathbf{F}^p(G)$ is a surjective isometry. In particular, $\{\tau_a : a \in G\} \subseteq \mathcal{L}\big(\mathbf{F}^p(G)\big)$.*

(iii) *For each $a_0 \in G$ and $f \in \mathbf{F}^p(G)$ we have $\lim_{a\to a_0} \tau_a f = \tau_{a_0} f$ in the norm of $\mathbf{F}^p(G)$.*

In particular, each $\mathbf{F}^p(G)$ is a homogeneous Banach space on G.

Proof. For (i) we refer to Remark 7.20(ii).

(ii) Each $\varphi \in \ell^p(\Gamma)$ is of the form $\varphi = \widehat{h}$ for some unique $h \in V^p(G)$. Via (7.28) and (7.37) we see that

$$\big\langle m_{F_p},\, \overline{(a,\cdot)}\,\varphi\big\rangle(A) \;=\; \big\langle \widehat{\chi_A},\, (\tau_a \overset{\vee}{\varphi})\widehat{\ }\big\rangle \;=\; \int_A R_G(\tau_a \overset{\vee}{\varphi})\, d\mu,$$

for each $A \in \mathcal{B}(G)$ and $a \in G$. It then follows from this identity and (7.37) that, for each $f \in \mathbf{F}^p(G)$, we have (with $\|\cdot\|_p$ denoting $\|\cdot\|_{\ell^p(\Gamma)}$)

$$\begin{aligned}
\|\tau_a f\|_{\mathbf{F}^p(G)} &= \sup_{\|\varphi\|_p\le 1} \int_G |\tau_a f|\, d|\langle m_{F_p}, \varphi\rangle| = \sup_{\|\varphi\|_p\le 1} \int_G |\tau_a f| \cdot |R_G \overset{\vee}{\varphi}|\, d\mu \\
&= \sup_{\|\varphi\|_p\le 1} \int_G |f| \cdot |\tau_{-a}(R_G \overset{\vee}{\varphi})|\, d\mu = \sup_{\|\varphi\|_p\le 1} \int_G |f| \cdot |R_G(\tau_a \overset{\vee}{\varphi})|\, d\mu \\
&= \sup_{\|\varphi\|_p\le 1} \int_G |f|\, d|\langle m_{F_p}, \overline{(a,\cdot)}\,\varphi\rangle| = \sup_{\|\xi\|_p\le 1} \int_G |f|\, d|\langle m_{F_p}, \xi\rangle| = \|f\|_{\mathbf{F}^p(G)}.
\end{aligned}$$

(iii) By part (ii) it suffices to show that $\lim_{a\to 0} \tau_a f = f$ in $\mathbf{F}^p(G)$. But, $\{\tau_a : a \in G\}$ is uniformly bounded in $\mathcal{L}\big(\mathbf{F}^p(G)\big)$ and so we only need to consider f coming from a dense subspace of $\mathbf{F}^p(G)$. According to (7.21) and the density of $\mathcal{T}(G)$ in $L^p(G)$, [140, p. 24], the space $\mathcal{T}(G)$ is also dense in $\mathbf{F}^p(G)$. Since each trigonometric polynomial is a finite linear combination of functions of the form $(\cdot,\gamma)$, for $\gamma \in \Gamma$, it suffices to show that $\lim_{a\to 0} \tau_a\big((\cdot,\gamma)\big) = (\cdot,\gamma)$ in $\mathbf{F}^p(G)$, for each $\gamma \in \Gamma$. But,

$$(\cdot,\gamma) - \tau_a\big((\cdot,\gamma)\big) \;=\; \big[1 - (a,\gamma)\big]\,(\cdot,\gamma)$$

with $|(\cdot,\gamma)| \equiv \chi_G$. Since $\|\cdot\|_{\mathbf{F}^p(G)}$ is a lattice norm it follows that

$$\big\|(\cdot,\gamma) - \tau_a\big((\cdot,\gamma)\big)\big\|_{\mathbf{F}^p(G)} \;=\; |1 - (a,\gamma)| \cdot \|\chi_G\|_{\mathbf{F}^p(G)}.$$

Continuity of the function $(\cdot,\gamma)$ on G implies that $\lim_{a\to 0}(a,\gamma) = 1$ and hence, $\lim_{a\to 0} \tau_a\big((\cdot,\gamma)\big) = (\cdot,\gamma)$ in $\mathbf{F}^p(G)$. $\square$

As pointed out in [84, Ch. I. Sections 2 & 7], homogeneous Banach spaces B over $\mathbb{T}$ always admit convolution with measures from $M(\mathbb{T})$ as continuous operators in B. An examination of the proofs given there (see pages 11 and 39) shows that B-valued Bochner integrals are involved. For groups more general than $\mathbb{T}$, extra care needs to be taken in this regard. This we now do, with the aim of extending Theorem 7.7 of [84, Ch. I] to the homogeneous Banach spaces $\mathbf{F}^p(G)$. To this end we require the following formula, which follows routinely from Fubini's Theorem and the definitions of Fourier transform and convolution. Namely, for fixed $f \in L^1(G)$ and $\lambda \in M(G)$, the element $\big(\chi_A(f * \lambda)\big)\widehat{\ } \in c_0(\Gamma)$ is given by

$$\big(\chi_A(f * \lambda)\big)\widehat{\ }(\gamma) = \int_G \overline{(x,\gamma)} \cdot \big(f\tau_{-x}(\chi_A)\big)\widehat{\ }(\gamma)\, d\lambda(x), \tag{7.44}$$

for each $\gamma \in \Gamma$ and $A \in \mathcal{B}(G)$.

Proposition 7.26. *Let $1 \le p \le 2$ and G be metrizable. For each $\lambda \in M(G)$, the convolution operator $C_\lambda^{(1)} \in \mathcal{L}\big(L^1(G)\big)$ as given by (7.7) maps $\mathbf{F}^p(G)$ continuously into itself and has operator norm, as an element of $\mathcal{L}\big(\mathbf{F}^p(G)\big)$, at most $4\|\lambda\|_{M(G)}$.*

Proof. The cases $p = 1, 2$ are clear; see Remark 7.9(i). So, we may assume that $1 < p < 2$. Fix $f \in \mathbf{F}^p(G)$ and $A \in \mathcal{B}(G)$. Define the vector-valued function $H_{f,A} : G \to \ell^{p'}(\Gamma)$ by

$$H_{f,A}(x) : \gamma \longmapsto \overline{(x,\gamma)} \cdot \big(f\tau_{-x}(\chi_A)\big)\widehat{\ }(\gamma)\frac{d\lambda}{d|\lambda|}(x), \qquad \gamma \in \Gamma,$$

for $|\lambda|$-a.e. $x \in G$.

Claim 1. *The function $H_{f,A} : G \to \ell^{p'}(\Gamma)$ is strongly $|\lambda|$-measurable (i.e., strongly $\mathcal{B}(G)$-measurable).*

To see this, note first that Γ is countable (by the metrizability of G, [140, Theorem 2.2.6]) and hence, $\ell^{p'}(\Gamma)$ is separable. By the Pettis Measurability Theorem, [42, p. 42], it suffices to show that $H_{f,A}$ is scalarly $|\lambda|$-measurable. So, let $\xi \in \ell^{p'}(\Gamma)^* = \ell^p(\Gamma)$. Then $\langle H_{f,A}, \xi\rangle$ is the scalar function given by the (countable) series

$$x \longmapsto \sum_{\gamma\in\Gamma} \overline{(x,\gamma)}\xi(\gamma) \cdot \big(f\tau_{-x}(\chi_A)\big)\widehat{\ }(\gamma)\frac{d\lambda}{d|\lambda|}(x), \tag{7.45}$$

for $|\lambda|$-a.e. $x \in G$. Since Γ is countable, the function $x \mapsto \overline{(x,\gamma)}$ is continuous, and $\frac{d\lambda}{d|\lambda|}$ is Borel measurable, it suffices to show that the $\mathbb{C}$-valued function $x \mapsto \big(f\tau_{-x}(\chi_A)\big)\widehat{\ }(\gamma)$, for $x \in G$, is $\mathcal{B}(G)$-measurable for each fixed $\gamma \in \Gamma$, that is,

$$x \longmapsto \int_G \overline{(t,\gamma)}f(t) \cdot \big(\tau_{-x}(\chi_A)\big)(t)\, d\mu(t), \qquad x \in G,$$

is $\mathcal{B}(G)$-measurable. Direct calculation shows that this integral equals

$$\begin{aligned}\int_G \overline{(t,\gamma)}f(t)\chi_A(x+t)\, d\mu(t) &= \int_G \overline{(t,\gamma)}f(t)(R_G\chi_A)(-x-t)\, d\mu(t)\\ &= R_G\big(\overline{(\cdot,\gamma)}f * (R_G\chi_A)(\cdot)\big)(x).\end{aligned}$$

Since $\overline{(\cdot,\gamma)}f \in L^1(G)$ and $(R_G\chi_A)(\cdot) \in L^\infty(G)$, the convolution of these two functions is continuous, [95, p. 250, F3]. Hence, $R_G\big(\overline{(\cdot,\gamma)}f * (R_G\chi_A)(\cdot)\big)$ is also continuous and thus, $\mathcal{B}(G)$-measurable. This establishes Claim 1.

Claim 2. $\int_G \big\|H_{f,A}(x)\big\|_{\ell^{p'}(\Gamma)}\, d|\lambda|(x) \le \|\lambda\|_{M(G)}\, \|f\|_{\mathcal{F}^p(G)} < \infty$.

Indeed, since $|\overline{(x,\gamma)}| = 1$, for all $x \in G$ and $\gamma \in \Gamma$, and $\big|\frac{d\lambda}{d|\lambda|}(x)\big| = 1$ for $|\lambda|$-a.e. $x \in G$, it follows that

$$\begin{aligned}\big\|H_{f,A}(x)\big\|_{\ell^{p'}(\Gamma)} &= \big\|\,\big(f\tau_{-x}(\chi_A)\big)\widehat{\ }\big\|_{\ell^{p'}(\Gamma)} \le \sup_{B\in\mathcal{B}(G)} \|(f\chi_B)\widehat{\ }\|_{\ell^{p'}(\Gamma)}\\ &= \sup_{B\in\mathcal{B}(G)} \Big\|\int_B f\, dm_{F_p}\Big\|_{\ell^{p'}(\Gamma)} \le \|f\|_{L^1(m_{F_p})} = \|f\|_{\mathbf{F}^p(G)}.\end{aligned}$$

This inequality easily implies Claim 2.

Claims 1 and 2 show that $H_{f,A} : G \to \ell^{p'}(\Gamma)$ is Bochner $|\lambda|$-integrable. Accordingly, there exists a vector (B)-$\int_G H_{f,A}\, d|\lambda| \in \ell^{p'}(\Gamma)$ which, for each element $\xi \in \ell^p(\Gamma)$, satisfies

$$\Big\langle (\mathrm{B})\text{-}\!\int_G H_{f,A}\, d|\lambda|,\ \xi\Big\rangle = \int_G \big\langle H_{f,A}(x),\ \xi\big\rangle\, d|\lambda|(x).$$

The choice $\xi = \chi_{\{\gamma\}}$, for each $\gamma \in \Gamma$, yields via (7.45), that

$$\begin{aligned}\Big((\mathrm{B})\text{-}\!\int_G H_{f,A}\, d|\lambda|\Big)(\gamma) &= \Big\langle (\mathrm{B})\text{-}\!\int_G H_{f,A}\, d|\lambda|,\ \chi_{\{\gamma\}}\Big\rangle\\ &= \int_G \overline{(x,\gamma)}\,\big(f\tau_{-x}(\chi_A)\big)\widehat{\ }(\gamma)\frac{d\lambda}{d|\lambda|}(x)\, d|\lambda|(x) = \big(\chi_A(f*\lambda)\big)\widehat{\ }(\gamma),\end{aligned}$$

where the last equality follows from (7.44). Since $\gamma \in \Gamma$ is arbitrary and $c_0(\Gamma)$ is a space of $\mathbb{C}$-valued functions on Γ, we have (B)-$\int_G H_{f,A}\, d|\lambda| = \big(\chi_A(f*\lambda)\big)\widehat{\ }$ as elements of $c_0(\Gamma)$. Hence, the element $\big(\chi_A(f*\lambda)\big)\widehat{\ }$ of $c_0(\Gamma)$ actually belongs to $\ell^{p'}(\Gamma)$. But, $A \in \mathcal{B}(G)$ is arbitrary and so $(f*\lambda) \in \Phi^p(G) = \mathbf{F}^p(G)$; see Proposition 7.13. This establishes that $C_\lambda^{(1)}\big(\mathbf{F}^p(G)\big) \subseteq \mathbf{F}^p(G)$.

Finally, by (3.4) (with $\nu := m_{F_p}$) and the inequality of Claim 2, we have

$$\begin{aligned}\|f*\lambda\|_{\mathbf{F}^p(G)} &\le 4 \sup_{A\in\mathcal{B}(G)} \Big\|\int_A (f*\lambda)\, dm_{F_p}\Big\|_{\ell^{p'}(\Gamma)}\\ &= 4 \sup_{A\in\mathcal{B}(G)} \big\|\big(\chi_A(f*\lambda)\big)\widehat{\ }\big\|_{\ell^{p'}(\Gamma)} = 4 \sup_{A\in\mathcal{B}(G)} \Big\|(\mathrm{B})\text{-}\!\int_G H_{f,A}\, d|\lambda|\ \Big\|_{\ell^{p'}(\Gamma)}\\ &\le 4 \sup_{A\in\mathcal{B}(G)} \int_G \big\|H_{f,A}(x)\big\|_{\ell^{p'}(\Gamma)}\, d|\lambda|(x) \le 4\|\lambda\|_{M(G)}\, \|f\|_{\mathbf{F}^p(G)}.\end{aligned}$$

This establishes that the norm of the operator $f \mapsto f*\lambda$, as an element of $\mathcal{L}\big(\mathbf{F}^p(G)\big)$, is at most $4\|\lambda\|_{M(G)}$. □

Remark 7.27. Let $1 \le p \le 2$. Given $1 \le r \le 2$ and $h \in \mathbf{F}^r(G) \subseteq L^1(G)$ we can consider the measure $\lambda = \mu_h$ as given by (7.3). It follows from Proposition 7.26 that h convolves $\mathbf{F}^p(G)$ continuously into $\mathbf{F}^p(G)$ and that the norm of this operator, as an element of $\mathcal{L}\big(\mathbf{F}^p(G)\big)$, is at most

$$4\|\mu_h\|_{M(G)} \;=\; 4\|h\|_{L^1(G)} \;\le\; 4\|h\|_{\mathbf{F}^r(G)}. \qquad \square$$

Finally, we come to the most delicate question: *Is* $\mathbf{F}^p(G)$ *genuinely larger than* $L^p(G)$ *whenever* $1 < p < 2$? This point already occurs in Proposition 7.23. It turns out that the question has an affirmative answer for the classical groups $\mathbb{T}^d$, for $d \in \mathbb{N}$. The proof, based on Fourier restriction theory, uses results from the theory of Salem measures as developed in [110], [111], [143]. We formally record the result: for the details see [113].

Proposition 7.28. *Let* $1 < p < 2$ *and* $d \in \mathbb{N}$. *Then both the inclusions*

$$L^p(\mathbb{T}^d) \;\subseteq\; \mathbf{F}^p(\mathbb{T}^d) \;\subseteq\; L^1(\mathbb{T}^d) \tag{7.46}$$

are proper.

Remark 7.29. (i) The second inclusion in (7.46) is actually proper for *every* G, that is, always $\mathbf{F}^p(G) \subsetneqq L^1(G)$. This was already established in the proof of Proposition 7.23. It is the strictness of the first inclusion in (7.46) which is delicate but, of course, it is precisely this fact which shows that $\{\mathbf{F}^p(G)\}_{1<p<2}$ is a *new* class of B.f.s.' Moreover, the results of this section show that the properties of the spaces $\mathbf{F}^p(G)$ make them of some interest for classical harmonic analysis.

(ii) For each $1 < p < 2$, it turns out that

$$L^r(\mathbb{T}^d) \;\nsubseteq\; \mathbf{F}^p(\mathbb{T}^d), \qquad 1 \le r < p.$$

This follows from the fact that $f(t) = t^{-1/p}$, for $|t| < \pi$, belongs to $L^r(\mathbb{T}) \setminus L^p(\mathbb{T})$ for $1 \le r < p$ and satisfies $\widehat{f}(n) = O(|n|^{-1/p'})$ for $|n| \to \infty$. The construction used in the proof of Proposition 7.28, as given in [113], shows that

$$\mathbf{F}^p(\mathbb{T}^d) \;\nsubseteq\; L^r(\mathbb{T}^d), \qquad 1 < r \le p. \qquad \square$$

7.2 Convolution operators acting in $L^1(G)$

In this section we make a detailed investigation of various operator theoretic, measure theoretic and topological properties of the convolution operators

$$C_\lambda^{(1)} : L^1(G) \to L^1(G)$$

as given by (7.7) for $p = 1$. We distinguish two cases, namely whether $\lambda \ll \mu$ or not. The importance of the collection $\mathcal{C}_1(G) := \{C_\lambda^{(1)} : \lambda \in M(G)\}$ is that it coincides with the class of all $S \in \mathcal{L}\big(L^1(G)\big)$ satisfying $S \circ \tau_a = \tau_a \circ S$ for all $a \in G$, [140, p. 75]. Since $\mathcal{C}_1(G)$ is also an involutive semisimple commutative Banach algebra with unit, it has received a great deal of attention from harmonic analysts and Banach algebraists alike; see for example [73], [96], [140] and the references therein. A point of some importance concerns the identification of its closed subalgebra $\big\{C_\lambda^{(1)} : \lambda = \mu_h \text{ for some } h \in L^1(G)\big\}$. One of our aims in this section is to establish some characterizations of this closed subalgebra.

For $\lambda \in M(G)$, a result of Akemann, [1, Theorem 4], states that $\lambda \ll \mu$ if and only if $C_\lambda^{(1)} \in \mathcal{L}\big(L^1(G)\big)$ is a compact operator if and only if $C_\lambda^{(1)}$ is a weakly compact operator. By Costé's Theorem, [42, pp. 90–92], these three equivalent statements are in turn equivalent to $C_\lambda^{(1)}$ being Bochner representable. Further additional equivalences (of a different nature) are given by the following

Theorem 7.30. *For $\lambda \in M(G)$, each of the following conditions is equivalent to λ being absolutely continuous with respect to μ.*

(i) *The operator $C_\lambda^{(1)} : L^1(G) \to L^1(G)$ is Pettis representable.*

(ii) *The function $K_\lambda : G \to M(G)$ defined by*

$$K_\lambda : x \mapsto \delta_x * \lambda, \qquad x \in G, \tag{7.47}$$

is continuous for the norm topology in $M(G)$.

(iii) *The $M(G)$-valued function K_λ is weakly continuous.*

(iv) *The range of K_λ is norm compact in the Banach space $M(G)$.*

(v) *The range of K_λ is weakly compact in $M(G)$.*

(vi) *There exists $A \in \mathcal{B}(G)$ with $\mu(A) > 0$ such that its image $K_\lambda(A)$ is norm separable in $M(G)$.*

Concerning statement (i) of Theorem 7.30 and Costé's Theorem, it is relevant to note that $L^1(G)$ does *not* have the Radon–Nikodým property, [42, p. 219].

An important subalgebra of $M(G)$ (see [73], [96]) is

$$M_0(G) := \big\{\lambda \in M(G) : \widehat{\lambda} \in c_0(\Gamma)\big\}.$$

Since G is infinite, the inclusions

$$L^1(G) \subseteq M_0(G) \subseteq M(G)$$

are proper, [96, p. 422]. For an interesting history of $M_0(\mathbb{T})$ we refer to [103] and the references therein. It is known that *every* $\lambda = \mu_h$ with $h \in L^1(G)$ has *natural spectrum* (in the sense of Zafran [167]), meaning that the spectrum $\sigma(C_\lambda^{(1)})$ of $C_\lambda^{(1)} \in \mathcal{L}\big(L^1(G)\big)$ coincides with $\overline{\widehat{\lambda}(\Gamma)} = \overline{\widehat{h}(\Gamma)}$, where the bar indicates "closure in $\mathbb{C}$". The situation for elements of the larger space $M_0(G)$ is rather different: there always exists a measure $\lambda \in M_0(G) \setminus L^1(G)$ *without* natural spectrum, that is, the containment $\overline{\widehat{\lambda}(\Gamma)} \subseteq \sigma(C_\lambda^{(1)})$ is proper, [96, p. 422]. Let $N_0(G)$ denote the subset of $M_0(G)$ consisting of all elements of $M(G)$ which have natural spectrum. Characterizations of $N_0(G)$ are known. For instance, $\lambda \in N_0(G)$ if and only if $C_\lambda^{(1)}$ is a *decomposable operator* if and only if $C_\lambda^{(1)}$ is a *Riesz operator*, [96, Theorem 4.11.8]. In view of the above discussion we note that the containment $N_0(G) \subseteq M_0(G)$ is proper, [96, Theorem 4.11.9]. Moreover, $N_0(G)$ itself contains $L^1(G)$ as a *proper* subspace, [167, Proposition 2.9]. So, we have

$$L^1(G) \subseteq N_0(G) \subseteq M_0(G) \subseteq M(G)$$

with all inclusions strict. Characterizations of $M_0(G)$ are also known. Define the Fourier–Stieltjes transform map

$$F_S : M(G) \to \ell^\infty(\Gamma)$$

by $\lambda \mapsto \widehat{\lambda}$ with $\widehat{\lambda}$ as defined by (7.2). Then it is known that $\lambda \in M(G)$ belongs to $M_0(G)$ if and only if $F_S \circ K_\lambda : G \to \ell^\infty(\Gamma)$, with K_λ as in (7.47), is continuous for the sup-norm in $\ell^\infty(\Gamma)$, [68]. Our second main aim of this section is to establish the following result which presents some further (and diverse) characterizations of $M_0(G)$ in terms of the operator $C_\lambda^{(1)}$ and/or the function K_λ.

Theorem 7.31. *For $\lambda \in M(G)$, the following assertions are equivalent.*

(i) *The measure λ belongs to $M_0(G)$.*

(ii) *The vector measure $m_\lambda^{(1)} : \mathcal{B}(G) \to L^1(G)$ defined by*

$$m_\lambda^{(1)}(A) := C_\lambda^{(1)}(\chi_A) = \chi_A * \lambda, \qquad A \in \mathcal{B}(G), \tag{7.48}$$

has relatively compact range in the Banach space $L^1(G)$.

(iii) *The convolution operator $C_\lambda^{(1)} : L^1(G) \to L^1(G)$ is completely continuous.*

(iv) *The map $F_{1,0} \circ C_\lambda^{(1)} : L^1(G) \to c_0(\Gamma)$ is completely continuous.*

(v) *The map $F_{1,0} \circ C_\lambda^{(1)} : L^1(G) \to c_0(\Gamma)$ is compact.*

(vi) *The map $F_{1,0} \circ C_\lambda^{(1)} : L^1(G) \to c_0(\Gamma)$ is weakly compact.*

(vii) *The map $F_{1,0} \circ C_\lambda^{(1)} : L^1(G) \to c_0(\Gamma)$ is Bochner representable.*

(viii) *The map $F_{1,0} \circ C_\lambda^{(1)} : L^1(G) \to c_0(\Gamma)$ is Pettis representable.*

(ix) *The function $F_S \circ K_\lambda$ is weakly continuous in $\ell^\infty(\Gamma)$.*

(x) *The range of $F_S \circ K_\lambda$ is norm compact in $\ell^\infty(\Gamma)$.*

(xi) *The range of $F_S \circ K_\lambda$ is weakly compact in $\ell^\infty(\Gamma)$.*

(xii) *The function $F_S \circ K_\lambda$ is Bochner μ-integrable in $\ell^\infty(\Gamma)$.*

(xiii) *The function $F_S \circ K_\lambda$ is Pettis μ-integrable in $\ell^\infty(\Gamma)$.*

(xiv) *The $c_0(\Gamma)$-valued vector measure $F_{1,0} \circ m_\lambda^{(1)}$ has relatively compact range.*

(xv) *There exists* $H \in \mathbb{P}\big(\mu, c_0(\Gamma)\big)$ *such that*

$$\left(F_{1,0} \circ m_\lambda^{(1)}\right)(A) = (\mathrm{P})\text{-}\int_A H\, d\mu, \qquad A \in \mathcal{B}(G).$$

(xvi) *There exists* $H \in \mathbb{B}\big(\mu, c_0(\Gamma)\big)$ *such that*

$$\left(F_{1,0} \circ m_\lambda^{(1)}\right)(A) = (\mathrm{B})\text{-}\int_A H\, d\mu, \qquad A \in \mathcal{B}(G).$$

Remark 7.32. (i) In relation to (vii), (viii), (xii) and (xiii) of Theorem 7.31, we point out that *neither* $c_0(\Gamma)$ *nor* $\ell^\infty(\Gamma)$ has the Radon–Nikodým property, [42, p. 219]

(ii) A result of Pigno and Saeki, [128, Corollary 3], provides a connection between Theorems 7.30 and 7.31. It states that $\lambda \in M(G)$ belongs to $L^1(G)$ if and only if $\lambda * M_0(G) \subseteq L^1(G)$. Equivalently, $C_\lambda^{(1)} \circ C_\eta^{(1)}$ is compact for every completely continuous operator $C_\eta^{(1)} \in \mathcal{C}_1(G)$ if and only if $C_\lambda^{(1)}$ is itself a compact operator.

(iii) Let

$$J : L^1(G) \to M(G)$$

be the linear map which assigns to each $h \in L^1(G)$ the finite measure μ_h as given by (7.3). Since $\|\mu_h\|_{M(G)} = \|h\|_{L^1(G)}$, the map J is an isometry. So, we may regard $L^1(G)$ as a closed subspace of $M(G)$. Let

$$\Lambda : c_0(\Gamma) \to \ell^\infty(\Gamma)$$

denote the natural inclusion. Then $F_S : M(G) \to \ell^\infty(\Gamma)$ and $F_{1,0} : L^1(G) \to c_0(\Gamma)$ satisfy

$$F_S \circ J \;=\; \Lambda \circ F_{1,0}, \tag{7.49}$$

that is, the following diagram commutes:

$$\begin{tikzcd} L^1(G) \arrow[r, "F_{1,0}"] \arrow[d, "J"'] & c_0(\Gamma) \arrow[d, "\Lambda"] \\ M(G) \arrow[r, "F_S"] & \ell^\infty(\Gamma). \end{tikzcd}$$

Both $F_{1,0}$ and F_S are continuous, because of (7.5) and the inequality

$$\|\widehat{\lambda}\|_{\ell^\infty(\Gamma)} \;\le\; \|\lambda\|_{M(G)}, \qquad \lambda \in M(G),$$

[140, p. 15], respectively. In relation to various statements in Theorem 7.31 we note that both $F_{1,0}$ and F_S are themselves *not* completely continuous and *not* weakly compact. This follows from Lemma 7.1, Remark 7.2 and (7.49). □

The proofs of both Theorem 7.30 and Theorem 7.31 will require various preliminary results. We begin with an analysis of the vector measures $m_\lambda^{(1)}$, for $\lambda \in M(G)$, as given by (7.48). It follows from (7.48) and (7.8), for $p = 1$, that

$$\big\|m_\lambda^{(1)}(A)\big\|_{L^1(G)} \le \|\lambda\|_{M(G)} \cdot \|\chi_A\|_{L^1(G)} = \|\lambda\|_{M(G)}\, \mu(A), \tag{7.50}$$

for each $A \in \mathcal{B}(G)$. Accordingly, $m_\lambda^{(1)}$ is surely σ-additive and has finite variation. Actually, the variation measure can be precisely described.

Lemma 7.33. *For each $\lambda \in M(G)$, the variation measure $\big|m_\lambda^{(1)}\big|$ is a multiple of Haar measure. Namely,*

$$\big|m_\lambda^{(1)}\big|(A) = \big|m_\lambda^{(1)}\big|(G) \cdot \mu(A), \qquad A \in \mathcal{B}(G).$$

In particular, $\big|m_\lambda^{(1)}\big|$ is translation invariant. Moreover, $L^1\big(\big|m_\lambda^{(1)}\big|\big) = L^1(G)$ whenever $\lambda \neq 0$.

Proof. It follows from (7.50) and the definition of variation measure that

$$\big|m_\lambda^{(1)}\big|(A) \le \|\lambda\|_{M(G)} \cdot \mu(A), \qquad A \in \mathcal{B}(G).$$

Fix $x \in G$. For each $A \in \mathcal{B}(G)$ we have

$$\begin{aligned} \big\|m_\lambda^{(1)}(A+x)\big\|_{L^1(G)} &= \big\|(\delta_x * \chi_A) * \lambda\big\|_{L^1(G)} = \big\|\delta_x * (\chi_A * \lambda)\big\|_{L^1(G)} \\ &= \|\chi_A * \lambda\|_{L^1(G)} = \big\|m_\lambda^{(1)}(A)\big\|_{L^1(G)}, \end{aligned} \tag{7.51}$$

where we have used the identities $\delta_x * \chi_A = \chi_{A+x}$ and

$$f * \delta_x = \tau_x f, \qquad f \in L^1(G).$$

It follows from (7.51) and the definition of the variation measure $|m_\lambda^{(1)}|$ that $\big|m_\lambda^{(1)}\big|(A+x) = \big|m_\lambda^{(1)}\big|(A)$ for $A \in \mathcal{B}(G)$. Since $x \in G$ is arbitrary, we can conclude that there exists a constant $c \ge 0$ such that $\big|m_\lambda^{(1)}\big| = c\mu$, [140, p. 2]. In particular, $c = c\mu(G) = \big|m_\lambda^{(1)}\big|(G)$. □

Since the $m_\lambda^{(1)}$-null and $\big|m_\lambda^{(1)}\big|$-null sets coincide, it follows from Lemma 7.33 that the convolution operator $C_\lambda^{(1)} \in \mathcal{L}\big(L^1(G)\big)$ is *μ-determined* for every $\lambda \in M(G) \setminus \{0\}$. To identify its optimal domain $L^1\big(m_\lambda^{(1)}\big)$ we require a further preliminary result. Recall, for $\lambda \in M(G)$ given, that its *reflection* $R\lambda \in M(G)$ is defined by

$$R\lambda : A \mapsto \lambda(-A), \qquad A \in \mathcal{B}(G),$$

and satisfies both (see [140, p. 16])

$$(\overline{R\lambda})\widehat{\ } = \overline{\widehat{\lambda}} \quad \text{and} \quad (R\lambda)\widehat{\ } = \widehat{\lambda} \circ R_\Gamma, \qquad \lambda \in M(G). \tag{7.52}$$

Here $\overline{\eta}(A) := \overline{\eta(A)}$, for $A \in \mathcal{B}(G)$, whenever $\eta \in M(G)$.

Lemma 7.34. *Let $\lambda \in M(G)$. Then*

$$\big\langle C_\lambda^{(1)}(f),\ \varphi\big\rangle = \big\langle f,\ \varphi * R\lambda\big\rangle, \qquad f \in L^1(G), \quad \varphi \in L^\infty(G). \tag{7.53}$$

Proof. The convolution $\varphi * R\lambda$ belongs to $L^\infty(G)$, [75, Theorem 20.13]. According to Fubini's Theorem and translation invariance of μ we have

$$\begin{aligned}
&\langle f, \varphi * R\lambda\rangle = \int_G f(t)\Big(\int_G \varphi(t-z)\, dR\lambda(z)\Big)\, d\mu(t)\\
&= \int_G f(t)\Big(\int_G \varphi(t+s)\, d\lambda(s)\Big)\, d\mu(t) = \int_G \Big(\int_G f(t)\varphi(t+s)\, d\mu(t)\Big)\, d\lambda(s)\\
&= \int_G \Big(\int_G f(y-s)\varphi(y)\, d\mu(y)\Big)\, d\lambda(s) = \int_G \varphi(y)\Big(\int_G f(y-s)\, d\lambda(s)\Big)\, d\mu(y)\\
&= \int_G \varphi(y)\cdot(f*\lambda)(y)\, d\mu(y) = \big\langle C_\lambda^{(1)}(f),\ \varphi\big\rangle. \qquad \square
\end{aligned}$$

Proposition 7.35. *Let $\lambda \in M(G)\setminus\{0\}$. Then $L^1\big(m_\lambda^{(1)}\big) = L^1(G)$ with their given norms being equivalent. In particular,*

$$L^1\big(|m_\lambda^{(1)}|\big) = L^1\big(m_\lambda^{(1)}\big) = L^1(G), \tag{7.54}$$

and hence,

$$L^r\big(|m_\lambda^{(1)}|\big) = L^r\big(m_\lambda^{(1)}\big) = L^r(G), \qquad 1 \le r \le \infty.$$

Proof. It follows from (7.53) with $f = \chi_A$ that

$$\big\langle m_\lambda^{(1)}(A),\ \varphi\big\rangle = \int_A \varphi * R\lambda\, d\mu, \qquad A \in \mathcal{B}(G),$$

for each $\varphi \in L^\infty(G) = L^1(G)^*$, and hence, that

$$\big|\langle m_\lambda^{(1)}, \varphi\rangle\big|(A) = \int_A |\varphi * R\lambda|\, d\mu, \qquad A \in \mathcal{B}(G). \tag{7.55}$$

For each $\gamma \in \Gamma$, the function $\varphi := (\cdot,\gamma) \in L^\infty(G)$ satisfies $\|\varphi\|_{L^\infty(G)} = 1$. Fix $f \in L^1\big(m_\lambda^{(1)}\big)$. Then it follows from (7.10), (7.52) and (7.55) that

$$\begin{aligned}
|\widehat{\lambda}(\gamma)|\int_G |f|\, d\mu &= \big|\overline{\widehat{\lambda}(\gamma)}\big|\int_G |f|\, d\mu = \big|(\overline{R\lambda})\widehat{\ }(\gamma)\big|\int_G |f|\, d\mu\\
&= \int_G |f|\cdot\big|(\cdot,\gamma) * \overline{R\lambda}\big|\, d\mu = \int_G |f|\cdot\big|\overline{(\cdot,\gamma)} * R\lambda\big|\, d\mu\\
&= \int_G |f|\, d|\langle m_\lambda^{(1)}, \overline{(\cdot,\gamma)}\rangle| \le \|f\|_{L^1(m_\lambda^{(1)})},
\end{aligned}$$

where we have used $\overline{h * \eta} = \overline{h} * \overline{\eta}$ whenever $h \in L^1(G)$ and $\eta \in M(G)$. Since $\gamma \in \Gamma$ is arbitrary, we can conclude that

$$\|\widehat{\lambda}\|_{\ell^\infty(\Gamma)} \, \|f\|_{L^1(G)} \le \|f\|_{L^1(m_\lambda^{(1)})}. \tag{7.56}$$

But, $\lambda \neq 0$ implies that $\|\widehat{\lambda}\|_{\ell^\infty(\Gamma)} > 0$, [140, p. 29], and hence, $L^1\big(m_\lambda^{(1)}\big) \subseteq L^1(G)$ continuously. Moreover. Lemma 7.33 gives $L^1(G) = L^1\big(|m_\lambda^{(1)}|\big)$ and we always have $L^1\big(|m_\lambda^{(1)}|\big) \subseteq L^1\big(m_\lambda^{(1)}\big)$ continuously. From these comments, (7.54) is immediate. $\square$

Remark 7.36. (i) Since $L^1(G)$ is weakly sequentially complete, [46, Ch. IV, Theorem 8.6], it cannot contain an isomorphic copy of c_0. So, we have that

$$L^1_{\mathrm{w}}\big(m_\lambda^{(1)}\big) = L^1\big(m_\lambda^{(1)}\big) = L^1\big(|m_\lambda^{(1)}|\big) = L^1(G), \qquad \lambda \in M(G) \setminus \{0\}.$$

(ii) Proposition 7.35 shows that the optimal domain of $C_\lambda^{(1)}$ is $L^1(G)$ itself, that is, $C_\lambda^{(1)}$ is already defined on its optimal domain with $I_{m_\lambda^{(1)}} = C_\lambda^{(1)}$. Accordingly, Theorems 7.30 and 7.31 are also statements about the integration operator $I_{m_\lambda^{(1)}}$ corresponding to the vector measure $m_\lambda^{(1)}$. In particular, the discussion immediately prior to Theorem 7.30 shows that $I_{m_\lambda^{(1)}}$ is compact if and only if it is weakly compact if and only if $\lambda \ll \mu$. Furthermore, Corollary 2.43 (applied to $T := C_\lambda^{(1)}$) and the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 7.31 show that $I_{m_\lambda^{(1)}}$ is completely continuous if and only if $\lambda \in M_0(G)$. Theorems 7.30 and 7.31 also provide examples of bounded operators in Banach spaces which are completely continuous but fail to be compact or weakly compact. Indeed, all convolution operators $C_\lambda^{(1)} \in \mathcal{L}\big(L^1(G)\big)$ with $\lambda \in M_0(G) \setminus L^1(G)$ are of this kind. For the existence of such measures λ we refer to the discussion immediately prior to Theorem 7.31. $\square$

It is time to prove Theorems 7.30 and 7.31. As usual, some preliminary results are needed.

Let $\lambda \in M(G)$. Fubini's Theorem yields $(f * \lambda)\widehat{\ } = \widehat{f} \cdot \widehat{\lambda}$ (an equality of functions on Γ) for every $f \in L^1(G)$. This is a special case of the general result that the convolution $\lambda * \eta = \eta * \lambda$ of any measure $\eta \in M(G)$ with λ, as defined in [140, Section 1.3.1], is again an element of $M(G)$ and satisfies $(\lambda * \eta)\widehat{\ } = \widehat{\lambda} \cdot \widehat{\eta}$, [140, p. 15]. For the special case of $\eta = \delta_x$ with $x \in G$ we have

$$(\delta_x * \lambda)(A) = \int_G \delta_x(A - y) \, d\lambda(y) = \lambda(A - x), \qquad A \in \mathcal{B}(G), \tag{7.57}$$

[140, p. 17], where we have used the identity $\delta_x(A - y) = \chi_{A-x}(y)$ for $y \in G$. That is, $\delta_x * \lambda$ is the x-translate of λ. Now, let $\gamma \in \Gamma$. In view of (7.47),

$$[K_\lambda(x)]\widehat{\ }(\gamma) = (\delta_x * \lambda)\widehat{\ }(\gamma) = \widehat{\delta}_x(\gamma)\widehat{\lambda}(\gamma) = \overline{(x, \gamma)} \, \widehat{\lambda}(\gamma), \tag{7.58}$$

for every $x \in G$. So, for $f \in L^1(G)$, it follows that

$$\int_G f(x)\big[K_\lambda(x)\big]\widehat{\ }(\gamma)\,d\mu(x) = \widehat{f}(\gamma)\widehat{\lambda}(\gamma) = (f * \lambda)\widehat{\ }(\gamma). \tag{7.59}$$

Applying (7.59) with $f := \chi_A$ and recalling the definition of $C_\lambda^{(1)}$ yield

$$\int_A \big[K_\lambda(x)\big]\widehat{\ }(\gamma)\,d\mu(x) = \big[C_\lambda^{(1)}(\chi_A)\big]\widehat{\ }(\gamma), \qquad A \in \mathcal{B}(G). \tag{7.60}$$

Let Z be a Banach space. A function $F : G \to Z^*$ will be called *Gelfand μ-integrable* if the scalar function $\langle z, F(\cdot)\rangle : x \mapsto \langle z, F(x)\rangle$, for $x \in G$, is μ-integrable for every $z \in Z$. In this case, given $A \in \mathcal{B}(G)$, there is a unique vector (w*)-$\int_A F\,d\mu \in Z^*$ satisfying

$$\Big\langle z,\ (\mathrm{w}^*)\text{-}\int_A F\,d\mu\Big\rangle = \int_A \langle z,\ F(\cdot)\rangle\,d\mu, \qquad z \in Z,$$

[42, p. 53]. The vector (w*)-$\int_A F\,d\mu \in Z^*$ is called the Gelfand integral of F over A (with respect to μ). If a function $F : G \to Z^*$ happens to be continuous with respect to the weak* topology $\sigma(Z^*, Z)$ on Z^*, then F is necessarily Gelfand μ-integrable because, for each $z \in Z$, the scalar function $\langle z, F(\cdot)\rangle$ is continuous on the compact space G and hence, is bounded.

Lemma 7.37. *Let Z and Y be Banach spaces and $T : Z^* \to Y^*$ be a linear map which is continuous for the weak* topologies on Z^* and Y^*. If $F : G \to Z^*$ is Gelfand μ-integrable, then so is the composition $T \circ F : G \to Y^*$ and*

$$(\mathrm{w}^*)\text{-}\int_A T\circ F\,d\mu = T\left((\mathrm{w}^*)\text{-}\int_A F\,d\mu\right), \qquad A \in \mathcal{B}(G).$$

Proof. According to duality theory, there exists a linear operator $S : Y \to Z$ (continuous for $\sigma(Y, Y^*)$ on Y and $\sigma(Z, Z^*)$ on Z) which satisfies $S^* = T$, [88, §20.4]. Fix $y \in Y$, in which case $S(y) \in Z$. Then

$$\langle y,\ (T\circ F)(x)\rangle = \langle S(y),\ F(x)\rangle, \qquad x \in G.$$

Accordingly, the Gelfand μ-integrability of F ensures that $\langle y,\ (T\circ F)(\cdot)\rangle \in L^1(G)$. Moreover, for $A \in \mathcal{B}(G)$, the vector $T\big((\mathrm{w}^*)\text{-}\int_A F\,d\mu\big) \in Y^*$ satisfies

$$\begin{aligned}\Big\langle y,\ T\Big((\mathrm{w}^*)\text{-}\int_A F\,d\mu\Big)\Big\rangle &= \Big\langle S(y),\ (\mathrm{w}^*)\text{-}\int_A F\,d\mu\Big\rangle \\ &= \int_A \langle S(y),\ F(\cdot)\rangle\,d\mu = \int_A \langle y,\ (T\circ F)(\cdot)\rangle\,d\mu,\end{aligned}$$

for all $y \in Y$. That is, (w*)-$\int_A T\circ F\,d\mu = T\big((\mathrm{w}^*)\text{-}\int_A F\,d\mu\big)$. ∎

We point out, for a Banach space Z, that every Pettis μ-integrable function $H : G \to Z^*$ is Gelfand μ-integrable and (P)-$\int_A H\, d\mu =$ (w*)-$\int_A H\, d\mu$ for $A \in \mathcal{B}(G)$. Hence, if $F \in \mathbb{B}(\mu, Z^*)$, then

$$\text{(B)-}\int_A F\, d\mu = \text{(P)-}\int_A F\, d\mu = \text{(w*)-}\int_A F\, d\mu, \qquad A \in \mathcal{B}(G).$$

Relevant to this section are Gelfand integrals in the particular dual spaces $M(G) = C(G)^*$ and $\ell^\infty(\Gamma) = \ell^1(\Gamma)^*$. The duality between $C(G)$ and $M(G)$ is given by

$$\langle \varphi, \lambda \rangle := \int_G \varphi\, d\lambda, \qquad \varphi \in C(G), \quad \lambda \in M(G).$$

Fix $\gamma \in \Gamma$. Let $\pi_\gamma : \ell^\infty(\Gamma) \to \mathbb{C}$ denote the coordinate functional at γ, that is, $\pi_\gamma(\xi) := \xi(\gamma)$ for $\xi \in \ell^\infty(\Gamma)$. In terms of $\chi_{\{\gamma\}} \in \ell^1(\Gamma) \subseteq \ell^\infty(\Gamma)^*$ we have $\pi_\gamma(\xi) = \langle \chi_{\{\gamma\}}, \xi \rangle$ for $\xi \in \ell^\infty(\Gamma)$. Similarly, $\pi_\gamma \circ F_S : M(G) \to \mathbb{C}$ corresponds to $\overline{(\cdot, \gamma)} \in C(G) \subseteq M(G)^*$ via

$$\big\langle \overline{(\cdot, \gamma)},\, \lambda \big\rangle = (\pi_\gamma \circ F_S)(\lambda) = \big\langle \chi_{\{\gamma\}},\, F_S(\lambda) \big\rangle, \qquad \lambda \in M(G). \tag{7.61}$$

The following result collects together various facts about Gelfand integrals which are needed in the sequel.

Proposition 7.38. *Let $\lambda \in M(G)$. The following assertions hold.*

(i) *The function $K_\lambda : G \to M(G)$ defined by (7.47) is continuous when $M(G)$ is equipped with its weak* topology. In particular, K_λ is Gelfand μ-integrable in $M(G) = C(G)^*$.*

(ii) *The Fourier–Stieltjes transform map $F_S : M(G) \to \ell^\infty(\Gamma)$ is continuous when $M(G) = C(G)^*$ and $\ell^\infty(\Gamma) = \ell^1(\Gamma)^*$ have their weak* topologies. Hence, $F_S \circ K_\lambda : G \to \ell^\infty(\Gamma)$ is weak* continuous and, consequently, $F_S \circ K_\lambda$ is Gelfand μ-integrable in $\ell^\infty(\Gamma) = \ell^1(\Gamma)^*$.*

(iii) *Suppose that $\lambda \neq 0$. For a Borel measurable function $f : G \to \mathbb{C}$ the following conditions are equivalent.*

  (a) *$f \in L^1(G)$.*

  (b) *The pointwise product $fK_\lambda : G \to M(G)$ is Gelfand μ-integrable in $M(G) = C(G)^*$.*

  (c) *The pointwise product $f \cdot (F_S \circ K_\lambda) : G \to \ell^\infty(\Gamma)$ is Gelfand μ-integrable in $\ell^\infty(\Gamma) = \ell^1(\Gamma)^*$.*

  In this case we have the identities

$$\left.\begin{aligned} &\big(J \circ C_\lambda^{(1)}\big)(f) = (\text{w}^*)\text{-}{\textstyle\int_G} fK_\lambda\, d\mu \quad \textit{and} \\ &\big(\Lambda \circ F_{1,0} \circ C_\lambda^{(1)}\big)(f) = (\text{w}^*)\text{-}{\textstyle\int_G} f \cdot (F_S \circ K_\lambda)\, d\mu \end{aligned}\right\} \tag{7.62}$$

and hence, for each $A \in \mathcal{B}(G)$, also

$$\left.\begin{array}{l} \big(J \circ m_\lambda^{(1)}\big)(A) = (\mathrm{w}^*)\text{-}\!\int_A K_\lambda \, d\mu \quad \textit{and} \\ \big(\Lambda \circ F_{1,0} \circ m_\lambda^{(1)}\big)(A) = (\mathrm{w}^*)\text{-}\!\int_A F_S \circ K_\lambda \, d\mu. \end{array}\right\} \tag{7.63}$$

Proof. (i) Observe that $K_\lambda : G \to M(G)$ is weak* continuous if and only if, for each $\varphi \in C(G)$, the scalar function $\big\langle \varphi,\, K_\lambda(\cdot)\big\rangle$ is continuous on G. The case $\lambda = 0$ is trivial. So, assume that $\lambda \neq 0$. Fix $\varphi \in C(G)$. Then, (7.57) implies that

$$\big\langle \varphi,\, K_\lambda(x)\big\rangle = \int_G \varphi(t)\, d(\delta_x * \lambda)(t) = \int_G \varphi(x+s)\, d\lambda(s), \qquad x \in G.$$

Let $\varepsilon > 0$. Since φ is uniformly continuous on G, choose a neighbourhood V of 0 such that $\big|\varphi(u) - \varphi(w)\big| < \varepsilon/|\lambda|(G)$ whenever $(u-w) \in V$. Hence, given $a \in G$, it follows that

$$\big|\langle \varphi,\, K_\lambda(x)\rangle - \langle \varphi,\, K_\lambda(a)\rangle\big| \le \int_G \big|\varphi(x+s) - \varphi(a+s)\big|\, d|\lambda|(s) < \varepsilon$$

for $x \in a + V$. So, $\big\langle \varphi,\, K_\lambda(\cdot)\big\rangle$ is continuous, that is, K_λ is weak* continuous.

(ii) Let $\rho \in \ell^1(\Gamma)$, in which case the support $\rho^{-1}(\mathbb{C}\setminus\{0\})$ of ρ is a countable subset of Γ. So, $H(\rho) : G \to \mathbb{C}$ defined by $x \mapsto \sum_{\gamma\in\Gamma} \overline{(x,\gamma)}\rho(\gamma)$ for $x \in G$ is continuous. It follows from (7.61) and the Dominated Convergence Theorem that $\big\langle H(\rho),\, \eta\big\rangle = \big\langle \rho,\, F_S(\eta)\big\rangle$ for $\eta \in M(G)$, which implies that $F_S : M(G) \to \ell^\infty(\Gamma)$ is continuous for the respective weak* topologies on $M(G)$ and $\ell^\infty(\Gamma)$. By this and part (i) the composition $F_S \circ K_\lambda : G \to \ell^\infty(\Gamma)$ is continuous for the weak* topology in $\ell^\infty(\Gamma)$. In particular, $F_S \circ K_\lambda$ is Gelfand μ-integrable in $\ell^\infty(\Gamma) = \ell^1(\Gamma)^*$.

(iii) Assume (a). For every $\varphi \in C(G)$, the scalar function $\big\langle \varphi, K_\lambda(\cdot)\big\rangle$ is bounded and continuous on G (see the proof of part (i)) and hence, we have that $\big\langle \varphi,\, (fK_\lambda)(\cdot)\big\rangle = f(\cdot)\big\langle \varphi,\, K_\lambda(\cdot)\big\rangle \in L^1(G)$. So, (b) holds.

(b) $\Rightarrow$ (c). This follows from part (ii) and Lemma 7.37 applied to $T := F_S$ because $F_S\big(f(x)K_\lambda(x)\big) = f(x) \cdot (F_S \circ K_\lambda)(x)$ for $x \in G$. Lemma 7.37 also gives

$$F_S\Big((\mathrm{w}^*)\text{-}\!\int_G fK_\lambda \, d\mu\Big) = (\mathrm{w}^*)\text{-}\!\int_G f \cdot (F_S \circ K_\lambda)\, d\mu. \tag{7.64}$$

(c) $\Rightarrow$ (a). Choose $\gamma \in \Gamma$ such that $\widehat{\lambda}(\gamma) \neq 0$. Since $f \cdot (F_S \circ K_\lambda)$ is Gelfand μ-integrable in $\ell^\infty(\Gamma) = \ell^1(\Gamma)^*$ and $\chi_{\{\gamma\}} \in \ell^1(\Gamma)$, we have

$$\Big\langle \chi_{\{\gamma\}},\; f(\cdot) \cdot (F_S \circ K_\lambda)(\cdot)\Big\rangle \in L^1(G). \tag{7.65}$$

On the other hand, for every $x \in G$, it follows from (7.58) that

$$\Big\langle \chi_{\{\gamma\}},\; f(x) \cdot (F_S \circ K_\lambda)(x)\Big\rangle = f(x) \cdot \big[K_\lambda(x)\big]^\wedge(\gamma) = \overline{(x,\gamma)} f(x)\widehat{\lambda}(\gamma). \tag{7.66}$$

Since $\left|\overline{(x,\gamma)}\,\widehat{\lambda}(\gamma)\right| = |\widehat{\lambda}(\gamma)| \neq 0$ for every $x \in G$, we conclude from (7.65) and (7.66) that $f \in L^1(G)$. Thus (a), (b) and (c) are equivalent.

It remains to verify (7.62) and (7.63). Since $\left[C_\lambda^{(1)} f\right]^\frown = (f * \lambda)^\frown = \widehat{f} \cdot \widehat{\lambda}$, it follows from (7.59) and (7.64) that

$$\Big[(\mathrm{w}^*)\text{-}\int_G fK_\lambda\, d\mu\Big]^\frown(\gamma) = \int_G f(x)\,\big[K_\lambda(x)\big]^\frown(\gamma)\, d\mu(x) = \big[\big(J \circ C_\lambda^{(1)}\big)(f)\big]^\frown(\gamma)$$

for every $\gamma \in \Gamma$.

Then the injectivity of F_S, [140, p. 29], establishes that $(\mathrm{w}^*)\text{-}\int_G fK_\lambda\, d\mu = \big(J \circ C_\lambda^{(1)}\big)(f)$ as elements of $M(G)$, which is the first identity in (7.62). This identity and (7.64) then give the second identity in (7.62) because $F_S \circ J \circ C_\lambda^{(1)} = \Lambda \circ F_{1,0} \circ C_\lambda^{(1)}$ (see (7.49)).

Finally, (7.63) follows from (7.62) with $f := \chi_A$ for $A \in \mathcal{B}(G)$. □

Our final preliminary result is the following one.

Lemma 7.39. *For any $\lambda \in M(G)$, the following assertions hold.*

(i) *If a character $\gamma \in \Gamma$ satisfies $\int_A \big[K_\lambda(x)\big]^\frown(\gamma)\, d\mu(x) = 0$ for every $A \in \mathcal{B}(G)$, then $\widehat{\lambda}(\gamma) = 0$.*

(ii) *Let $Q : G \to \ell^\infty(\Gamma)$ be a Pettis μ-integrable function satisfying*

$$\int_A \big[K_\lambda(x)\big]^\frown(\gamma)\, d\mu(x) = \int_A [Q(x)](\gamma)\, d\mu(x), \qquad A \in \mathcal{B}(G), \tag{7.67}$$

for every $\gamma \in \Gamma$. Then the following statements hold.

(a) *Given any countable subset $\Gamma_0 \subseteq \Gamma$, there exists a set $A_0 \in \mathcal{B}(G)$ with $\mu(A_0) = 1$ such that $\big[K_\lambda(x)\big]^\frown(\gamma) = [Q(x)](\gamma)$ for all $x \in A_0$ and $\gamma \in \Gamma_0$.*

(b) *If there is a countable subset $\Gamma_0 \subseteq \Gamma$ such that $[Q(x)](\gamma) = 0$ for all $x \in G$ and $\gamma \in \Gamma \setminus \Gamma_0$, then $(F_S \circ K_\lambda)(x) = Q(x)$ for μ-a.e. $x \in G$.*

Proof. (i) The assumption implies that $\big[K_\lambda(x)\big]^\frown(\gamma) = 0$ for μ-a.e. $x \in G$. Then (7.58) implies that $\widehat{\lambda}(\gamma) = 0$ because $\overline{(x,\gamma)} \neq 0$ for every $x \in G$.

(ii) To prove (a) we may assume that Γ_0 is infinite. Let $\{\gamma_n : n \in \mathbb{N}\}$ be an enumeration of Γ_0. For each $n \in \mathbb{N}$, applying (7.67) with $\gamma := \gamma_n$ we can find a set $A_n \in \mathcal{B}(G)$ such that $\mu(A_n) = 1$ and $\big[K_\lambda(x)\big]^\frown(\gamma_n) = [Q(x)](\gamma_n)$ for all $x \in A_n$. Then the set $A_0 := \bigcap_{n=1}^\infty A_n$ satisfies the requirement of (a).

To prove (b), choose any $\gamma \in \Gamma \setminus \Gamma_0$. For each $A \in \mathcal{B}(G)$, the identity (7.67) gives $\int_A \big[K_\lambda(x)\big]^\frown(\gamma)\, d\mu(x) = 0$ because $[Q(x)](\gamma) = 0$ for all $x \in G$. Apply part (i) to deduce that $\widehat{\lambda}(\gamma) = 0$ which then implies, via (7.58), that

$$\big[K_\lambda(x)\big]^\frown(\gamma) = \overline{(x,\gamma)}\,\widehat{\lambda}(\gamma) = 0 = [Q(x)](\gamma)$$

for all $x \in G$ and $\gamma \in \Gamma \setminus \Gamma_0$. Now choose any set $A_0 \in \mathcal{B}(G)$ satisfying the conclusion of part (a). The properties of A_0 then imply, for every $x \in A_0$, that

$$\big[(F_S \circ K_\lambda)(x)\big](\gamma) = \big[K_\lambda(x)\big]\hat{}\,(\gamma) = [Q(x)](\gamma), \qquad \gamma \in \Gamma,$$

so that $(F_S \circ K_\lambda)(x) = Q(x)$. Since $\mu(A_0) = 1$, part (b) is proved. $\square$

Before starting the proof of Theorem 7.30, observe that the Banach space $L^1(G)$ is generated by $\{\chi_A : A \in \mathcal{B}(G)\}$, that is, the linear span of this set is dense in $L^1(G)$. Since $\{\chi_A : A \in \mathcal{B}(G)\}$ is the range of the $L^1(G)$-valued vector measure $A \mapsto \chi_A$, for $A \in \mathcal{B}(G)$, and such a range is always relatively weakly compact (see Lemma 3.3), we see that $L^1(G)$ is *weakly compactly generated.*

Proof of Theorem 7.30. By the classical equivalences mentioned immediately prior to Theorem 7.30, it suffices to prove that (i)–(vi) are equivalent to $\lambda \ll \mu$.

Now, by Costé's Theorem and the fact that Bochner representability implies Pettis representability, we see that $\lambda \ll \mu$ implies (i). Conversely, assume that (i) holds. Let $f \in L^1(G)$. Then the $L^1(G)$-valued vector measure given by the formula $A \mapsto C_\lambda^{(1)}(f\chi_A) = \lambda * (f\chi_A)$ has finite variation, because

$$\big\|\lambda * (f\chi_A)\big\|_{L^1(G)} \le \|\lambda\|_{M(G)}\|f\chi_A\|_{L^1(G)} = \|\lambda\|_{M(G)} \int_A |f|\, d\mu, \quad A \in \mathcal{B}(G),$$

(see (7.8) with $p = 1$) and $A \mapsto \int_A |f|\, d\mu$ is a finite measure. Since $L^1(G)$ is weakly compactly generated and $C_\lambda^{(1)}$ is Pettis representable (by hypothesis), it follows from Proposition 3.46 with $E = L^1(G)$ that $T := C_\lambda^{(1)}$ is Bochner representable. By Costé's Theorem $\lambda \ll \mu$.

To see that $\lambda \ll \mu$ implies (ii), let $h \in L^1(G)$ satisfy $\lambda = \mu_h = J(h)$. The function $F_h : G \to L^1(G)$ defined by

$$F_h(x) := \mu_h * \delta_x = h * \delta_x = \tau_x h, \quad x \in G, \tag{7.68}$$

is continuous, [140, Theorem 1.1.5], and satisfies $K_\lambda = J \circ F_h$. Since $J : L^1(G) \to M(G)$ is continuous (even an isometry), (ii) follows.

(ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) $\Rightarrow$ (vi) are clear.

To see that (iii) implies $\lambda \ll \mu$, let $A \in \mathcal{B}(G)$. The linear functional given by $\alpha_A : \eta \mapsto \eta(A)$ for $\eta \in M(G)$ is norm continuous and hence, belongs to $M(G)^*$. So, by (iii) and (7.57) the function

$$\alpha_A \circ K_\lambda : x \mapsto (\lambda * \delta_x)(A) = \lambda(A - x), \qquad x \in G,$$

is continuous. Then $\lambda \ll \mu$ follows from [139, Theorem, p. 230].

(iii) $\Leftrightarrow$ (v). Clearly (iii) $\Rightarrow$ (v). So, assume (v). Since the range $K_\lambda(G)$ is then compact for both the weak (by hypothesis) and weak* (by Proposition 7.38(i)) topologies, these Hausdorff topologies actually coincide on $K_\lambda(G)$. But, by Proposition 7.38(i), $K_\lambda : G \to M(G)$ *is* weak* continuous and hence, it is weakly continuous, that is, (iii) holds.

(vi) $\Rightarrow$ (i). If $\lambda = 0$ there is nothing to prove. So, suppose that $\lambda \neq 0$. Assume that (i) fails. Then, $\lambda = \eta + \mu_h$ for some singular measure $\eta \in M(G)$ and some function $h \in L^1(G)$. Let $F_h : G \to L^1(G)$ be the continuous function defined by (7.68). The range of the continuous function $J \circ F_h$ on G is compact and hence, separable in the metric space $M(G)$. Since $K_\lambda = K_\eta + J \circ F_h$ it follows that $K_\lambda(A)$ is separable if and only if $K_\eta(A)$ is separable. So, we may further assume that $h = 0$, that is, λ $(= \eta)$ is singular. Hence, there is a set $C \in \mathcal{B}(G)$ such that $|\lambda|(C) = 0$ and $\mu(G \setminus C) = 0$.

Since $K_\lambda(A)$ is separable in $M(G)$, there exists a sequence $\{x_n\}_{n=1}^\infty$ in A such that the vectors $K_\lambda(x_n) = \lambda * \delta_{x_n}$, for $n \in \mathbb{N}$, form a countable dense subset of $K_\lambda(A)$. Let $B := \bigcap_{n=1}^\infty (C + x_n)$. Then

$$|\lambda|(B - x_n) \leq |\lambda|\big((C + x_n) - x_n\big) = |\lambda|(C) = 0, \qquad n \in \mathbb{N}.$$

Given $x \in G$, there exists a subsequence $K_\lambda(x_{n(k)}) \to K_\lambda(x)$ in $M(G)$ as $k \to \infty$. Since $|\delta_z * \lambda| = \delta_z * |\lambda|$, for $z \in A$, and

$$\big|\, |\lambda_1|(B) - |\lambda_2|(B) \big| \leq |\lambda_1 - \lambda_2|(B), \qquad \lambda_1, \lambda_2 \in M(G),$$

it follows that $\lim_{k\to\infty} |\lambda|(B - x_{n(k)}) = |\lambda|(B - x)$. In particular, we have $|\lambda|(B - x) = 0$. This and Fubini's Theorem give

$$0 = \int_A |\lambda|(B - x)\, d\mu(x) = \int_G \mu\big(A \cap (B - t)\big)\, d|\lambda|(t). \tag{7.69}$$

On the other hand, $\mu(B) = 1$ as $\mu(C + x_n) = 1$ for every $n \in \mathbb{N}$. So, $\mu(B - t) = 1$ and hence, $\mu\big(A \cap (B - t)\big) = \mu(A)$ for all $t \in G$. This and (7.69) imply that $0 = \mu(A)\, |\lambda|(G)$, which contradicts the fact that both $\mu(A) > 0$ and $|\lambda|(G) = \|\lambda\|_{M(G)} > 0$. So, (i) must hold.

This completes the proof of Theorem 7.30. ☐

Remark 7.40. (i) The equivalences $(\lambda \ll \mu) \Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) rely on Rudin, [139], as indicated in the proof. The proof of (vi) $\Rightarrow$ (i) was suggested by L. Rodríguez-Piazza.

(ii) For $G = \mathbb{R}^n$ and $\lambda \in M(\mathbb{R}^n)$ it is shown in [51] that if there exists a set $A \in \mathcal{B}(\mathbb{R}^n)$ with $\mu(A) > 0$ such that either, $K_\lambda(A)$ is relatively compact in $M(\mathbb{R}^n)$ (see Corollary 2 in [51]) or, $K_\lambda(A)$ is norm separable in $M(\mathbb{R}^n)$ (see Corollary 1(b) in [51]), then $\lambda \ll \mu$. It is stated that the same conclusions hold for a general locally compact abelian group G but, no details are provided.

Let G again be a *compact* abelian group. A measure $\lambda \in M(G)$ has a *separable orbit* if there exists a countable set $D_\lambda \subseteq G$ with the property that, for every $x \in G$ and $\varepsilon > 0$, there exists $z \in D_\lambda$ such that $\|K_\lambda(x) - K_\lambda(z)\|_{M(G)} < \varepsilon$. Of course, in this case $\mu(D_\lambda) = 0$. It is known that if $\lambda \in M(G)$ has a separable orbit, then $K_\lambda : G \to M(G)$ is norm continuous, [159, Lemma], and that $\lambda \ll \mu$, [159, Theorem 1]. Actually, $\lambda \ll \mu$ if and only if λ has a separable orbit, [159, Theorem 3].

(iii) Since G is infinite, there always exists a singular measure $\lambda \in M_0(G)$, [73, Remark 10.2.15]. Then $\lambda \notin L^1(G)$ and so $K_\lambda(G) \subseteq M(G)$ is non-separable by Theorem 7.30. Hence, $M_0(G)$ is *always* non-separable. This is of particular interest when G is metrizable. For, in this case, $L^1(G)$ is actually a *separable* closed subspace of $M_0(G)$, [140, Appendix E8]. So, the gap between $L^1(G)$ and $M_0(G)$ may be rather large. For metrizable G, a different proof of the non-separability of $M_0(G)$ is also possible; this is part of Menchoff's Theorem, [163, III.C.6].

(iv) It has already been noted that $L^1(G)$ is always weakly compactly generated. Suppose that the larger Banach space $M(G)$ is also weakly compactly generated. Since $M(G) = C(G)^*$ is a dual Banach space, Kuo's Theorem would imply that it has the Radon–Nikodým property, [42, Ch. III, Corollary 3.7]. It would then follow that its closed subspace $L^1(G)$ also has the Radon–Nikodým property, [42, Ch. III, Theorem 3.2], which is not the case, [42, p. 219]. Accordingly, $M(G)$ is neither weakly compactly generated nor has the Radon–Nikodým property. Since separable Banach spaces are necessarily weakly compactly generated it follows that $M(G)$ is non-separable. Of course, this also follows from part (iii) above or, directly from the observations that $\|\delta_x - \delta_z\|_{M(G)} = 2$ for distinct $x, z \in G$ and that G is uncountable. Since $L^1(G)$ is a closed subspace of $M_0(G)$ it follows, again from [42, Ch. III, Theorem 3.2], that $M_0(G)$ also *fails* to have the Radon–Nikodým property.

More can be said about $M(G)$. Indeed $M(G)$ is closed (for the variation norm) within the Banach space $\mathrm{ca}\big(\mathcal{B}(G)\big)$ of all $\mathbb{C}$-valued σ-additive measures, [46, Ch. III, Section 7]. Let $M^+(G)$ denote the cone of all non-negative measures $\lambda \in M(G)$, that is, $\lambda(A) \geq 0$ for all $A \in \mathcal{B}(G)$. Then the $\mathbb{R}$-valued measures in $M(G)$, denoted by $M_{\mathbb{R}}(G)$, can be ordered via $\lambda_1 \geq \lambda_2$ if and only if $(\lambda_1 - \lambda_2) \in M^+(G)$. Equipped with this order, $M_{\mathbb{R}}(G)$ is a Dedekind complete Banach lattice (over $\mathbb{R}$). If $\{\lambda_n\}_{n=1}^\infty \subseteq M_{\mathbb{R}}(G)$ has the property that $\lim_{n\to\infty} \lambda_n(A)$ exists in $\mathbb{R}$ for each $A \in \mathcal{B}(G)$, then $\lambda : \mathcal{B}(G) \to \mathbb{R}$ defined by $A \mapsto \lim_{n\to\infty} \lambda_n(A)$ is an element of $M_{\mathbb{R}}(G)$; see [46, Ch. III, Corollary 7.4]. It follows that if $\{\lambda_n\}_{n=1}^\infty \subseteq M^+(G)$ satisfies $\lambda_n \downarrow 0$ in the order of $M_{\mathbb{R}}(G)$, then $\lim_{n\to\infty} \lambda_n(A) = 0$ for all $A \in \mathcal{B}(G)$. In particular, for $A = G$, it follows that $\|\lambda_n\|_{M_{\mathbb{R}}(G)} \to 0$ as $n \to \infty$. Accordingly, the Banach lattice $M_{\mathbb{R}}(G)$ has σ-o.c. norm and, due to Dedekind completeness, even o.c. norm. Hence, so does its complexification $M(G) = M_{\mathbb{R}}(G) + iM_{\mathbb{R}}(G)$, after noting that the variation measure $|\lambda|$ of $\lambda \in M(G)$ is exactly the modulus of λ formed as an element of $M_{\mathbb{R}}(G) + iM_{\mathbb{R}}(G)$ via

$$\sup_{0\leq\theta<2\pi} \big|(\cos\theta)\mathrm{Re}(\lambda) + (\sin\theta)\mathrm{Im}(\lambda)\big|.$$

So, $M(G)$ is a *Dedekind complete*, complex *Banach lattice* with *o.c. norm*. Since $M(G)$ is *not* weakly compactly generated, it follows that $M(G)$ cannot have a *weak order unit* (see [108, p. 18] for the definition). This follows from Theorem 2 and Theorem 8 of [21], for example. The positive and negative parts of $\lambda \in M_{\mathbb{R}}(G)$, in the sense of Banach lattices (see [108, p. 2], for the definition), are precisely those corresponding to the *Hahn decomposition* $\lambda = \lambda^+ - \lambda^-$ of λ in the sense of measure theory. It follows that every convolution operator $C_\lambda^{(1)}$, for $\lambda \in M(G)$, is a *regular operator* in $L^1(G)$; see [149, p. 233] for the definition. This remains true for the operators $C_\lambda^{(p)}$ (as given by (7.7)) for $\lambda \in M(G)$ and $1 \le p < \infty$. For more precise details of the "lattice theoretic" aspects of the class of regular operators $C_\lambda^{(p)}$ acting in the complex Banach lattices $L^p(G)$, $1 \le p < \infty$, we refer to [4], [5], [7], for example. $\square$

Proof of Theorem 7.31. Since all statements are trivially true for $\lambda = 0$, we may assume that $\lambda \neq 0$.

(i) $\Rightarrow$ (ii). As (i) holds, the operator $T_\lambda : f \mapsto f * \lambda$ is compact from $L^2(G)$ into itself, [42, p. 93]. So, the set $\{T_\lambda(\chi_A) : A \in \mathcal{B}(G)\}$ is relatively compact in $L^2(G)$. Since $L^2(G) \subseteq L^1(G)$ continuously, this set is also relatively compact in $L^1(G)$. So, (ii) holds because the range of $m_\lambda^{(1)}$ equals this set in $L^1(G)$.

(ii) $\Leftrightarrow$ (iii). This is immediate from Corollary 2.43 with $T := C_\lambda^{(1)}$.

(iii) $\Rightarrow$ (iv) is clear.

(iv) $\Rightarrow$ (i). Let $\{\gamma_n\}_{n=1}^\infty$ be any sequence of distinct characters from Γ. By the Riemann–Lebesgue Lemma, $\lim_{n\to\infty}(\cdot, \gamma_n) = 0$ weakly in $L^1(G)$. By (7.10), with $\gamma := \gamma_n$ there, we have

$$\begin{aligned}\lim_{n\to\infty} \widehat{\lambda}(\gamma_n)\chi_{\{\gamma_n\}} &= \lim_{n\to\infty} F_{1,0}\big((\cdot,\gamma_n)\widehat{\lambda}(\gamma_n)\big) \\ &= \lim_{n\to\infty} \big(F_{1,0} \circ C_\lambda^{(1)}\big)(\cdot,\gamma_n) = 0\end{aligned}$$

with convergence in the *norm* of $c_0(\Gamma)$, because of hypothesis (iv). Because $\big|\widehat{\lambda}(\gamma_n)\big| = \big\|\widehat{\lambda}(\gamma_n)\chi_{\{\gamma_n\}}\big\|_{\ell^\infty(\Gamma)}$ for each $n \in \mathbb{N}$, it follows that $\lim_{n\to\infty} \widehat{\lambda}(\gamma_n) = 0$. So, $\widehat{\lambda} \in \ell^\infty(\Gamma)$ has the property that $\lim_{n\to\infty} \widehat{\lambda}(\gamma_n) = 0$ for every sequence of distinct elements $\{\gamma_n\}_{n=1}^\infty \subseteq \Gamma$. This easily implies that $\widehat{\lambda} \in c_0(\Gamma)$, that is, $\lambda \in M_0(G)$.

(i) $\Rightarrow$ (v). For $\lambda \in M_0(G)$, the function $F_S \circ K_\lambda : G \to \ell^\infty(\Gamma)$ is norm continuous, [68, p. 158], and so has compact range. Hence, $F_S \circ K_\lambda \in \mathbb{B}\big(\mu, \ell^\infty(\Gamma)\big)$. By Proposition 7.38(iii) we have

$$\Big(\Lambda \circ F_{1,0} \circ C_\lambda^{(1)}\Big)(f) = (\mathrm{w}^*)\text{-}\int_G f \cdot (F_S \circ K_\lambda)\, d\mu = (\mathrm{B})\text{-}\int_G f \cdot (F_S \circ K_\lambda)\, d\mu$$

for every $f \in L^1(G)$. It then follows from Proposition 3.47 that the linear operator $\Lambda \circ (F_{1,0} \circ C_\lambda^{(1)}) : L^1(G) \to \ell^\infty(\Gamma)$ is compact. Consequently, so is the operator $F_{1,0} \circ C_\lambda^{(1)} : L^1(G) \to c_0(\Gamma)$ because $c_0(\Gamma)$ is a closed subspace of $\ell^\infty(\Gamma)$ and the operator $\Lambda : c_0(\Gamma) \to \ell^\infty(\Gamma)$ is the natural inclusion.

(v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii). Clearly (v) $\Rightarrow$ (vi) and (vii) $\Rightarrow$ (viii). For the implication (vi) $\Rightarrow$ (vii) apply [42, Ch. III, Theorem 2.12].

(viii) $\Rightarrow$ (i). Choose a μ-scalarly bounded function $H \in \mathbb{P}\big(\mu, c_0(\Gamma)\big)$ such that $(F_{1,0} \circ C_\lambda^{(1)})(f) = (\mathrm{P})\text{-}\int_G fH\,d\mu$ for $f \in L^1(G)$. Fix any $\gamma \in \Gamma$ and recall the definition of the coordinate functional $\pi_\gamma : \ell^\infty(\Gamma) \to \mathbb{C}$ (just prior to Proposition 7.38). It is continuous on $\ell^\infty(\Gamma)$ and hence, its restriction to the closed subspace $c_0(\Gamma)$ is also continuous. Since $H \in \mathbb{P}\big(\mu, c_0(\Gamma)\big)$, it follows that

$$\big[C_\lambda^{(1)}(\chi_A)\big]\hat{\ }(\gamma) = \Big\langle (\mathrm{P})\text{-}\int_A H\,d\mu,\ \pi_\gamma \Big\rangle = \int_A [H(x)](\gamma)\,d\mu(x),$$

for $A \in \mathcal{B}(G)$. This and (7.60) yield

$$\int_A \big[K_\lambda(x)\big]\hat{\ }(\gamma)\,d\mu(x) = \int_A [H(x)](\gamma)\,d\mu(x), \qquad A \in \mathcal{B}(G).$$

Now, let $\{\gamma_n\}_{n=1}^\infty$ be any infinite sequence of distinct characters in Γ. Apply Lemma 7.39(ii)(a) to $Q := H$ (or, strictly speaking, to $Q := \Lambda \circ H$) to find a set $A_0 \in \mathcal{B}(G)$ such that $\mu(A_0) = 1$ and $\big[K_\lambda(x)\big]\hat{\ }(\gamma_n) = [H(x)](\gamma_n)$ for all $x \in A_0$ and $n \in \mathbb{N}$. Take any point $a \in A_0$. Then

$$|\widehat{\lambda}(\gamma_n)| = \big|\overline{(a,\gamma_n)}\,\widehat{\lambda}(\gamma_n)\big| = \big|\,\big[K_\lambda(a)\big]\hat{\ }(\gamma_n)\big| \;=\; \big|\,[H(a)](\gamma_n)\big| \to 0$$

as $n \to \infty$ because $H(a) \in c_0(\Gamma)$. For the same reason as in the proof of (iv) $\Rightarrow$ (i) it follows that the element $\widehat{\lambda} \in \ell^\infty(\Gamma)$ actually belongs to $c_0(\Gamma)$, that is, $\lambda \in M_0(G)$.

(i) $\Rightarrow$ (x) $\Rightarrow$ (xi). The implication (i) $\Rightarrow$ (x) has already been established in the proof of (i) $\Rightarrow$ (v) whereas (x) $\Rightarrow$ (xi) is clear.

(ix) $\Leftrightarrow$ (xi). Recalling that $F_S \circ K_\lambda : G \to \ell^\infty(\Gamma)$ is weak* continuous (see Proposition 7.38(ii)), one can argue as in the proof of (iii) $\Leftrightarrow$ (v) in Theorem 7.30.

(ix) $\Rightarrow$ (iv). In this case $F_S \circ K_\lambda$ has weakly compact range in $\ell^\infty(\Gamma)$ and hence, takes its values in a weakly compactly generated (closed) subspace of $\ell^\infty(\Gamma)$. So, it is Pettis μ-integrable by D.R. Lewis' Theorem, [42, p. 88]. As μ is a *perfect measure* (see, for example [162, Proposition A.4]), it follows by a result of C. Stegall, [162, Proposition 5.7], that the Pettis indefinite μ-integral $A \mapsto (\mathrm{P})\text{-}\int_A F_S \circ K_\lambda\,d\mu$ on $\mathcal{B}(G)$ has relatively compact range in $\ell^\infty(\Gamma)$. On the other hand, Proposition 7.38(iii) implies that

$$\Big(\Lambda \circ F_{1,0} \circ m_\lambda^{(1)}\Big)(A) = (\mathrm{w}^*)\text{-}\int_A F_S \circ K_\lambda\,d\mu = (\mathrm{P})\text{-}\int_A F_S \circ K_\lambda\,d\mu,$$

for $A \in \mathcal{B}(G)$. So, the vector measure $\Lambda \circ (F_{1,0} \circ m_\lambda^{(1)}) : \mathcal{B}(G) \to \ell^\infty(\Gamma)$ has relatively compact range. Since the range $\mathcal{R}(F_{1,0} \circ m_\lambda^{(1)}) \subseteq c_0(\Gamma)$ with $c_0(\Gamma)$ a closed subspace of $\ell^\infty(\Gamma)$, the range of the vector measure $F_{1,0} \circ m_\lambda^{(1)}$ is relatively compact in

$c_0(\Gamma)$. It follows from Corollary 2.43, with $T := F_{1,0} \circ C_\lambda^{(1)} : L^1(G) \to c_0(\Gamma)$, that $F_{1,0} \circ C_\lambda^{(1)}$ is completely continuous.

(i) $\Rightarrow$ (xii). This has already been established in the proof of (i) $\Rightarrow$ (v).

(xii) $\Rightarrow$ (iv). The indefinite Bochner μ-integral $A \mapsto$ (B)-$\int_A F_S \circ K_\lambda \, d\mu$ on $\mathcal{B}(G)$ has relatively compact range in $\ell^\infty(\Gamma)$, [42, Ch. II, Corollary 3.9]. Hence, the range of the vector measure $\Lambda \circ F_{1,0} \circ m_\lambda^{(1)} : \mathcal{B}(G) \to \ell^\infty(\Gamma)$ is also relatively compact because

$$\left(\Lambda \circ F_{1,0} \circ m_\lambda^{(1)}\right)(A) = (\mathrm{w}^*)\text{-}\int_A F_S \circ K_\lambda \, d\mu = (\mathrm{B})\text{-}\int_A F_S \circ K_\lambda \, d\mu,$$

for $A \in \mathcal{B}(G)$; see (7.63) in Proposition 7.38(iii). So, we can obtain (iv) as in the proof of (ix) $\Rightarrow$ (iv).

This establishes the equivalences (i)–(xii). The equivalence of these with each of (xiii) and (xiv) has been established along the way.

(xv) $\Rightarrow$ (xiv). If there exists $H \in \mathbb{P}\big(\mu, c_0(\Gamma)\big)$ as in (xv), then the argument of (ix) $\Rightarrow$ (iv) shows that $F_{1,0} \circ m_\lambda^{(1)}$ has relatively compact range in $c_0(\Gamma)$.

(viii) $\Rightarrow$ (xv) is clear.

Finally, (xvi) $\Rightarrow$ (xv) is clear, as is (vii) $\Rightarrow$ (xvi).

This completes the proof of Theorem 7.31. □

Remark 7.41. As a consequence of Theorems 7.30 and 7.31 we see that every $\lambda \in M_0(G) \setminus L^1(G)$ provides an example of a vector measure, namely the measure $m_\lambda^{(1)} : \mathcal{B}(G) \to L^1(G)$, which has relatively compact range but, whose associated integration operator $I_{m_\lambda^{(1)}} = C_\lambda^{(1)}$ *fails* to be a compact operator. □

7.3 Operators acting in $L^p(G)$ via convolution with functions

For $\lambda \in M(G)$ we now consider the convolution operators $C_\lambda^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ for $1 < p < \infty$, as given by (7.7), from the viewpoint of their optimal domain. Of course, the corresponding vector measure $m_\lambda^{(p)} : \mathcal{B}(G) \to L^p(G)$ is defined by

$$m_\lambda^{(p)} : A \mapsto C_\lambda^{(p)}(\chi_A) = \lambda * \chi_A, \qquad A \in \mathcal{B}(G). \tag{7.70}$$

Both the properties and the identification of the optimal domain $L^1(m_\lambda^{(p)})$ of the extended operator $I_{m_\lambda^{(p)}} : L^1(m_\lambda^{(p)}) \to L^p(G)$ of $C_\lambda^{(p)}$ are important as well as certain operator theoretic properties of $I_{m_\lambda^{(p)}}$. The subclass corresponding to $\lambda \ll \mu$ has been investigated in [123]. As will be seen, the situation for $\lambda \ll \mu$ with $1 < p < \infty$ is very different from that of $p = 1$ as treated in Section 7.2. The situation for general measures λ acting in $L^p(G)$ (i.e., not necessarily $\lambda \in L^1(G)$), which is treated in [124], is quite different from that of absolutely continuous measures. Accordingly, this case will be considered separately in Section 7.4. So, here we will deal exclusively with the measures $\lambda = \mu_h$ (see (7.3)) for $h \in L^1(G)$. In this case, we will simply write $C_h^{(p)}$ for the convolution operator $C_{\mu_h}^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ and $m_h^{(p)}$ for the corresponding vector measure $m_{\mu_h}^{(p)}$ as given by (7.70). Since the results for $1 < p < \infty$ are (mostly) already available in the literature (unlike for $p = 1$ as treated in Section 7.2), we will restrict ourselves to a careful and detailed summary of the relevant definitions and theorems, together with various remarks and examples. So, let us begin.

Let $1 \le p < \infty$ and fix $h \in L^1(G)$. Then the operator

$$C_h^{(p)} : f \mapsto f * h, \qquad f \in L^p(G),$$

is continuous (i.e., belongs to $\mathcal{L}(L^p(G))$) because of the inequality

$$\big\|C_h^{(p)}(f)\big\|_{L^p(G)} \le \|f\|_{L^p(G)} \|h\|_{L^1(G)}, \qquad f \in L^p(G); \tag{7.71}$$

see (7.8) with $\lambda := \mu_h$. Moreover, $C_h^{(p)}$ is always a *compact* operator, [48, Corollary 6], [63, Theorem 4.2.2]. The set function $m_h^{(p)} : \mathcal{B}(G) \to L^p(G)$ defined by

$$m_h^{(p)} : A \mapsto C_h^{(p)}(\chi_A) = \chi_A * h, \qquad A \in \mathcal{B}(G), \tag{7.72}$$

is surely finitely additive. Actually, the inequality

$$\big\|m_h^{(p)}(A)\big\|_{L^p(G)} \le \|\chi_A\|_{L^p(G)}\, \|h\|_{L^1(G)} = \big(\mu(A)\big)^{1/p}\, \|h\|_{L^1(G)}, \qquad A \in \mathcal{B}(G),$$

which follows from (7.71) and (7.72) with $f := \chi_A$, shows that $m_h^{(p)}$ is σ-additive. For each $\varphi \in L^{p'}(G) = L^p(G)^*$ it turns out that

$$\langle m_h^{(p)}, \varphi\rangle(A) = \int_A (\varphi * R_G h)\, d\mu, \qquad A \in \mathcal{B}(G); \tag{7.73}$$

see [123, p. 531]. The proof is similar to that of Lemma 7.34, which is the case $p = 1$. The variation measure is then given by

$$\big|\langle m_h^{(p)}, \varphi\rangle\big|(A) = \int_A |\varphi * R_G h|\, d\mu, \qquad A \in \mathcal{B}(G). \tag{7.74}$$

It is a consequence of Hölder's inequality, (7.71) with p' in place of p, and $\|R_G h\|_{L^1(G)} = \|h\|_{L^1(G)}$ that

$$\begin{aligned}\int_G |f| \cdot |\varphi * R_G h|\, d\mu &\le \|f\|_{L^p(G)}\, \|\varphi * R_G h\|_{L^{p'}(G)} \\ &\le \|f\|_{L^p(G)}\, \|\varphi\|_{L^{p'}(G)}\, \|h\|_{L^1(G)} < \infty,\end{aligned} \tag{7.75}$$

for each $f \in L^p(G)$.

The next result, [123, Lemma 2.2], collects together some basic properties of the vector measure $m_h^{(p)}$. Part (i) relies on the compactness of the operator $C_h^{(p)} \in \mathcal{L}\big(L^p(G)\big)$. The formula (7.76) in part (ii) follows from the definition of the semivariation of $m_h^{(p)}$ and (7.74). Then (7.76), together with the inequality (for $\varphi \in L^{p'}(G)$)

$$\int_A |\varphi * R_G h|\, d\mu \leq \big(\mu(A)\big)^{1/p} \|\varphi\|_{L^{p'}(G)}\, \|h\|_{L^1(G)}, \qquad A \in \mathcal{B}(G),$$

which is a consequence of (7.75) with $f := \chi_A$, form the basis of the proof of (7.77). Part (iii) is then immediate from (7.77). For the precise details we refer to [123].

Lemma 7.42. *Let $1 \leq p < \infty$ and fix $h \in L^1(G)$.*

(i) *The range $\mathcal{R}\big(m_h^{(p)}\big)$ of $m_h^{(p)}$ is a relatively compact subset of $L^p(G)$.*

(ii) *Given any set $A \in \mathcal{B}(G)$, its semivariation equals*

$$\big\|m_h^{(p)}\big\|(A) \;=\; \sup\left\{\int_A |\varphi * R_G h|\, d\mu : \|\varphi\|_{L^{p'}(G)} \leq 1\right\}, \tag{7.76}$$

and satisfies

$$\|\widehat{h}\|_{c_0(\Gamma)}\, \mu(A) \;\leq\; \big\|m_h^{(p)}\big\|(A) \;\leq\; \|h\|_{L^1(G)}\big(\mu(A)\big)^{1/p}. \tag{7.77}$$

(iii) *The vector measure $m_h^{(p)}$ is always absolutely continuous with respect to μ. Conversely, if $h \neq 0$, then μ is absolutely continuous with respect to $m_h^{(p)}$.*

Remark 7.43. (i) It follows from Lemma 7.42(iii) that the convolution operator $C_h^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ is *μ-determined* whenever $1 \leq p < \infty$ and $h \in L^1(G) \setminus \{0\}$.

(ii) Lemma 7.42(i) should be compared with Proposition 7.6(iii). □

The following result, which is a combination of Theorem 1.1, Lemma 3.1 and Proposition 3.4 of [123], summarizes the essential properties of the optimal domain space $L^1\big(m_h^{(p)}\big)$ of $C_h^{(p)}$.

Theorem 7.44. *Let $1 \leq p < \infty$ and fix $h \in L^1(G) \setminus \{0\}$.*

(i) *A $\mathcal{B}(G)$-measurable function $f : G \to \mathbb{C}$ is $m_h^{(p)}$-integrable if and only if*

$$\int_G |f| \cdot |\varphi * R_G h|\, d\mu \;<\; \infty, \qquad \varphi \in L^{p'}(G) = L^p(G)^*. \tag{7.78}$$

Moreover, the norm of $f \in L^1\big(m_h^{(p)}\big)$ is given by

$$\|f\|_{L^1(m_h^{(p)})} = \sup\left\{\int_G |f| \cdot |\varphi * R_G h|\, d\mu :\; \varphi \in L^{p'}(G),\; \|\varphi\|_{L^{p'}(G)} \leq 1\right\}.$$

(ii) *The inclusions*

$$L^p(G) \subseteq L^1\big(m_h^{(p)}\big) = L^1_{\mathrm{w}}\big(m_h^{(p)}\big) \subseteq L^1(G) \tag{7.79}$$

hold and are continuous. Indeed,

$$\|f\|_{L^1(m_h^{(p)})} \le \|h\|_{L^1(G)}\, \|f\|_{L^p(G)}, \qquad f \in L^p(G),$$

and also

$$\|f\|_{L^1(G)} \le \|\widehat{h}\|^{-1}_{c_0(\Gamma)}\, \|f\|_{L^1(m_h^{(p)})}, \qquad f \in L^1\big(m_h^{(p)}\big). \tag{7.80}$$

(iii) *For each $1 \le q \le p$ we have $L^1\big(m_h^{(p)}\big) \subseteq L^1\big(m_h^{(q)}\big)$ and*

$$\|f\|_{L^1(m_h^{(q)})} \le \|f\|_{L^1(m_h^{(p)})}, \qquad f \in L^1\big(m_h^{(p)}\big).$$

(iv) *The subspaces $\mathcal{T}(G)$ and $L^p(G)$ are both dense in $L^1\big(m_h^{(p)}\big)$.*

(v) *$L^1\big(m_h^{(p)}\big)$ is a translation invariant subspace of $L^1(G)$ which is stable under formation of reflections and complex conjugates. Moreover, for each $x \in G$, we have*

$$\Big(I_{m_h^{(p)}} \circ \tau_x\Big)(f) = \Big(\tau_x \circ I_{m_h^{(p)}}\Big)(f), \qquad f \in L^1\big(m_h^{(p)}\big),$$

where the equality is between elements of $L^p(G)$. The range of $I_{m_h^{(p)}}$ and its closure are both translation invariant subspaces of $L^p(G)$.

(vi) *The extension $I_{m_h^{(p)}} : L^1\big(m_h^{(p)}\big) \to L^p(G)$ of $C_h^{(p)}$ to its optimal domain $L^1\big(m_h^{(p)}\big)$ is given by*

$$I_{m_h^{(p)}}(f) = h * f, \qquad f \in L^1\big(m_h^{(p)}\big). \tag{7.81}$$

Remark 7.45. (i) Since the weakly sequentially complete space $L^1(G)$ and the reflexive spaces $L^p(G)$, for $1 < p < \infty$, cannot contain an isomorphic copy of c_0, the equality in (7.79) is clear. Moreover, (7.78) is based on (7.74).

(ii) The inequality (7.80) is a consequence of [123, Lemma 3.1] and (7.56) in the proof of Proposition 7.35. Indeed, fix $f \in L^1(m_h^{(p)})$. Then it follows from [123, Lemma 3.1] that

$$\|f\|_{L^1(m_h^{(1)})} \le \|f\|_{L^1(m_h^{(p)})}.$$

On the other hand, (7.56) gives

$$\|\widehat{h}\|_{c_0(\Gamma)}\, \|f\|_{L^1(G)} = \|\widehat{\mu}_h\|_{\ell^\infty(\Gamma)}\, \|f\|_{L^1(G)} \le \|f\|_{L^1(m_{\mu_h}^{(1)})} = \|f\|_{L^1(m_h^{(1)})}.$$

Therefore, (7.80) holds.

(iii) It follows from (7.81) that $\big(I_{m_h^{(p)}}(f)\big)^{\widehat{}} = \widehat{h}\cdot\widehat{f}$ for all $f \in L^1\big(m_h^{(p)}\big)$. Of course, also $\big(C_h^{(p)}(f)\big)^{\widehat{}} = \widehat{h}\cdot\widehat{f}$ for all $f \in L^p(G)$. By considering the functions $f := (\cdot,\gamma)$, for $\gamma\in\Gamma$, and (7.10) it is clear that $C_h^{(p)}$ is injective if and only if its extension $I_{m_h^{(p)}}$ is injective if and only if $\widehat{h}(\gamma)\neq 0$ for every $\gamma\in\Gamma$.

(iv) If $h \in L^1(G)^+$, then it follows from (7.78) that

$$L^1\big(m_h^{(p)}\big) \;=\; M_h^{(p)} \;=\; N_h^{(p)}, \tag{7.82}$$

where

$$M_h^{(p)} := \big\{f \in L^1(G) : (f\chi_A) * h \in L^p(G) \text{ for all } A \in \mathcal{B}(G)\big\}$$

and

$$N_h^{(p)} := \big\{f \in L^1(G) : |f| * h \in L^p(G)\big\},$$

[123, Proposition 3.2]. Actually, for arbitrary $h \in L^1(G)$,

$$N_{|h|}^{(p)} \;\subseteq\; L^1\big(m_h^{(p)}\big), \tag{7.83}$$

[123, Remark 3.3(i)]. These alternate descriptions of $L^1\big(m_h^{(p)}\big)$ as given by (7.82), for $h \geq 0$, have some interesting consequences. Indeed, let $1 < p < \infty$. For any r satisfying $1 < r < p$, choose any function $h \in L^r(G)\setminus L^p(G)$; see the discussion immediately after (7.1). If $q \in (1,p)$ satisfies

$$\frac{1}{r} + \frac{1}{q} = \frac{1}{p} + 1$$

(i.e., $q = pr/(pr-p+r)$), then it follows from [75, Theorem 20.18] that $|f| * |h| \in L^p(G)$ for every $f \in L^q(G)$. That is, $L^q(G) \subseteq N_{|h|}^{(p)}$. Combining this with (7.83), we conclude that

$$L^p(G) \;\subseteq\; L^{pr/(pr-p+r)}(G) \;\subseteq\; N_{|h|}^{(p)} \;\subseteq\; L^1\big(m_h^{(p)}\big), \qquad 1 < r < p < \infty, \tag{7.84}$$

for every $h \in L^r(G)\setminus L^p(G)$, where the last inclusion in (7.84) is an equality whenever $h \in L^1(G)^+$. Since $1 < q < p$, this shows (for $C_h^{(p)}$) that there exists an L^q-space which is *strictly larger* than $L^p(G)$ and lies between $L^p(G)$ and the optimal domain space $L^1\big(m_h^{(p)}\big)$. This is a quite different phenomena to that for the optimal domain space $\mathbf{F}^p(\mathbb{T}^d)$, for $1 < p < 2$, of the Fourier transform map $F_p : L^p(G) \to \ell^{p'}(\Gamma)$, where there is *no* strictly larger space $L^q(\mathbb{T}^d)$ than $L^p(\mathbb{T}^d)$ which lies between $L^p(\mathbb{T}^d)$ and $\mathbf{F}^p(\mathbb{T}^d)$; see Remark 7.29(ii).

As a consequence, we can extend Example 6.26(ii). Suppose that $1 < p < \infty$ and fix $1 < r < p$ and $h \in L^r(G)\setminus L^p(G)$. If q satisfies $(1/r) + (1/q) = (1/p) + 1$, then (7.84) shows that $L^q(G) \subseteq L^1\big(m_h^{(p)}\big)$. So, we can apply Lemma 6.24(ii) to obtain

$$C_h^{(p)} \;\in\; \mathcal{A}_{t,v}\big(L^p(G),\, L^p(G)\big)$$

whenever t, v satisfy $1 \leq q \leq tq \leq v \leq p$.

(v) What about the function $h \in L^1(G)$ itself? If it happens that $h \in L^p(G)$, then (7.79) implies that $h \in L^1\big(m_h^{(p)}\big)$. It can also happen that the function $h \in L^1\big(m_h^{(p)}\big) \setminus L^p(G)$. Indeed, let $1<p<\infty$ and define $r:=2p/(p+1)$, in which case $1<r<\min\{2,p\}$. Choose any $h \in L^r(G)\setminus L^p(G)$. Since $pr/(pr-p+r)$ now equals r, it follows from (7.84) that $h \in L^r(G) = L^{pr/(pr-p+r)}(G) \subseteq L^1\big(m_h^{(p)}\big)$. $\square$

We record the conclusion of Remark 7.45(v) formally.

Proposition 7.46. *For each $1<p<\infty$ and every $h \in L^{2p/(p+1)}(G)$ the inclusions*

$$L^p(G) \subseteq L^{2p/(p+1)}(G) \subseteq L^1\big(m_h^{(p)}\big)$$

are valid, with the first inclusion proper.

It turns out that the first equality in (7.82) is actually valid for *arbitrary* $h \in L^1(G)$.

Proposition 7.47. *Let $1 \le p < \infty$ and $h \in L^1(G) \setminus \{0\}$. Then the optimal domain $L^1\big(m_h^{(p)}\big)$ of $C_h^{(p)}$ is given by*

$$L^1\big(m_h^{(p)}\big) = \big\{f \in L^1(G) : \big((\chi_A f) * h\big) \in L^p(G) \ \ \textit{for all} \ \ A \in \mathcal{B}(G)\big\}\,.$$

Proof. Let $f \in L^1\big(m_h^{(p)}\big)$. Since $L^1\big(m_h^{(p)}\big)$ is a B.f.s., also $\chi_A f \in L^1\big(m_h^{(p)}\big)$ for all $A \in \mathcal{B}(G)$, that is, $\big((\chi_A f) * h\big) \in L^p(G)$ for all $A \in \mathcal{B}(G)$; see Theorem 7.44(vi).

Conversely, suppose that $f \in L^1(G)$ satisfies $\big((\chi_A f) * h\big) \in L^p(G)$ for all $A \in \mathcal{B}(G)$. Then the set function $\eta^f : \mathcal{B}(G) \to L^p(G)$ defined by

$$A \mapsto \eta^f(A) := (\chi_A f) * h, \qquad A \in \mathcal{B}(G),$$

is surely finitely additive. Let $H := \big\{\overline{(\cdot,\gamma)} : \gamma \in \Gamma\big\}$, considered as a subset of the dual space $L^{p'}(G) = L^p(G)^*$. Fix $\gamma \in \Gamma$. Then

$$\begin{aligned}\big\langle \eta^f(A),\, \overline{(\cdot,\gamma)}\big\rangle &= \int_G \overline{(\cdot,\gamma)} \cdot \big((\chi_A f) * h\big)\, d\mu \\ &= (\chi_A f)\widehat{\ }(\gamma)\,\widehat{h}(\gamma) = \widehat{h}(\gamma) \int_G \overline{(x,\gamma)}\, \chi_A(x) f(x)\, d\mu(x).\end{aligned}$$

So, the $\mathbb{C}$-valued set function $\big\langle \eta^f,\, \overline{(\cdot,\gamma)}\big\rangle \mapsto \big\langle \eta^f(A),\, \overline{(\cdot,\gamma)}\big\rangle$ on $\mathcal{B}(G)$ equals the indefinite integral of $\big(\widehat{h}(\gamma)\overline{(\cdot,\gamma)} f\big) \in L^1(G)$ with respect to μ and hence, is σ-additive. Since the weakly sequentially complete space $L^1(G)$ and the reflexive spaces $L^p(G)$, for $1<p<\infty$, cannot contain an isomorphic copy of ℓ^∞ and $H \subseteq L^p(G)^* = L^{p'}(G)$ is a total set of functionals for $L^p(G)$, it follows from Lemma 3.2 that η^f is σ-additive.

Let $\varphi \in L^{p'}(G)$. Then it was just shown that

$$A \mapsto \big\langle \eta^f(A),\, \varphi\big\rangle = \big\langle (\chi_A f) * h,\, \varphi\big\rangle, \qquad A \in \mathcal{B}(G),$$

is σ-additive. Define $A_n := |f|^{-1}([0,n])$ for $n \in \mathbb{N}$, in which case $(A \cap A_n) \uparrow A$ for $A \in \mathcal{B}(G)$ fixed. By σ-additivity of $\langle \eta^f, \varphi \rangle$ we have

$$\big\langle \eta^f(A),\, \varphi \big\rangle = \lim_{n\to\infty} \big\langle (\chi_{A\cap A_n} f) * h,\, \varphi \big\rangle.$$

But, $\chi_{A\cap A_n} f \in L^\infty(G) \subseteq L^1\big(m_h^{(p)}\big)$ and so by Theorem 7.44 and (7.73) we have

$$\big\langle (\chi_{A\cap A_n} f) * h,\, \varphi \big\rangle = \Big\langle I_{m_h^{(p)}}\big(\chi_{A\cap A_n} f\big),\, \varphi \Big\rangle = \int_A \chi_{A_n} f \, d\langle m_h^{(p)}, \varphi \rangle.$$

That is,

$$\big\langle \eta^f(A),\, \varphi \big\rangle = \lim_{n\to\infty} \int_A f_n \, d\langle m_h^{(p)}, \varphi \rangle,$$

where the functions $f_n := \chi_{A_n} f \in L^\infty(G) \subseteq L^\infty\big(\langle m_h^{(p)}, \varphi \rangle\big)$ converge pointwise on G to f and $\langle m_h^{(p)}, \varphi \rangle$ is a complex measure. By Lemma 2.17 we conclude that f is $\langle m_h^{(p)}, \varphi \rangle$-integrable, that is, $\int_G |f| \, d|\langle m_h^{(p)}, \varphi \rangle| < \infty$. Because of (7.74), it follows from Theorem 7.44(i) that $f \in L^1\big(m_h^{(p)}\big)$. $\square$

The same arguments used in the proof of Proposition 7.21 and Remark 7.22 also apply to yield the following Banach space properties of the optimal domain spaces $L^1\big(m_h^{(p)}\big)$.

Proposition 7.48. *Let $1 \le p < \infty$. Each of the following assertions is valid for every $h \in L^1(G)$.*

(i) *$L^1\big(m_h^{(p)}\big)$ is weakly sequentially complete and hence, its associate space $L^1\big(m_h^{(p)}\big)' = L^1\big(m_h^{(p)}\big)^*$.*

(ii) *$L^\infty(G) \subseteq L^1\big(m_h^{(p)}\big)'$ and $L^1\big(m_h^{(p)}\big)$ is also sequentially complete relative to $\sigma\big(L^1(m_h^{(p)}),\, L^\infty(G)\big)$.*

(iii) *$L^1\big(m_h^{(p)}\big)$ has σ-o.c. norm and satisfies the σ-Fatou property.*

(iv) *$L^1\big(m_h^{(p)}\big)$ is weakly compactly generated.*

(v) *Whenever G is metrizable, the Banach space $L^1\big(m_h^{(p)}\big)$ is separable.*

Theorem 7.44(iv),(v) together with next result show that the optimal domain spaces $L^1\big(m_h^{(p)}\big)$ are well suited for harmonic analysis.

Theorem 7.49. *Let $1 \le p < \infty$ and $h \in L^1(G)\setminus\{0\}$. For each $a \in G$ the translation operator $\tau_a : L^1\big(m_h^{(p)}\big) \to L^1\big(m_h^{(p)}\big)$ is a surjective isometry. Moreover, for each $a_0 \in G$ and $f \in L^1\big(m_h^{(p)}\big)$ we have $\lim_{a\to a_0} \tau_a f = \tau_{a_0} f$ in the norm of $L^1\big(m_h^{(p)}\big)$. In particular, $L^1\big(m_h^{(p)}\big)$ is a homogeneous Banach space on G, up to an equivalent lattice norm.*

Proof. Fix $a \in G$ and $f \in L^1\big(m_h^{(p)}\big)$. Then, for $\varphi \in L^{p'}(G)$,

$$\begin{aligned}\int_G |\tau_a f| \cdot |\varphi * R_G h| \, d\mu &= \int_G |f| \cdot \big|\tau_{-a}(\varphi * R_G h)\big| \, d\mu \\ &= \int_G |f| \cdot \big|(\tau_{-a}\varphi) * R_G h\big| \, d\mu.\end{aligned}$$

Since $\tau_{-a} : L^{p'}(G) \to L^{p'}(G)$ is a surjective isometry, it follows from (7.74) and the definition of the norm in $L^1\big(m_h^{(p)}\big)$ that

$$\begin{aligned}\|\tau_a f\|_{L^1(m_h^{(p)})} &= \sup_{\|\varphi\|_{p'} \le 1} \int_G |\tau_a f| \cdot |\varphi * R_G h| \, d\mu \\ &= \sup_{\|\varphi\|_{p'} \le 1} \int_G |f| \cdot \big|(\tau_{-a}\varphi) * R_G h\big| \, d\mu \\ &= \sup_{\|\psi\|_{p'} \le 1} \int_G |f| \cdot |\psi * R_G h| \, d\mu = \|f\|_{L^1(m_h^{(p)})},\end{aligned}$$

where $\|\cdot\|_{p'}$ denotes $\|\cdot\|_{L^{p'}(G)}$. Accordingly, $\tau_a \in \mathcal{L}\big(L^1\big(m_h^{(p)}\big)\big)$ is a surjective isometry.

The analogous argument as in the proof of Proposition 7.25(iii), together with the density of $\mathcal{T}(G)$ in $L^1\big(m_h^{(p)}\big)$ (see Theorem 7.44(iv)), shows that $\lim_{a \to a_0} \tau_a f = \tau_{a_0} f$ in $L^1\big(m_h^{(p)}\big)$ for each $a_0 \in G$ and $f \in L^1\big(m_h^{(p)}\big)$. $\square$

The reader will have noticed that, up to now, it has not been discussed whether the inclusion $L^1\big(m_h^{(p)}\big) \subseteq L^1(G)$ is proper or not. Moreover, the variation measure $\big|m_h^{(p)}\big|$ has not been considered. We now address these topics, for which some preparation is required.

Fix $1 \le p < \infty$ and let $h \in L^p(G)$. The "translation function"

$$F_h^{(p)} : G \to L^p(G)$$

defined by

$$F_h^{(p)} : x \mapsto \tau_x h, \qquad x \in G, \tag{7.85}$$

is continuous, [140, Theorem 1.1.5], and so has compact range in $L^p(G)$. Hence, $F_h^{(p)}$ assumes its values in a separable (closed) subspace of $L^p(G)$. Moreover, $\big\|F_h^{(p)}(x)\big\|_{L^p(G)} = \|h\|_{L^p(G)}$ for each $x \in G$, that is, the scalar-valued function $x \mapsto \big\|F_h^{(p)}(x)\big\|_p$ is constant on G. Hence, $F_h^{(p)}$ is *strongly μ-measurable* by the Pettis Measurability Theorem, [42, Ch. II, Theorem 2.2], and has bounded range. Accordingly, $F_h^{(p)} \in \mathbb{B}\big(\mu, L^p(G)\big)$. For the following result we refer to [123, Theorem 1.2].

Theorem 7.50. *Let $1 \le p < \infty$ and fix $h \in L^1(G) \setminus \{0\}$. Then the following assertions are equivalent.*

(i) $h \in L^p(G)$.

(ii) *The extended operator $I_{m_h^{(p)}} : L^1\big(m_h^{(p)}\big) \to L^p(G)$ of $C_h^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ to its optimal domain space $L^1\big(m_h^{(p)}\big)$ is a compact operator.*

(iii) *There exists $H \in \mathbb{B}\big(\mu, L^p(G)\big)$ such that*

$$m_h^{(p)}(A) = (\mathrm{B})\text{-}\int_A H\, d\mu, \qquad A \in \mathcal{B}(G).$$

(iv) *The vector measure $m_h^{(p)} : \mathcal{B}(G) \to L^p(G)$ has finite variation.*

(v) *There exists $A \in \mathcal{B}(G)$ such that $0 < \big|m_h^{(p)}\big|(A) < \infty$.*

(vi) $L^1\big(\big|m_h^{(p)}\big|\big) = L^1(G)$.

(vii) $L^1\big(m_h^{(p)}\big) = L^1(G)$.

(viii) $L^1\big(\big|m_h^{(p)}\big|\big) = L^1\big(m_h^{(p)}\big)$.

If any one of (i)–(viii) *holds, then for each $f \in L^1\big(m_h^{(p)}\big) = L^1(G)$ the function $fF_h^{(p)} \in \mathbb{B}(\mu, L^p(G))$ (with $F_h^{(p)}$ as given by* (7.85)) *and*

$$I_{m_h^{(p)}}(f) = (\mathrm{B})\text{-}\int_G fF_h^{(p)}\, d\mu.$$

Moreover, $\big|m_h^{(p)}\big|$ is given by

$$\big|m_h^{(p)}\big|(A) = \|h\|_{L^p(G)}\, \mu(A), \qquad A \in \mathcal{B}(G).$$

The following fact is an extension of Example 6.26(i).

Example 7.51. Let $1 < p < \infty$ and fix $1 \le r < \infty$ and $0 < q < \infty$. Given any $h \in L^p(G)$, Theorem 7.50 implies that $L^1\big(m_h^{(p)}\big) = L^1(G)$. Since μ is non-atomic (see Lemma 7.97) it follows from Lemma 6.24(iii) that

$$C_h^{(p)} \in \mathcal{A}_{r,q}\big(L^p(G), L^p(G)\big) \quad \text{if and only if} \quad 1 \le r \le q \le p. \qquad \square$$

We have already noted that $C_h^{(p)} : L^p(G) \to L^p(G)$ is a compact operator for every $1 \le p < \infty$ and $h \in L^1(G)$. In particular, $C_h^{(p)}$ is also completely continuous. What about its optimal extension $I_{m_h^{(p)}} : L^1\big(m_h^{(p)}\big) \to L^p(G)$? Since the codomain space $L^p(G)$ is reflexive (whenever $p \neq 1$) this extension is always weakly compact. Moreover, Theorem 7.50 shows that the extended operator $I_{m_h^{(p)}}$ is compact if and only if $h \in L^p(G)$. The following result, essentially Proposition 4.1 of [123], shows that more can be said about $I_{m_h^{(p)}}$. Because the proof is quite illuminating we reproduce it here.

Proposition 7.52. *Let $1 < p < \infty$ and $h \in L^1(G) \setminus \{0\}$.*

(i) *Whenever $h \notin L^p(G)$, the extended operator $I_{m_h^{(p)}} : L^1\big(m_h^{(p)}\big) \to L^p(G)$ is not compact, and both inclusions*
$$L^p(G) \subseteq L^1\big(m_h^{(p)}\big) \subseteq L^1(G) \tag{7.86}$$
are proper.

(ii) *The first inclusion in (7.86) is proper for arbitrary $h \in L^1(G) \setminus \{0\}$. In particular, the optimal domain space $L^1\big(m_h^{(p)}\big)$ of $C_h^{(p)}$ is always genuinely larger than $L^p(G)$.*

(iii) *If $h \notin L^p(G)$, then the variation measure $\big|m_h^{(p)}\big| : \mathcal{B}(G) \to [0,\infty]$ is totally infinite, that is, $\big|m_h^{(p)}\big|(A) \in \{0,\infty\}$ for every $A \in \mathcal{B}(G)$. In particular, $L^1\big(\big|m_h^{(p)}\big|\big) = \{0\}$.*

(iv) *There exists $h \in L^1(G)$ with the inclusion $\bigcup_{1<p<\infty} L^1\big(m_h^{(p)}\big) \subseteq L^1(G)$ proper.*

Proof. (i) and (ii). Suppose that $h \notin L^p(G)$. The non-compactness of $I_{m_h^{(p)}}$ is a consequence of (i) $\Leftrightarrow$ (ii) in Theorem 7.50. Since $I_{m_h^{(p)}}$ is an extension of the compact operator $C_h^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ and $L^p(G) \subseteq L^1\big(m_h^{(p)}\big)$ continuously, the domain $L^1\big(m_h^{(p)}\big)$ of $I_{m_h^{(p)}}$ must be strictly larger than $L^p(G)$. That $L^1\big(m_h^{(p)}\big)$ is a proper subspace of $L^1(G)$ follows from (i) $\Leftrightarrow$ (vii) in Theorem 7.50.

If $h \in L^p(G)$, then $L^1\big(m_h^{(p)}\big) = L^1(G)$ by (i) $\Leftrightarrow$ (vii) in Theorem 7.50. Hence, $L^p(G) \subsetneqq L^1(G)$ shows that the second containment in (7.86) is also proper in this case.

(iii) This is clear from (i) $\Leftrightarrow$ (v) in Theorem 7.50.

(iv) By [55, p. 160], there exists $h \in L^1(G)$ with $\widehat{h}$ belonging to the set $c_0(\Gamma) \setminus \bigcup_{1\le r<\infty} \ell^r(\Gamma)$. Consider first $1 < p \le 2$. If it is the case that $h \in L^1\big(m_h^{(p)}\big)$, then $I_{m_h^{(p)}}(h) = h * h \in L^p(G)$ by (7.81). Then the Hausdorff–Young inequality (7.6) implies that $(\widehat{h})^2 = (h*h)\widehat{\ } \in \ell^{p'}(\Gamma)$, that is, $\widehat{h} \in \ell^{2p'}(\Gamma)$. This contradicts the choice of h and so $h \notin L^1\big(m_h^{(p)}\big)$. Now let $2 < p < \infty$. Assume that $h \in L^1\big(m_h^{(p)}\big)$. Again by (7.81) it follows that $I_{m_h^{(p)}}(h) = h * h \in L^p(G) \subseteq L^2(G)$ and hence, $(\widehat{h})^2 = (h * h)\widehat{\ } \in \ell^2(\Gamma)$. That is, $\widehat{h} \in \ell^4(\Gamma)$, which again contradicts the choice of h. So, $h \notin L^1\big(m_h^{(p)}\big)$. $\square$

Remark 7.53. (i) In the proof of part (iv) of Proposition 7.52 it is necessary to consider the cases $1<p\le 2$ and $2<p<\infty$ separately. Indeed, the Hausdorff–Young inequality fails in $L^p(G)$ for every (infinite) G and every $2 < p < \infty$, [55, p. 151].

(ii) As noted immediately after Example 7.51, every convolution operator $C_h^{(p)}$ for $1\le p<\infty$ and $h\in L^1(G)$ is completely continuous. According to Theorem 7.50, each extended operator $I_{m_h^{(p)}}$ (being compact) is completely continuous whenever $h\in L^p(G)$.

Question. *Is $I_{m_h^{(p)}}$ completely continuous for all $h \in L^1(G)$?*

(iii) The gaps between the proper inclusions in (7.86) are rather "large". That is, if $1 < p < \infty$ and $h \in L^1(G) \setminus L^p(G)$, then both $L^1\big(m_h^{(p)}\big) \setminus L^p(G)$ and $L^1(G) \setminus L^1\big(m_h^{(p)}\big)$, considered as subsets of $L^1(G)$, contain *infinite, linearly independent* subsets, [123, Proposition 4.2].

By Proposition 7.52(iv), there exists $h \in L^1(G)$ with $h \notin \bigcup_{1<p<\infty} L^1\big(m_h^{(p)}\big)$. For such functions h, the set $L^1(G) \setminus \bigcup_{1<p<\infty} L^1\big(m_h^{(p)}\big)$ also contains an infinite, linearly independent subset, [123, p. 539].

(iv) Let $1 < p < \infty$ and fix $h \in L^1(G) \setminus L^p(G)$. Then the compact operator $C_h^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ does not have any Bochner μ-integral representation, that is, there is no $H \in \mathbb{B}(\mu, L^p(G))$ such that $C_h^{(p)}(f) = (\mathrm{B})\text{-}\int_G fH\, d\mu$ for $f \in L^p(G)$, [123, Remark 4.3]. This contrasts with the fact that $C_h^{(1)} \in \mathcal{L}\big(L^1(G)\big)$ is Bochner representable (due to Costé's Theorem; see Theorem 7.30).

The above example of the compact operators $C_h^{(p)}$, with $h \in L^1(G) \setminus L^p(G)$, shows that the Dunford-Pettis Integral Representation Theorem (see Proposition 3.47), stating that every compact operator $T \in \mathcal{L}(L^1(\mu), Z)$ with μ a finite measure and Z a Banach space is Bochner representable, does not have any natural extension to compact operators $T \in \mathcal{L}(L^p(\mu), Z)$ for $1 < p < \infty$. In fact, such operators need not be Pettis representable either! This follows from the facts that the vector measure $m_h^{(p)}$ fails to have σ-finite variation (see Proposition 7.52(iii)) and that vector measures arising as Pettis indefinite integrals necessarily have σ-finite variation (see the discussion immediately prior to Example 3.45). □

Precise information about the surjectivity and injectivity of the extended operator $I_{m_h^{(p)}}$ (beyond that of Remark 7.45(iii)) is also known, [123, Theorem 1.3]. Namely, we have

Proposition 7.54. *Let $1 \le p < \infty$ and $h \in L^1(G) \setminus \{0\}$.*

(i) *The range $I_{m_h^{(p)}}\big(L^1(m_h^{(p)})\big)$ of the extended operator $I_{m_h^{(p)}}$ of $C_h^{(p)}$ is always a proper subspace of $L^p(G)$.*

(ii) *The following assertions are equivalent.*

  (a) *The range $I_{m_h^{(p)}}\big(L^1(m_h^{(p)})\big)$ is a dense subspace of $L^p(G)$.*

  (b) *$\widehat{h}(\gamma) \neq 0$ for every $\gamma \in \Gamma$.*

  (c) *The extended operator $I_{m_h^{(p)}} : L^1\big(m_h^{(p)}\big) \to L^p(G)$ is injective.*

If any one of (a)–(c) *holds in part* (ii) *and $1 < p < \infty$, then the inclusion*

$$C_h^{(p)}\big(L^p(G)\big) \subseteq I_{m_h^{(p)}}\big(L^1(m_h^{(p)})\big)$$

is proper.

Remark 7.55. If there exists $h \in L^1(G)$ such that $I_{m_h^{(p)}}$ (or $C_h^{(p)}$) is injective, then necessarily G is metrizable. Indeed, since $\widehat{h} \in c_0(\Gamma)$, we have that the set $\{\gamma \in \Gamma : \widehat{h}(\gamma) \neq 0\}$ is countable. By (b) $\Leftrightarrow$ (c) in Proposition 7.54(ii) it follows that Γ itself is countable and hence, G is metrizable, [75, Theorem 24.15]. In particular, if G is non-metrizable, then $I_{m_h^{(p)}}$ fails to be injective for every $h \in L^1(G)$ and $1 \leq p < \infty$. $\square$

Let $1 \leq p < \infty$. An operator $T \in \mathcal{L}\big(L^p(G)\big)$ is called a *p-multiplier operator* if it commutes with all translations, that is, $\tau_x \circ T = T \circ \tau_x$ for all $x \in G$. In this case, there exists a unique function $\psi \in \ell^\infty(\Gamma)$ such that

$$(Tf)\widehat{\ } = \psi \widehat{f}, \qquad f \in L^p(G);$$

see [16, Theorem 4.4] or [95, Corollary 4.1.2], for example. The unique function ψ is called the *p-multiplier* corresponding to T; we also denote T by $T_\psi^{(p)}$. The space of all such p-multipliers ψ is denoted by $\mathcal{M}_p(G)$. It was already noted at the beginning of Section 7.2 that $\mathcal{M}_1(G) = \big\{\widehat{\lambda} : \lambda \in M(G)\big\}$ and, for each $\widehat{\lambda} \in \mathcal{M}_1(G)$, that the corresponding 1-multiplier operator $T_{\widehat{\lambda}}^{(1)}$ equals $C_\lambda^{(1)}$. For $p = 2$ it is known that $\mathcal{M}_2(G) = \ell^2(\Gamma)$, [95, Theorem 4.1.1]. No characterizations of $\mathcal{M}_p(G)$ are known for $p \neq 1, 2$. Of course, in this case, $\mathcal{M}_p(G)$ is a *proper* subset of $\ell^\infty(\Gamma)$. Since the p-multiplier operators form a commutative, unital subalgebra of $\mathcal{L}\big(L^p(G)\big)$, a p-multiplier operator $T \in \mathcal{L}\big(L^p(G)\big)$ is idempotent if and only if its corresponding p-multiplier is equal to χ_A for some subset $A \subseteq \Gamma$. In this case, A is called a *p-multiplier set*.

A linear map $P : Z \to Z$, with Z a Banach space, is a *projection* if $P^2 = P$. Let W and Y be linear subspaces of Z satisfying $W \cap Y = \{0\}$ and $Z = W + Y$. Then we say that Z admits an *algebraic direct sum decomposition* via W and Y, and write $Z = W \dotplus Y$. In this case there is a unique projection $P : Z \to Z$ satisfying $P(Z) = W$ and $(I - P)(Z) = Y$. If, in addition, P is continuous, we write $Z = W \oplus Y$ and say that Z admits a *topological* direct sum decomposition via W and Y. In this case, both W and Y are closed because $Y = P^{-1}(\{0\})$ and $W = (I - P)^{-1}(\{0\})$. A closed subspace V of Z is said to be *complemented* in Z if there exists a closed subspace U of Z satisfying $Z = V \oplus U$.

Our final result of this section (see [123, Theorem 1.4]) makes an important connection between the range of the extended operator $I_{m_h^{(p)}} : L^1\big(m_h^{(p)}\big) \to L^p(G)$ and elements of $\mathcal{M}_p(G)$. For any subset $A \subseteq \Gamma$, the linear span of those trigonometric polynomials on G whose Fourier transform has its support in A is denoted by $\mathcal{T}(G, A)$. Recall, for $\xi \in \ell^\infty(\Gamma)$, that its *support* is the subset of Γ given by

$$\operatorname{supp}(\xi) := \{\gamma \in \Gamma : \xi(\gamma) \neq 0\}.$$

Proposition 7.56. *Let $1 \le p < \infty$ and $h \in L^1(G) \setminus \{0\}$. The following assertions are equivalent.*

(i) $\chi_{\mathrm{supp}\,(\widehat{h})} \in \mathcal{M}_p(G)$.

(ii) *The Banach space $L^p(G)$ admits the algebraic direct sum decomposition*

$$L^p(G) = \overline{I_{m_h^{(p)}}\big(L^1(m_h^{(p)})\big)} \dotplus \overline{\mathcal{T}\big(G,\ \Gamma \setminus \mathrm{supp}(\widehat{h})\big)}.$$

(iii) *The Banach space $L^p(G)$ admits the topological direct sum decomposition*

$$L^p(G) = \overline{I_{m_h^{(p)}}\big(L^1(m_h^{(p)})\big)} \oplus \overline{\mathcal{T}\big(G,\ \Gamma \setminus \mathrm{supp}(\widehat{h})\big)}.$$

For the restricted case $1 < p < \infty$, any one of (i)–(iii) *is equivalent to:*

(iv) *The closed subspace $\overline{I_{m_h^{(p)}}\big(L^1(m_h^{(p)})\big)}$ is complemented in $L^p(G)$.*

We end with a remark concerning harmonic analysis. Given any countable set $A \subseteq \Gamma$ there always exists $h \in L^1(G)$ satisfying $\mathrm{supp}(\widehat{h}) = A$. Indeed, if $\{\gamma_n\}_{n=1}^{\infty}$ is any enumeration of A, then the continuous function $h := \sum_{n=1}^{\infty} n^{-2}(\cdot, \gamma_n)$ has the required property. Restrict attention now to the circle group $G := \mathbb{T}$ (hence, $\Gamma = \mathbb{Z}$) and let $1 < p < r \le 2$. Then there exists an r-multiplier set $A \subseteq \mathbb{Z}$ which is *not* a p-multiplier set, [112]. Choose $h \in L^1(\mathbb{T})$ with $\mathrm{supp}(\widehat{h}) = A$. By Proposition 7.56 we have the topological direct sum decomposition

$$L^r(\mathbb{T}) = \overline{I_{m_h^{(r)}}\big(L^1(m_h^{(r)})\big)} \oplus \overline{\mathcal{T}(\mathbb{T},\ \mathbb{Z} \setminus A)},$$

whereas such a decomposition does not hold for $L^p(\mathbb{T})$.

Indeed, the space $\overline{I_{m_h^{(p)}}\big(L^1(m_h^{(p)})\big)}$ is not complemented in $L^p(\mathbb{T})$ at all!

7.4 Operators acting in $L^p(G)$ via convolution with measures

The aim of this section is to study the class of convolution operators $C_\lambda^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ as given by (7.7), for $1 \le p < \infty$ and *arbitrary* $\lambda \in M(G)$, from the viewpoint of their optimal domain and the extended operator $I_{m_\lambda^{(p)}}$. For the corresponding vector measure $m_\lambda^{(p)} : \mathcal{B}(G) \to L^p(G)$ as given by (7.70), the properties and identification of the optimal domain space $L^1\big(m_\lambda^{(p)}\big)$ are important as well as certain operator theoretic properties of the extended operator $I_{m_\lambda^{(p)}}$ of $C_\lambda^{(p)}$. The case when $\lambda \ll \mu$ was treated in Section 7.3. Here we are mainly interested in the measures $\lambda \in M(G) \setminus L^1(G)$. Although there are several similar features to the results of Section 7.3, there are also many differences, some quite surprising. For instance, it can happen that $L^1\big(m_\lambda^{(p)}\big) = L^p(G)$ (i.e., no further extension of $C_\lambda^{(p)}$ is possible), a phenomenon which is impossible for $\lambda \ll \mu$; see Proposition 7.52(ii).

So, some new techniques and arguments will be required. Some of the results of this section occur in the recent paper [124] and hence, will only be (appropriately) summarized. However, many others are new; these will (of course!) come with complete proofs.

We begin by characterizing compactness of the convolution operators $C_\lambda^{(p)}$. If $p = 1$, then $C_\lambda^{(1)} \in \mathcal{L}\big(L^1(G)\big)$ is compact if and only if $\lambda \ll \mu$; see the discussion immediately prior to Theorem 7.30. For $1 < p < \infty$ some preparatory facts are needed.

For each measure $\lambda \in M(G)$, we recall that its reflection is denoted by $R\lambda$ and satisfies (7.52). The following result occurs as Lemma 2.2 in [124].

Lemma 7.57. *Let $1 \le p < \infty$. For each $\lambda \in M(G)$ the following assertions hold.*

(i) *If $\varphi \in L^{p'}(G) = L^p(G)^*$, then $\varphi * R\lambda \in L^{p'}(G)$ and*

$$\big\langle C_\lambda^{(p)}(f),\, \varphi\big\rangle = \big\langle f,\, \varphi * R\lambda\big\rangle, \qquad f \in L^p(G). \tag{7.87}$$

(ii) *The dual operator $\big(C_\lambda^{(p)}\big)^*$ of $C_\lambda^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ is the convolution operator, from $L^{p'}(G)$ into $L^{p'}(G)$, defined by $\varphi \mapsto \varphi * R\lambda$. In particular, if $1 < p < \infty$, then $\big(C_\lambda^{(p)}\big)^* = C_{R\lambda}^{(p')}$.*

Those measures $\lambda \in M(G)$ for which $C_\lambda^{(p)}$ is actually compact can now be described; see [124, Proposition 2.3] for the equivalences (i)–(iii) below, where the given proof is based on Lemma 7.57 above and Schauder's Theorem. Since $\mathcal{R}\big(m_\lambda^{(p)}\big) = \big\{I_{m_\lambda^{(p)}}(\chi_A) : A \in \mathcal{B}(G)\big\}$ and $X(\mu) := L^1\big(m_\lambda^{(p)}\big)$ is a B.f.s. over $(G, \mathcal{B}(G), \mu)$ (see Lemma 7.59 below) with σ-o.c. norm, the equivalence (iii) $\Leftrightarrow$ (iv) below follows from Proposition 2.41.

Proposition 7.58. *For each $\lambda \in M(G)$, the following statements are equivalent.*

(i) *$\lambda \in M_0(G)$.*

(ii) *The convolution operator $C_\lambda^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ is compact for some (every) value of $1 < p < \infty$.*

(iii) *The range $\mathcal{R}\big(m_\lambda^{(p)}\big)$ of the vector measure $m_\lambda^{(p)}$ is a relatively compact subset of $L^p(G)$ for some (every) $1 < p < \infty$.*

(iv) *$I_{m_\lambda^{(p)}} : L^1\big(m_\lambda^{(p)}\big) \to L^p(G)$ maps every bounded, uniformly μ-integrable subset of $L^1\big(m_\lambda^{(p)}\big)$ to a relatively compact subset of $L^p(G)$.*

We point out, in particular, that $C_h^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ is a compact operator for every $h \in L^1(G) \subseteq M_0(G)$. This was already noted at the beginning of Section 7.3. Proposition 7.58 should be compared with Lemma 7.42(i).

Various properties of the vector measures $m_\lambda^{(p)}$, for $\lambda \in M(G)$ and $1 \le p < \infty$, will also be required. The σ-additivity of $m_\lambda^{(p)}$ is immediate from the continuity of $C_\lambda^{(p)}$ (see (7.8)) and the definition of $m_\lambda^{(p)}$ (see (7.70)). For each $\varphi \in L^{p'}(G) = L^p(G)^*$, it follows from (7.87) that

$$\langle m_\lambda^{(p)}, \varphi\rangle(A) = \int_A (\varphi * R\lambda)\, d\mu, \qquad A \in \mathcal{B}(G), \tag{7.88}$$

and hence, that

$$\big|\langle m_\lambda^{(p)}, \varphi\rangle\big|(A) = \int_A |\varphi * R\lambda|\, d\mu, \qquad A \in \mathcal{B}(G). \tag{7.89}$$

These formulae, which should be compared with (7.73) and (7.74), are crucial for the proof of the following result (see [124, Proposition 2.4]).

Lemma 7.59. *Let $1 \le p < \infty$ and fix $\lambda \in M(G)$.*

(i) *Given any set $A \in \mathcal{B}(G)$, its semivariation equals*

$$\big\|m_\lambda^{(p)}\big\|(A) = \sup\left\{ \int_A |\varphi * R\lambda|\, d\mu : \varphi \in L^{p'}(G),\ \|\varphi\|_{L^{p'}(G)} \le 1 \right\}$$

and satisfies

$$\|\widehat{\lambda}\|_{\ell^\infty(\Gamma)}\, \mu(A) \le \big\|m_\lambda^{(p)}\big\|(A) \le \|\lambda\|_{M(G)}\, \big(\mu(A)\big)^{1/p}.$$

(ii) *The vector measure $m_\lambda^{(p)}$ is always absolutely continuous with respect to μ. Conversely, if $\lambda \ne 0$, then μ is absolutely continuous with respect to $m_\lambda^{(p)}$.*

Remark 7.60. (i) It follows from part (ii) of Lemma 7.59 that the convolution operator $C_\lambda^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ is *μ-determined* whenever $1 \le p < \infty$ and the measure $\lambda \in M(G) \setminus \{0\}$.

(ii) Lemma 7.59 should be compared with parts (ii) and (iii) of Lemma 7.42. Note that part (i) of Lemma 7.42 has no analogue in Lemma 7.59, for a good reason; see Proposition 7.58. □

The following result (see Theorem 1.1 and Corollary 3.2 of [124]) summarizes the essential properties of the optimal domain space $L^1\big(m_\lambda^{(p)}\big)$ of $C_\lambda^{(p)}$. It should be compared with Theorem 7.44 which corresponds to the case when $\lambda \ll \mu$.

Theorem 7.61. *Let $1 \le p < \infty$ and fix $\lambda \in M(G) \setminus \{0\}$.*

(i) *A $\mathcal{B}(G)$-measurable function $f : G \to \mathbb{C}$ is $m_\lambda^{(p)}$-integrable if and only if*

$$\int_G |f| \cdot |\varphi * R\lambda|\, d\mu < \infty, \qquad \varphi \in L^{p'}(G) = L^p(G)^*.$$

Moreover, the norm of $f \in L^1\big(m_\lambda^{(p)}\big)$ is given by

$$\|f\|_{L^1(m_\lambda^{(p)})} = \sup\left\{ \int_G |f| \cdot |\varphi * R\lambda|\, d\mu : \varphi \in L^{p'}(G),\ \|\varphi\|_{L^{p'}(G)} \le 1 \right\}. \tag{7.90}$$

(ii) *The natural inclusions*

$$L^p(G) \subseteq L^1\big(m_\lambda^{(p)}\big) = L^1_{\mathrm{w}}\big(m_\lambda^{(p)}\big) \subseteq L^1(G) \tag{7.91}$$

hold and are continuous. Moreover, if

$$J_\lambda^{(p)} : L^p(G) \to L^1\big(m_\lambda^{(p)}\big)$$

denotes the natural injection, then

$$\|\widehat{\lambda}\|_{\ell^\infty(\Gamma)} \le \big\|J_\lambda^{(p)}\big\| \le \|\lambda\|_{M(G)} \tag{7.92}$$

and, if $\Lambda_\lambda^{(p)} : L^1\big(m_\lambda^{(p)}\big) \to L^1(G)$ *denotes the natural injection, then*

$$\Big[\|m_\lambda^{(p)}\|(G)\Big]^{-1} \le \|\Lambda_\lambda^{(p)}\| \le \big[\|\widehat{\lambda}\|_{\ell^\infty(\Gamma)}\big]^{-1}.$$

(iii) *For each* $1 \le q \le p$ *we have* $L^1\big(m_\lambda^{(p)}\big) \subseteq L^1(m_\lambda^{(q)})$ *continuously and the natural injection* $Q_\lambda^{(q,p)} : L^1\big(m_\lambda^{(p)}\big) \to L^1(m_\lambda^{(q)})$ *satisfies*

$$\big\|m_\lambda^{(q)}\big\|(G) \cdot \big[\,\|\lambda\|_{M(G)}\big]^{-1} \le \big\|Q_\lambda^{(q,p)}\big\| \le 1.$$

(iv) *The subspaces* $\mathcal{T}(G)$ *and* $L^p(G)$ *are both dense in* $L^1\big(m_\lambda^{(p)}\big)$.

(v) $L^1\big(m_\lambda^{(p)}\big)$ *is a translation invariant subspace of* $L^1(G)$ *which is stable under formation of reflections and complex conjugates. Moreover, for each* $x \in G$, *we have*

$$\Big(I_{m_\lambda^{(p)}} \circ \tau_x\Big)(f) = \Big(\tau_x \circ I_{m_\lambda^{(p)}}\Big)(f), \qquad f \in L^1\big(m_\lambda^{(p)}\big),$$

where the equality is between elements of $L^p(G)$. *The range of* $I_{m_\lambda^{(p)}}$ *and its closure are both translation invariant subspaces of* $L^p(G)$.

(vi) *The extension of* $C_\lambda^{(p)}$ *to its optimal domain, namely the linear operator* $I_{m_\lambda^{(p)}} : L^1\big(m_\lambda^{(p)}\big) \to L^p(G)$, *has operator norm precisely* 1 *and is given by*

$$I_{m_\lambda^{(p)}}(f) = f * \lambda, \qquad f \in L^1\big(m_\lambda^{(p)}\big). \tag{7.93}$$

We point out that the equality $L^1\big(m_\lambda^{(p)}\big) = L^1_{\mathrm{w}}\big(m_\lambda^{(p)}\big)$ in (7.91) holds for the same reason as explained in Remark 7.45(i).

Define the spaces

$$M_\lambda^{(p)} := \big\{f \in L^1(G) : (f\chi_A) * \lambda \in L^p(G) \text{ for all } A \in \mathcal{B}(G)\big\}$$

and

$$N_\lambda^{(p)} := \big\{f \in L^1(G) : |f| * \lambda \in L^p(G)\big\}.$$

Then it turns out that $L^1\big(m_\lambda^{(p)}\big) = M_\lambda^{(p)} = N_\lambda^{(p)}$ for every *non-negative* measure $\lambda \in M(G)$ and that $N_{|\lambda|}^{(p)} \subseteq L^1\big(m_\lambda^{(p)}\big)$ for arbitrary measures $\lambda \in M(G)$, [124, Proposition 3.3]; this inclusion can be strict (see Proposition 7.105). For the case when $\lambda \ll \mu$ we refer to Remark 7.45(iii).

An examination of the proof of Proposition 7.47 shows that if η^f as given there is replaced with the set function

$$A \mapsto \eta^f(A) := (\chi_A f) * \lambda, \qquad A \in \mathcal{B}(G),$$

and we use (7.88) in place of (7.73), then the same argument can be adapted to prove the following result.

Proposition 7.62. *Let $1 \le p < \infty$ and $\lambda \in M(G) \setminus \{0\}$. Then the optimal domain $L^1\big(m_\lambda^{(p)}\big)$ of $C_\lambda^{(p)}$ is given by*

$$L^1\big(m_\lambda^{(p)}\big) \;=\; \big\{f \in L^1(G) : (f\chi_A) * \lambda \in L^p(G) \;\; \textit{for all} \;\; A \in \mathcal{B}(G)\big\}.$$

Some further comments concerning (7.92) are worthwhile. Indeed, the following result gives more precise details about the exact value of the operator norms $\big\|J_\lambda^{(p)}\big\|$.

Proposition 7.63. *Let $1 \le p < \infty$ and $J_\lambda^{(p)} \in \mathcal{L}\big(L^p(G),\, L^1(m_\lambda^{(p)})\big)$ be the embedding as given in the statement of* Theorem 7.61(ii)*.*

(i) *For each $\lambda \in M(G)$ we have $\big\|J_\lambda^{(p)}\big\| = \big\|C_\lambda^{(p)}\big\|$.*

(ii) *If $\lambda \ge 0$, then the second inequality in* (7.92) *is actually an equality, that is,*

$$\|J_\lambda^{(p)}\| \;=\; \lambda(G) \;=\; \|\widehat{\lambda}\|_{\ell^\infty(\Gamma)}.$$

(iii) *For each $\lambda \in M(G)\setminus\{0\}$ and $1 < p < \infty$, the operator $J_\lambda^{(p)}$ is weakly compact, but fails to be completely continuous or compact.*

Proof. (i) According to Lemma 7.57(ii) and the fact that the norm of an operator equals the norm of its dual operator, [46, Ch. VI, Lemma 2.1], it suffices to show that $\big\|J_\lambda^{(p)}\big\| = \|C_{R\lambda}^{(p')}\|$. Now, with $\|\cdot\|_p$ and $\|\cdot\|_{p'}$ denoting $\|\cdot\|_{L^p(G)}$ and $\|\cdot\|_{L^{p'}(G)}$ respectively, we have

$$\big\|J_\lambda^{(p)}\big\| = \sup_{\|f\|_p \le 1} \big\|J_\lambda^{(p)}(f)\big\|_{L^1(m_\lambda^{(p)})} = \sup_{\|f\|_p \le 1} \; \sup_{\|\varphi\|_{p'} \le 1} \int_G |f| \cdot |\varphi * R\lambda| \, d\mu,$$

where we have used (7.90). Exchanging suprema gives

$$\begin{aligned} \big\|J_\lambda^{(p)}\big\| &= \sup_{\|\varphi\|_{p'} \le 1} \; \sup_{\|f\|_p \le 1} \int_G |f| \cdot |\varphi * R\lambda| \, d\mu \\ &= \sup_{\|\varphi\|_{p'} \le 1} \|\varphi * R\lambda\|_{p'} = \big\|C_{R\lambda}^{(p')}\big\|. \end{aligned}$$

(ii) For $\lambda \geq 0$ we have $\|\lambda\|_{M(G)} = \lambda(G)$ and so $\left\|J_\lambda^{(p)}\right\| \leq \lambda(G)$ follows from (7.92).

On the other hand, since also $R\lambda \geq 0$, it is routine to check that $\chi_G * R\lambda = (R\lambda)(G)\chi_G = \lambda(G)\chi_G$ is a constant, non-negative function on G. By noting that the constant function $\mathbf{1}$ belongs to the unit ball of $L^{p'}(G) = L^p(G)^*$, it follows from (7.90) that

$$\begin{aligned}\left\|J_\lambda^{(p)}\right\| &\geq \left\|J_\lambda^{(p)}(\mathbf{1})\right\|_{L^1(m_\lambda^{(p)})} = \|\mathbf{1}\|_{L^1(m_\lambda^{(p)})} \\ &= \sup_{\|\varphi\|_{p'}\leq 1} \int_G |\mathbf{1}| \cdot |\varphi * R\lambda| \, d\mu \geq \int_G \mathbf{1} \cdot |\chi_G * R\lambda| \, d\mu = \lambda(G).\end{aligned}$$

So, we have established the first equality $\left\|J_\lambda^{(p)}\right\| = \lambda(G)$.

For the second equality in part (ii), it is clear from (7.2) that $\|\widehat{\lambda}\|_{\ell^\infty(\Gamma)} \leq |\lambda|(G) = \lambda(G)$. On the other hand, also

$$\|\widehat{\lambda}\|_{\ell^\infty(\Gamma)} \geq |\widehat{\lambda}(e)| = |\lambda(G)| = \lambda(G).$$

(iii) Since $L^p(G)$ is reflexive, it is clear that $J_\lambda^{(p)}$ is weakly compact. Also, the reflexivity of $L^p(G)$ implies that complete continuity and compactness of $J_\lambda^{(p)}$ coincide. So, it suffices to show that $J_\lambda^{(p)}$ is not completely continuous. Assume the contrary. Since $L^p(G) \subseteq L^1\big(m_\lambda^{(p)}\big) \subseteq L^1(G)$ continuously, it follows that the natural inclusion

$$W_p : L^p(G) \to L^1(G)$$

is also completely continuous. Choose a distinct sequence of elements $\{\gamma_n\}_{n=1}^\infty \subseteq \Gamma$, in which case the sequence $\big\{(\cdot, \gamma_n)\big\}_{n=1}^\infty$ converges weakly to 0 in $L^p(G)$ (by the Riemann–Lebesgue Lemma). By complete continuity of W_p it follows that $\big\{(\cdot, \gamma_n)\big\}_{n=1}^\infty$ is norm convergent (in $L^1(G)$) to some $h \in L^1(G)$. Since $\big\{(\cdot, \gamma_n)\big\}_{n=1}^\infty$ also converges weakly to 0 in $L^1(G)$ (again by the Riemann–Lebesgue Lemma), it follows that $h = 0$, that is $\big\|(\cdot, \gamma_n)\big\|_{L^1(G)} \to 0$ as $n \to \infty$. This is nonsense because $\big\|(\cdot, \gamma_n)\big\|_{L^1(G)} = 1$ for all $n \in \mathbb{N}$. So, W_p and hence, also $J_\lambda^{(p)}$, cannot be completely continuous. □

Remark 7.64. (i) Let $M_{\widehat{\lambda}} \in \mathcal{L}\big(\ell^2(\Gamma)\big)$ denote the operator of (coordinatewise) multiplication by $\widehat{\lambda} \in \ell^\infty(\Gamma)$, in which case $\|M_{\widehat{\lambda}}\| = \|\widehat{\lambda}\|_{\ell^\infty(\Gamma)}$. Since the Fourier transform map $F_2 : L^2(G) \to \ell^2(\Gamma)$ is an isometric isomorphism and satisfies $C_\lambda^{(2)} = F_2^{-1} \circ M_{\widehat{\lambda}} \circ F_2$, it follows that

$$\left\|J_\lambda^{(2)}\right\| = \left\|C_\lambda^{(2)}\right\| = \|M_{\widehat{\lambda}}\| = \|\widehat{\lambda}\|_{\ell^\infty(\Gamma)}$$

holds for *arbitrary* $\lambda \in M(G)$, not just for $\lambda \geq 0$.

Consider now $p = 1$ and those measures $\lambda = \mu_h$ with $h \in L^1(G)$; see (7.3). Then

$$C_h^{(1)}(f) : x \mapsto \int_G h(y-x)\, f(y)\, d\mu(y), \qquad x \in G,$$

for $f \in L^1(G)$, is a *kernel operator*. Hence, its absolute value $\big|C_h^{(1)}\big|$ in the Banach lattice of regular operators on $L^1(G)$ is precisely $C_{|h|}^{(1)}$, [149, pp. 282–283]. Since $L^1(G)$ is an abstract L^1-space, it has property (P) in the sense of [149, p. 251]. It then follows from Theorem 1.5 and Corollary 2 (p. 235) of [149, Ch. IV] that

$$\big\|C_{|h|}^{(1)}\big\| = \|h\|_{L^1(G)} = \big\|\,\big|C_h^{(1)}\big|\,\big\| = \big\|C_h^{(1)}\big\|.$$

In particular, it follows from the previous paragraph that

$$\big\|J_{\mu_h}^{(2)}\big\| = \big\|C_h^{(2)}\big\| = \|\widehat{h}\|_{\ell^\infty(\Gamma)} \le \|h\|_{L^1(G)} = \big\|C_h^{(1)}\big\| = \big\|J_{\mu_h}^{(1)}\big\|,$$

for every $h \in L^1(G)$. It would be interesting to know, even for those measures $\lambda = \mu_h$ with $h \in L^1(G)$, what the dependence of the operator norm $\big\|C_\lambda^{(p)}\big\|$ of $C_\lambda^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ is as a function of p and λ.

(ii) For each $1 < p < \infty$ and $\lambda \in M(G) \setminus \{0\}$, the natural injection

$$\Lambda_\lambda^{(p)} : L^1\big(m_\lambda^{(p)}\big) \to L^1(G)$$

fails to be completely continuous or compact. Indeed, if $\Lambda_\lambda^{(p)}$ is completely continuous (resp. compact), then also the inclusion $W_p : L^p(G) \to L^1(G)$ is completely continuous (resp. compact), which is surely not the case (see the proof of Proposition 7.63(iii)). □

The same arguments used in the proof of Proposition 7.21 and Remark 7.22 also apply to yield the following Banach space properties of the optimal domain spaces $L^1\big(m_\lambda^{(p)}\big)$. Note that the special case when $\lambda \ll \mu$ is precisely Proposition 7.48.

Proposition 7.65. *Let $1 \le p < \infty$. Each of the following assertions is valid for every $\lambda \in M(G)$.*

(i) $L^1\big(m_\lambda^{(p)}\big)$ *is weakly sequentially complete and hence, its associate space* $L^1\big(m_\lambda^{(p)}\big)' = L^1\big(m_\lambda^{(p)}\big)^*$.

(ii) $L^\infty(G) \subseteq L^1\big(m_\lambda^{(p)}\big)'$ *and* $L^1\big(m_\lambda^{(p)}\big)$ *is also sequentially complete for* $\sigma\big(L^1\big(m_\lambda^{(p)}\big), L^\infty(G)\big)$.

(iii) $L^1\big(m_\lambda^{(p)}\big)$ *has σ-o.c. norm and satisfies the σ-Fatou property.*

(iv) $L^1\big(m_\lambda^{(p)}\big)$ *is weakly compactly generated.*

(v) *Whenever G is metrizable, the Banach space* $L^1\big(m_\lambda^{(p)}\big)$ *is separable.*

The following result, which shows that the spaces $L^1\big(m_\lambda^{(p)}\big)$ are well suited for harmonic analysis, can be established by appropriately modifying the proof of Theorem 7.49. Indeed, wherever $R_G h$ occurs in that proof simply replace it with $R\lambda$, and replace the use of the identity (7.74) with the identity (7.89).

Theorem 7.66. *Let $1 \leq p < \infty$ and $\lambda \in M(G)\setminus\{0\}$. For each $a \in G$ the translation operator $\tau_a : L^1\big(m_\lambda^{(p)}\big) \to L^1\big(m_\lambda^{(p)}\big)$ is a surjective isometry. Moreover, for each $a_0 \in G$ and $f \in L^1\big(m_\lambda^{(p)}\big)$ we have $\lim_{a\to a_0} \tau_a f = \tau_{a_0} f$ in the norm of $L^1\big(m_\lambda^{(p)}\big)$. In particular, $L^1\big(m_\lambda^{(p)}\big)$ is a homogeneous Banach space on G, up to an equivalent lattice norm.*

The compactness of $C_\lambda^{(p)}$ is completely characterized by Proposition 7.58. The following result, which characterizes the compactness of the optimal extension $I_{m_\lambda^{(p)}} : L^1\big(m_\lambda^{(p)}\big) \to L^p(G)$ of $C_\lambda^{(p)}$, is a combination of Theorem 1.2, Remark 4.2(b) and Proposition 4.3 of [124]. The special case of $\lambda \ll \mu$ occurs in Theorem 7.50.

Theorem 7.67. *Let $1 < p < \infty$ and fix $\lambda \in M(G) \setminus \{0\}$. Then the following assertions are equivalent.*

(i) *There exists $h \in L^p(G)$ such that $\lambda = \mu_h$; see* (7.3).

(ii) *The extension $I_{m_\lambda^{(p)}} : L^1\big(m_\lambda^{(p)}\big) \to L^p(G)$ of $C_\lambda^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ to its optimal domain space $L^1\big(m_\lambda^{(p)}\big)$ is a compact operator.*

(iii) *There exists $H \in \mathbb{B}(\mu, L^p(G))$ such that*
$$m_\lambda^{(p)}(A) = (\mathrm{B})\text{-}\int_A H\,d\mu, \qquad A \in \mathcal{B}(G).$$

(iv) *There exists $A_0 \in \mathcal{B}(G)$ with $\mu(A_0) > 0$ and a Bochner integrable function $H_0 : A_0 \to L^p(G)$ with respect to μ restricted to A_0 such that*
$$m_\lambda^{(p)}(A \cap A_0) = (\mathrm{B})\text{-}\int_{A\cap A_0} H_0\,d\mu, \qquad A \in \mathcal{B}(G).$$

(v) *There exists $F \in \mathbb{P}\big(\mu, L^p(G)\big)$ such that*
$$m_\lambda^{(p)}(A) = (\mathrm{P})\text{-}\int_A F\,d\mu, \qquad A \in \mathcal{B}(G).$$

(vi) *The vector measure $m_\lambda^{(p)} : \mathcal{B}(G) \to L^p(G)$ has finite variation.*

(vii) *There is a set $A_0 \in \mathcal{B}(G)$ satisfying $0 < \big|m_\lambda^{(p)}\big|(A_0) < \infty$.*

(viii) $L^1\big(\big|m_\lambda^{(p)}\big|\big) = L^1(G)$.

(ix) $L^1\big(m_\lambda^{(p)}\big) = L^1(G)$.

(x) $L^1\big(\big|m_\lambda^{(p)}\big|\big) = L^1\big(m_\lambda^{(p)}\big)$.

Moreover, for $p = 1$, the statements (i)–(v) *are equivalent.*

Remark 7.68. (i) For $p = 1$, each of the statements (vi)–(x) always holds individually without assuming any extra conditions on λ; see Lemma 7.33 and Proposition 7.35.

(ii) Let $1 < p < \infty$ and $\lambda \in M_0(G) \setminus L^p(G)$. By Proposition 7.58 the convolution operator $C_\lambda^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ is compact, but its optimal extension $I_{m_\lambda^{(p)}} : L^1\big(m_\lambda^{(p)}\big) \to L^p(G)$ fails to be compact (cf. Theorem 7.67). However, the composition $F_{p,0} \circ I_{m_\lambda^{(p)}} : L^1\big(m_\lambda^{(p)}\big) \to c_0(\Gamma)$ is compact since

$$F_{p,0} \circ I_{m_\lambda^{(p)}} = F_{1,0} \circ C_\lambda^{(1)} \circ \Lambda_\lambda^{(p)}$$

and $F_{1,0} \circ C_\lambda^{(1)}$ is compact (cf. Theorem 7.31). $\square$

We now turn our attention to a closer look at the optimal domain spaces $L^1\big(m_\lambda^{(p)}\big)$ for arbitrary $\lambda \in M(G)$. Some preparatory results will be required.

Lemma 7.69. *Let $1 \le p < \infty$ and $\lambda, \eta \in M(G)$. Then*

$$m_{\lambda+\eta}^{(p)} = m_\lambda^{(p)} + m_\eta^{(p)} \tag{7.94}$$

and

$$L^p(G) \subseteq L^1\big(m_\lambda^{(p)}\big) \cap L^1\big(m_\eta^{(p)}\big) \subseteq L^1\big(m_{\lambda+\eta}^{(p)}\big). \tag{7.95}$$

If $L^1\big(m_\eta^{(p)}\big) = L^p(G)$ for some $p \in (1, \infty)$, then also

$$L^1\big(m_\lambda^{(p)}\big) \cap L^1\big(m_{\lambda+\eta}^{(p)}\big) = L^p(G). \tag{7.96}$$

Proof. The identity (7.94) follows from the formula $f * (\lambda + \eta) = (f * \lambda) + (f * \eta)$, valid for each $f \in L^p(G)$.

The first inclusion in (7.95) is clear as $L^p(G) \subseteq L^1\big(m_\kappa^{(p)}\big)$ for all $\kappa \in M(G)$. The second inclusion in (7.95) is immediate from the identity

$$(f\chi_A) * (\lambda + \eta) = \big((f\chi_A) * \lambda\big) + \big((f\chi_A) * \eta\big), \qquad A \in \mathcal{B}(G),$$

valid for each $f \in L^1(G)$, and Proposition 7.62.

Concerning (7.96), it is clear that $L^p(G) \subseteq L^1\big(m_\lambda^{(p)}\big) \cap L^1\big(m_{\lambda+\eta}^{(p)}\big)$. On the other hand, (7.95) implies that

$$L^1\big(m_\lambda^{(p)}\big) \cap L^1\big(m_{\lambda+\eta}^{(p)}\big) = L^1\big(m_{-\lambda}^{(p)}\big) \cap L^1\big(m_{\lambda+\eta}^{(p)}\big) \subseteq L^1\big(m_\eta^{(p)}\big) = L^p(G).$$

These two containments imply (7.96). $\square$

Corollary 7.70. *Let $1 \le p < \infty$ and $\lambda, \eta \in M(G)^+$. Then*

$$L^1\big(m_\lambda^{(p)}\big) \cap L^1\big(m_\eta^{(p)}\big) = L^1\big(m_{\lambda+\eta}^{(p)}\big). \tag{7.97}$$

Proof. Since $\lambda + \eta \in M(G)^+$, it follows from the discussion immediately after Theorem 7.61 that

$$L^1\big(m^{(p)}_{\lambda+\eta}\big) \;=\; N^{(p)}_{\lambda+\eta} \;=\; \big\{f \in L^1(G) : |f| * (\lambda+\eta) \in L^p(G)\big\}.$$

So, let $f \in L^1\big(m^{(p)}_{\lambda+\eta}\big)$, in which case $|f|*(\lambda+\eta) \in L^p(G)$. Since $|f|*\lambda \le |f|*(\lambda+\eta)$ and $|f| * \eta \le |f| * (\lambda+\eta)$, it follows that both functions $|f| * \lambda$ and $|f| * \eta$ belong to $L^p(G)$ and hence, that $f \in N^{(p)}_\lambda \cap N^{(p)}_\eta = L^1\big(m^{(p)}_\lambda\big) \cap L^1\big(m^{(p)}_\eta\big)$. That is, $L^1\big(m^{(p)}_{\lambda+\eta}\big) \subseteq L^1\big(m^{(p)}_\lambda\big) \cap L^1\big(m^{(p)}_\eta\big)$. The equality in (7.97) then follows from Lemma 7.69. $\square$

Fix $a \in G$. Then $C^{(p)}_{\delta_a} = \tau_a$ is an isomorphism of $L^p(G)$ onto itself. According to Proposition 4.18 we have

$$L^1\big(m^{(p)}_{\delta_a}\big) \;=\; L^p(G), \qquad 1 \le p < \infty, \quad a \in G. \tag{7.98}$$

This phenomenon cannot occur for measures which are absolutely continuous with respect to μ; see Proposition 7.52(ii). We will see soon that there exist many measures, other than just Dirac measures (and not necessarily purely atomic), which also exhibit this feature. We begin with some simple examples.

Lemma 7.71. *Let $1 \le p < \infty$. Given $h \in L^1(G)$ and $a \in G$, we have*

$$L^1\big(m^{(p)}_{\delta_a+\mu_h}\big) \cap L^1\big(m^{(p)}_{-\mu_h}\big) \;=\; L^p(G). \tag{7.99}$$

If either $h \in L^1(G)^+$ or $h \in L^p(G)$, then

$$L^1\big(m^{(p)}_{\delta_a+\mu_h}\big) \;=\; L^p(G). \tag{7.100}$$

Proof. Note that $(\delta_a + \mu_h) + (-\mu_h) \;=\; \delta_a$. So, (7.95) and (7.98) yield

$$L^p(G) \subseteq L^1(m^{(p)}_{\delta_a+\mu_h}) \cap L^1(m^{(p)}_{-\mu_h}) \subseteq L^1\big(m^{(p)}_{\delta_a}\big) \;=\; L^p(G).$$

This is precisely (7.99).

If $h \in L^p(G)$, then Theorem 7.67 (for $1 < p < \infty$) and Theorem 7.50 (for $p = 1$) imply that $L^1(m^{(p)}_{-\mu_h}) = L^1(G)$. In this case, (7.91) shows that the left-hand side of (7.99) equals $L^1(m^{(p)}_{\delta_a+\mu_h})$ and hence, (7.99) reduces to (7.100).

Suppose that $h \in L^1(G)^+$, in which case $\lambda := \delta_a + \mu_h$ belongs to $M(G)^+$. For $f \in L^1\big(m^{(p)}_\lambda\big) = N^{(p)}_\lambda$ we know that $|f| * \lambda \in L^p(G)$. Then the inequalities

$$|\tau_a f| \;=\; |f * \delta_a| \;=\; |f| * \delta_a \le |f| * \lambda \tag{7.101}$$

show that $\tau_a f \in L^p(G)$ and hence, also $f \in L^p(G)$. So, $L^1\big(m^{(p)}_\lambda\big) \subseteq L^p(G)$ which then implies that $L^1\big(m^{(p)}_\lambda\big) = L^p(G)$. This is precisely (7.100) again. $\square$

Remark 7.72. Let $1 < p < \infty$ and $\kappa \in M(G)^+$. Then, for each $a \in G$, we again have that

$$L^1\big(m^{(p)}_{\kappa+\delta_a}\big) \;=\; L^p(G). \tag{7.102}$$

This can be established as in the proof of Lemma 7.71. Indeed, now set $\lambda := \delta_a + \kappa$, in which case $0 \le \delta_a \le \lambda$, and again use inequality (7.101). ☐

The class of examples recorded in Lemma 7.71 and Remark 7.72 will shortly be enlarged by removing the restriction that κ (or μ_h) is a non-negative measure. But, the element $a \in G$ will be required to lie outside the support of κ. First we require a technical fact.

Lemma 7.73. *Let K be a compact subset of G with $0 \notin K$. Then there exists an open neighbourhood W of 0 in G satisfying $W \cap (K+W) = \emptyset$.*

Proof. For each $x \in K$, choose an open neighbourhood V_x of 0 for which we have $V_x \cap (x + V_x) = \emptyset$. Since $\{x + V_x : x \in K\}$ is an open cover of K there exist finitely many elements $x_1, \dots, x_n \in K$ with $K \subseteq \bigcup_{j=1}^n (x_j + V_{x_j})$. Then $V := \bigcap_{j=1}^n V_{x_j}$ is an open neighbourhood of 0 satisfying $V \cap K = \emptyset$. Now choose an open neighbourhood W of 0 such that $W - W \subseteq V$. Then it is routine to check that

$$W \cap \big(x_j + V_{x_j} + W\big) \;=\; \emptyset, \qquad 1 \le j \le n.$$

From these identities it follows that $W \cap (K + W) = \emptyset$. ☐

Recall that the *support* of $\lambda \in M(G)$, denoted by $\operatorname{supp}(\lambda)$, is the smallest closed set $K \subseteq G$ with the property that $|\lambda|(K) = |\lambda|(G)$. Its existence is a consequence of the regularity of λ.

Proposition 7.74. *Let $\lambda \in M(G)$ satisfy $0 \notin \operatorname{supp}(\lambda)$. Then*

$$L^1\big(m^{(p)}_{\lambda+\delta_0}\big) \;=\; L^p(G), \qquad 1 < p < \infty. \tag{7.103}$$

Proof. With $K := -\operatorname{supp}(\lambda)$, in which case $0 \notin K$, it follows from Lemma 7.73 that there exists an open neighbourhood U of 0 in G satisfying

$$\big(U - \operatorname{supp}(\lambda)\big) \cap U = \emptyset.$$

We proceed by contradiction. So, suppose that there exists a function $h \in L^1\big(m^{(p)}_{\lambda+\delta_0}\big) \setminus L^p(G)$. Since $\{x + U : x \in G\}$ is an open cover of G, there exist finitely many elements $x_1, \dots, x_n \in G$ satisfying $G = \bigcup_{j=1}^n (x_j + U)$. The inequality $|h| \le \sum_{j=1}^n |h|\chi_{x_j+U}$ together with $|h| \notin L^p(G)$ implies that $|h|\chi_{x_j+U} \notin L^p(G)$ for some $j \in \{1, \dots, n\}$ and hence, also $h\chi_{x_j+U} \notin L^p(G)$. Accordingly, there exists $x \in G$ such that $h\chi_{x+U} \notin L^p(G)$. Note that $h\chi_{x+U} = h\tau_x(\chi_U)$. Define

$$f := \tau_{-x}\big(h\chi_{x+U}\big) \;=\; \chi_U \tau_{-x}(h). \tag{7.104}$$

Since $h\chi_{x+U} \notin L^p(G)$ also $f = \tau_{-x}\big(h\chi_{x+U}\big) \notin L^p(G)$.

Now, $h \in L^1(m^{(p)}_{\lambda+\delta_0})$ implies that $\tau_{-x}(h) \in L^1(m^{(p)}_{\lambda+\delta_0})$ (see Theorem 7.61(v)) and hence, by the ideal property, also $\chi_U \tau_{-x}(h) \in L^1(m^{(p)}_{\lambda+\delta_0})$. That is, the function $f \in L^1(m^{(p)}_{\lambda+\delta_0})$. It follows that

$$(f * \lambda) + f \;=\; f * (\lambda + \delta_0) \;=\; I_{m^{(p)}_{\lambda+\delta_0}}(f) \in L^p(G) \tag{7.105}$$

and hence, also $\big(f * (\lambda + \delta_0)\big)\chi_U \in L^p(G)$. But, according to (7.104) and (7.105) we have

$$\big(f * (\lambda + \delta_0)\big)\chi_U \;=\; (f * \lambda)\chi_U + f\chi_U \;=\; (f * \lambda)\chi_U + f$$

and so $(f * \lambda)\chi_U + f \in L^p(G)$. If we can show that

$$(f * \lambda)\chi_U \equiv 0, \tag{7.106}$$

then $f \in L^p(G)$ will follow and we have the desired contradiction. That is, no such function $h \in L^1(m^{(p)}_{\lambda+\delta_0}) \setminus L^p(G)$ exists, from which (7.103) follows.

So, it remains to establish (7.106). Recall, for $u \in G$, that

$$(f * \lambda)(u) \;=\; \int_G f(u - y)\, d\lambda(y) \;=\; \int_{\operatorname{supp}(\lambda)} f(u - y)\, d\lambda(y). \tag{7.107}$$

If $u \in U$, then $\big(U - \operatorname{supp}(\lambda)\big) \cap U = \emptyset$ implies that $u - y \notin U$ for all $y \in \operatorname{supp}(\lambda)$. Accordingly, (7.104) implies that

$$f(u - y) \;=\; \chi_U(u - y) \cdot (\tau_{-x}h)(u - y) \;=\; 0, \qquad u \in U, \quad y \in \operatorname{supp}(\lambda),$$

and so (by (7.107)) we can conclude that $(f * \lambda)(u) = 0$ for all $u \in U$. It is then immediate that (7.106) is indeed valid. $\square$

Corollary 7.75. *Let $1 < p < \infty$ and $n \in \mathbb{N}$.*

(i) *Given distinct elements $\{a_j\}_{j=1}^n \subseteq G\setminus\{0\}$ and $\{\beta_j\}_{j=1}^n \subseteq \mathbb{C}\setminus\{0\}$, the measure $\lambda := \delta_0 + \sum_{j=1}^n \beta_j \delta_{a_j}$ satisfies $L^1(m^{(p)}_\lambda) = L^p(G)$.*

(ii) *Let $h \in L^1(G)$ satisfy $h\chi_K = h$ for some compact set $K \subseteq G$ with $0 \notin K$. Then $L^1(m^{(p)}_{\delta_0+\mu_h}) = L^p(G)$.*

In order to extend Proposition 7.74 from δ_0 to δ_a with $a \in G$ arbitrary, we require the following fact.

Lemma 7.76. *Let $1 \le p < \infty$ and $\lambda \in M(G)$. Then*

$$L^1(m^{(p)}_\lambda) \;=\; L^1(m^{(p)}_{\lambda*\delta_a}), \qquad a \in G. \tag{7.108}$$

Proof. Fix $a \in G$. It follows from $(\lambda * \delta_a)(A) = \lambda(A - a)$ that

$$R(\lambda * \delta_a)(A) \;=\; (R\lambda)(A + a), \qquad A \in \mathcal{B}(G).$$

Direct calculation gives

$$\varphi * R(\lambda * \delta_a) \;=\; (\tau_{-a}\varphi) * R\lambda, \qquad \varphi \in L^{p'}(G),$$

with equality as elements of $L^{p'}(G)$. Hence, for each Borel measurable function $f : G \to \mathbb{C}$, it follows that

$$\int_G |f| \, d|\varphi * R(\lambda * \delta_a)| < \infty \iff \int_G |f| \, d|(\tau_{-a}\varphi) * R\lambda| < \infty.$$

Since $\varphi \mapsto \tau_{-a}\varphi$ is an isometric isomorphism of $L^{p'}(G)$ onto itself, (7.108) follows from Theorem 7.61(i). □

Remark 7.77. (i) Let $1 < p < \infty$ and $\lambda \in M(G)$ satisfy $\mathrm{supp}(\lambda) \neq G$. Then

$$L^1\big(m^{(p)}_{\lambda+\delta_a}\big) \;=\; L^p(G), \qquad a \notin \mathrm{supp}(\lambda).$$

To see this, fix $a \notin \mathrm{supp}\,(\lambda)$. Let $\kappa := \lambda + \delta_a$ in which case $\delta_{-a} * \kappa = \delta_0 + (\delta_{-a} * \lambda)$. Since $\mathrm{supp}\,(\delta_{-a} * \lambda) = \mathrm{supp}\,(\lambda) - a$ and $0 \notin \mathrm{supp}\,(\lambda) - a$ (as $a \notin \mathrm{supp}\,(\lambda)$), we have $0 \notin \mathrm{supp}\,(\delta_{-a} * \lambda)$. By Proposition 7.74 we can conclude that $L^1\big(m^{(p)}_{\delta_0+(\delta_{-a}*\lambda)}\big) = L^p(G)$. But,

$$L^1\big(m^{(p)}_{\delta_0+(\delta_{-a}*\lambda)}\big) \;=\; L^1\big(m^{(p)}_{\delta_{-a}*\kappa}\big) \;=\; L^1\big(m^{(p)}_{\kappa}\big),$$

where the last equality follows from Lemma 7.76. Hence, $L^1\big(m^{(p)}_{\kappa}\big) = L^p(G)$ as required.

(ii) Let $\{a_j\}_{j=1}^{\infty}$ be a sequence of distinct elements of G with $\lim_{j\to\infty} a_j = 0$ and let $(\beta_j)_{j=1}^{\infty} \in \ell^1$. For $\eta := \sum_{j=1}^{\infty} \beta_j \delta_{a_j}$ we have

$$L^1\big(m^{(p)}_{\eta}\big) \;=\; L^p(G), \qquad 1 < p < \infty.$$

To see this, we may assume without loss of generality that $a_1 \neq 0$ and $\beta_1 \neq 0$, in which case $a := a_1 \notin \mathrm{supp}\,(\lambda)$ where $\lambda := \sum_{j=2}^{\infty} (\beta_j/\beta_1)\delta_{a_j}$. According to part (i) above we have $L^1\big(m^{(p)}_{\lambda+\delta_a}\big) = L^p(G)$. Then $\beta_1(\lambda + \delta_a) = \eta$ and the identity $L^1\big(m^{(p)}_{\beta\kappa}\big) = L^1\big(m^{(p)}_{\kappa}\big)$, valid for all $\kappa \in M(G)$ and $\beta \in \mathbb{C} \setminus \{0\}$, imply the stated claim.

(iii) If $\lambda \in M(G)^+$ satisfies $\lambda(\{a\}) \neq 0$ for some $a \in G$, then

$$L^1\big(m^{(p)}_{\lambda}\big) \;=\; L^p(G), \qquad 1 < p < \infty. \tag{7.109}$$

Indeed, simply note that $\lambda = \lambda(\{a\})\delta_a + (\lambda - \lambda(\{a\})\delta_a)$ and apply (7.102). As a special case, let $\{\beta_j\}_{j=1}^{\infty} \in (\ell^1)^+$ and $\{a_j\}_{j=1}^{\infty}$ be any sequence of distinct elements in G. Then (7.109) is satisfied for $\lambda := \sum_{j=1}^{\infty} \beta_j \delta_{a_j}$. □

Let $M_c(G)$ denote the subspace of $M(G)$ consisting of all *continuous measures* λ, that is, $\lambda(\{a\}) = 0$ for all $a \in G$. The following observation is the contrapositive statement to (7.109).

Corollary 7.78. *Let $\lambda \in M(G)^+$ satisfy $L^1\big(m_\lambda^{(p)}\big) \neq L^p(G)$ for some $1 < p < \infty$. Then $\lambda \in M_c(G)$.*

The following result follows from the same argument used to establish (7.98).

Proposition 7.79. *Let $a \in G$ and $\lambda \in M_0(G)$. Then*

$$L^1\big(m_{\lambda+\delta_a}^{(p)}\big) \;=\; L^p(G), \qquad 1 < p < \infty. \tag{7.110}$$

Proof. For each p, we note that

$$C_{\lambda+\delta_a}^{(p)} = C_\lambda^{(p)} + C_{\delta_a}^{(p)} \;=\; \tau_a + C_\lambda^{(p)}.$$

The operator $C_\lambda^{(p)}$ is compact because $\lambda \in M_0(G)$ (cf. Proposition 7.58). Consequently, since τ_a is an isomorphism of $L^p(G)$ onto itself, the operator $\tau_a + C_\lambda^{(p)}$ is Fredholm, [150, Theorem 5.10]. Then Proposition 4.18 implies that (7.110) is valid. $\square$

Lemma 7.80. *Let $\lambda, \eta \in M(G)$. Then*

$$L^1\big(m_\eta^{(p)}\big) \;\subseteq\; L^1\big(m_{\lambda*\eta}^{(p)}\big), \qquad 1 < p < \infty.$$

Proof. It follows from the formula

$$(\lambda * \eta)(A) \;=\; \int_G \int_G \chi_A(x+y)\, d\lambda(x) d\eta(y), \qquad A \in \mathcal{B}(G),$$

[140, p. 15], that $R(\lambda * \eta) = (R\lambda) * (R\eta)$ as measures on G. Let $f \in L^1\big(m_\eta^{(p)}\big)$. Given $\psi \in L^{p'}(G)$, also $\varphi := \psi * R\lambda \in L^{p'}(G)$. Accordingly,

$$\int_G |f|\, d|\varphi * R\eta| < \infty \qquad \text{(cf. Theorem 7.61(i))}.$$

But, direct calculation yields that

$$\int_G |f|\, d|\varphi * R\eta| \;=\; \int_G |f|\, d|\psi * R\lambda * R\eta| \;=\; \int_G |f|\, d|\psi * R(\lambda * \eta)|.$$

Hence, $\int_G |f|\, d|\psi * R(\lambda*\eta)| < \infty$ for all $\psi \in L^{p'}(G)$. Then Theorem 7.61(i) implies that $f \in L^1\big(m_{\lambda*\eta}^{(p)}\big)$. $\square$

Corollary 7.81. *Let $\lambda \in M(G)$. If there exists $\eta \in M(G)$ satisfying $\lambda * \eta = \delta_0$, then*

$$L^1\big(m_\lambda^{(p)}\big) \;=\; L^p(G), \qquad 1 < p < \infty.$$

Proof. According to Lemma 7.80 we have

$$L^p(G) \subseteq L^1\big(m_\eta^{(p)}\big) \subseteq L^1\big(m_{\lambda*\eta}^{(p)}\big) = L^1\big(m_{\delta_0}^{(p)}\big).$$

Since $L^1\big(m_{\delta_0}^{(p)}\big) = L^p(G)$ (see(7.98)), the desired equality follows. □

Remark 7.82. (i) If $\lambda \in M_0(G)$, then there is *no solution* to the equation $\lambda*\eta = \delta_0$ for $\eta \in M(G)$. For, any solution η must satisfy $\widehat{\lambda} \cdot \widehat{\eta} \equiv 1$ with $\widehat{\lambda} \in c_0(\Gamma)$ and $\widehat{\eta} \in \ell^\infty(\Gamma)$, which is impossible. So, Corollary 7.81 only applies in $M(G) \setminus M_0(G)$.

(ii) Let $\kappa \in M(G)$ satisfy $L^1\big(m_\kappa^{(p)}\big) = L^p(G)$ for some $1 < p < \infty$. Given $\lambda \in M(G)$, if the equation $\lambda * \eta = \kappa$ has a solution $\eta \in M(G)$, then the same argument as in the proof of Corollary 7.81 shows that $L^1\big(m_\lambda^{(p)}\big) = L^p(G)$. □

So far we have produced several results which exhibit measures $\lambda \in M(G)$ satisfying $L^1\big(m_\lambda^{(p)}\big) = L^p(G)$, that is, the μ-determined operator $C_\lambda^{(p)}$ is already defined on its optimal domain and so no further extension is possible. The following result, [124, Proposition 4.5], which is based on Theorems 7.61 and 7.67 above, shows that there also exist many measures λ for which the inclusion $L^p(G) \subseteq L^1\big(m_\lambda^{(p)}\big)$ is *proper.* □

Proposition 7.83. *Let $1 < p < \infty$. Then the inclusion*

$$L^p(G) \subseteq L^1\big(m_\lambda^{(p)}\big)$$

is proper for every $\lambda \in M_0(G) \setminus \{0\}$.

It is also possible to determine precisely when the optimal domain $L^1\big(m_\lambda^{(p)}\big)$ of $C_\lambda^{(p)}$ is the *largest possible*, that is, equals $L^1(G)$. Indeed, Theorem 7.67 shows that this is the case exactly when $\lambda = \mu_h$ for some $h \in L^p(G)$. Accordingly, the inclusion $L^1\big(m_\lambda^{(p)}\big) \subseteq L^1(G)$ is proper whenever $\lambda \in M(G) \setminus L^p(G)$. In particular, for every $\lambda \in M_0(G) \setminus L^p(G)$ we can conclude that

$$L^p(G) \subsetneqq L^1\big(m_\lambda^{(p)}\big) \subsetneqq L^1(G), \qquad 1 < p < \infty.$$

Our final aim in this section is to investigate the injectivity and surjectivity of the extended operator $I_{m_\lambda^{(p)}}$, along the lines of $\lambda = \mu_h$ (with $h \in L^1(G)$) as considered in Proposition 7.54.

Lemma 7.84. *Let $1 \le p < \infty$ and $\lambda \in M(G)$.*

(i) $\mathcal{T}\big(G, \operatorname{supp}(\widehat{\lambda})\big) = C_\lambda^{(p)}\big(\mathcal{T}(G)\big) = I_{m_\lambda^{(p)}}\big(\mathcal{T}(G)\big) \subseteq \mathcal{R}\big(C_\lambda^{(p)}\big) \subseteq \mathcal{R}\big(I_{m_\lambda^{(p)}}\big).$

(ii) $\overline{\mathcal{T}\big(G, \operatorname{supp}(\widehat{\lambda})\big)} = \overline{\mathcal{R}\big(C_\lambda^{(p)}\big)} = \overline{\mathcal{R}\big(I_{m_\lambda^{(p)}}\big)}$, *with closures in $L^p(G)$.*

Proof. (i) For each $\gamma \in \Gamma$, it follows from (7.10) that

$$C_\lambda^{(p)}\big((\cdot,\gamma)\big) = I_{m_\lambda^{(p)}}\big((\cdot,\gamma)\big) = (\cdot,\gamma) * \lambda = \widehat{\lambda}(\gamma)(\cdot,\gamma).$$

So,

$$C_\lambda^{(p)}\Big(\mathcal{T}\big(G, \operatorname{supp}(\widehat{\lambda})\big)\Big) \subseteq \mathcal{T}\big(G, \operatorname{supp}(\widehat{\lambda})\big)$$

and

$$I_{m_\lambda^{(p)}}\Big(\mathcal{T}\big(G, \operatorname{supp}(\widehat{\lambda})\big)\Big) \subseteq \mathcal{T}\big(G, \operatorname{supp}(\widehat{\lambda})\big).$$

Conversely, let $\gamma \in \operatorname{supp}(\widehat{\lambda})$. Then

$$(\cdot,\gamma) = \widehat{\lambda}(\gamma)^{-1} C_\lambda^{(p)}\big((\cdot,\gamma)\big) = \widehat{\lambda}(\gamma)^{-1} I_{m_\lambda^{(p)}}\big((\cdot,\gamma)\big).$$

This establishes the equality in (i). The stated inclusions are obvious.

(ii) Theorem 7.61(iv) shows that $\mathcal{T}(G)$ is dense in $L^1\big(m_\lambda^{(p)}\big)$, from which it follows that

$$\mathcal{T}\big(G, \operatorname{supp}(\widehat{\lambda})\big) \subseteq \mathcal{R}\big(I_{m_\lambda^{(p)}}\big) = I_{m_\lambda^{(p)}}\big(\overline{\mathcal{T}(G)}\big),$$

where the bar denotes "closure with respect to $\|\cdot\|_{L^1(m_\lambda^{(p)})}$". By continuity of the operator $I_{m_\lambda^{(p)}} : L^1\big(m_\lambda^{(p)}\big) \to L^p(G)$ and part (i) we have

$$I_{m_\lambda^{(p)}}\big(\overline{\mathcal{T}(G)}\big) \subseteq \overline{I_{m_\lambda^{(p)}}\big(\mathcal{T}(G)\big)} = \overline{\mathcal{T}\big(G, \operatorname{supp}(\widehat{\lambda})\big)},$$

where now the bar denotes "closure in $L^p(G)$". The previous two containments give $\overline{\mathcal{T}\big(G, \operatorname{supp}(\widehat{\lambda})\big)} = \overline{\mathcal{R}\big(I_{m_\lambda^{(p)}}\big)}$. Part (i) then yields that also

$$\overline{\mathcal{R}\big(C_\lambda^{(p)}\big)} = \overline{\mathcal{T}\big(G, \operatorname{supp}(\widehat{\lambda})\big)}. \qquad \square$$

Lemma 7.85. *Let $1 \le p < \infty$ and $\lambda \in M(G)$. The following assertions are equivalent.*

(i) $\dim\big(\ker I_{m_\lambda^{(p)}}\big) < \infty$.

(ii) $\dim\big(\ker C_\lambda^{(p)}\big) < \infty$.

(iii) $\Gamma \setminus \operatorname{supp}(\widehat{\lambda})$ *is a finite set.*

If any one of (i)–(iii) *holds, then*

$$\mathcal{T}\big(G,\ \Gamma \setminus \operatorname{supp}(\widehat{\lambda})\big) = \ker C_\lambda^{(p)} = \ker I_{m_\lambda^{(p)}}. \tag{7.111}$$

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii). Suppose that $\Gamma \setminus \operatorname{supp}(\widehat{\lambda})$ is an infinite set. Let $\{\gamma_n\}_{n=1}^\infty$ be any sequence of distinct elements in $\Gamma \setminus \operatorname{supp}(\widehat{\lambda})$. For each $\xi = (\xi_n)_{n=1}^\infty \in \ell^1$ the continuous function $h_\xi := \sum_{n=1}^\infty \xi_n(\cdot,\gamma_n)$ satisfies $(h_\xi * \lambda)\widehat{\ } = \widehat{h}_\xi\widehat{\lambda} \equiv 0$ on Γ. By injectivity of F_S we conclude that $h_\xi \in \ker C_\lambda^{(p)}$. Since $\xi \in \ell^1$ is arbitrary, this contradicts the assumption that $\ker C_\lambda^{(p)}$ is finite-dimensional.

(iii) $\Rightarrow$ (i). According to (7.93), a function $f \in L^1\big(m_\lambda^{(p)}\big)$ belongs to $\ker I_{m_\lambda^{(p)}}$ iff $f * \lambda = 0$ iff $\widehat{f}\,\widehat{\lambda} \equiv 0$ on Γ iff $\widehat{f} \equiv 0$ on $\operatorname{supp}(\widehat{\lambda})$ iff $\operatorname{supp}(\widehat{f}) \subseteq \Gamma \setminus \operatorname{supp}(\widehat{\lambda})$. The same argument shows that $f \in L^p(G)$ belongs to $\ker C_\lambda^{(p)}$ iff $\operatorname{supp}(\widehat{f}) \subseteq \Gamma \setminus \operatorname{supp}(\widehat{\lambda})$. From these two observations it is immediate that (iii) $\Rightarrow$ (i) and also that (7.111) holds whenever $\dim(\ker C_\lambda^{(p)}) < \infty$. $\square$

We have an immediate consequence of the previous two results.

Corollary 7.86. *Let $1 \le p < \infty$ and $\lambda \in M(G)$. The following statements are equivalent.*

(i) $\operatorname{supp}(\widehat{\lambda}) = \Gamma$.

(ii) $C_\lambda^{(p)} : L^p(G) \to L^p(G)$ *is injective.*

(iii) $I_{m_\lambda^{(p)}} : L^1\big(m_\lambda^{(p)}\big) \to L^p(G)$ *is injective.*

(iv) $\overline{\mathcal{R}\big(I_{m_\lambda^{(p)}}\big)} = \overline{\mathcal{R}\big(C_\lambda^{(p)}\big)} = L^p(G)$.

Proof. (i) $\Rightarrow$ (iii). Let $f \in \ker I_{m_\lambda^{(p)}}$. Then $(f * \lambda)\widehat{\ } = \widehat{f}\,\widehat{\lambda} = 0$ on Γ and so actually $\widehat{f} = 0$ on Γ (as $\operatorname{supp}(\widehat{\lambda}) = \Gamma$). Hence, also $f = 0$.

(iii) $\Rightarrow$ (ii). This is clear as $L^p(G) \subseteq L^1\big(m_\lambda^{(p)}\big)$.

(i) $\Leftrightarrow$ (iv). It is clear, from Lemma 7.84(ii) and the observation that $\mathcal{T}(G,\Gamma) = \mathcal{T}(G)$ is dense in $L^p(G)$, that (i) $\Rightarrow$ (iv).

Suppose that $\operatorname{supp}(\widehat{\lambda}) \ne \Gamma$. Then there exists $\gamma \notin \operatorname{supp}(\widehat{\lambda})$. Since $\widehat{\varphi}(\gamma) = 0$ for every $\varphi \in \mathcal{T}\big(G, \operatorname{supp}(\widehat{\lambda})\big)$ but $(\cdot,\gamma)\widehat{\ }(\gamma) = 1$, it follows from continuity of $F_{1,0}$ that $(\cdot,\gamma) \notin \overline{\mathcal{T}\big(G, \operatorname{supp}(\widehat{\lambda})\big)}$. According to Lemma 7.84(ii) we see that (iv) fails to hold.

(ii) $\Rightarrow$ (i). Assume to the contrary that there is $\gamma_0 \in \Gamma$ for which $\widehat{\lambda}(\gamma_0) = 0$. Then $C_\lambda^{(p)}\big((\cdot,\gamma_0)\big) = \widehat{\lambda}(\gamma_0)(\cdot,\gamma_0) = 0$ which contradicts (ii). $\square$

Perhaps more interesting is the question of surjectivity. First a preliminary result.

Lemma 7.87. *Let $\lambda \in M(G) \setminus \{0\}$.*

(i) $\operatorname{supp}(\widehat{\lambda})$ *is a finite subset of Γ if and only if $\lambda \in \mathcal{T}(G)$.*

(ii) *Suppose that $\lambda \in M_0(G)$. Then $\lambda \in \mathcal{T}(G)$ if and only if*

$$\beta_\lambda := \inf\big\{\, |\widehat{\lambda}(\gamma)| : \gamma \in \operatorname{supp}(\widehat{\lambda}) \,\big\} > 0. \tag{7.112}$$

Proof. (i) If $\lambda \in \mathcal{T}(G)$, that is, $\lambda = \mu_h$ for some $h \in \mathcal{T}(G)$, then clearly $\operatorname{supp}(\widehat{\lambda}) = \operatorname{supp}(\widehat{h})$ is a finite set. On the other hand, if $\operatorname{supp}(\widehat{\lambda}) = \{\gamma_1, \dots, \gamma_n\}$ is finite, then $g := \sum_{j=1}^n \widehat{\lambda}(\gamma_j)(\cdot, \gamma_j)$ belongs to $\mathcal{T}(G)$ and satisfies $\widehat{g} = \widehat{\lambda}$. By injectivity of F_S we conclude that $\lambda = \mu_g$ can be identified with $g \in \mathcal{T}(G)$.

(ii) If $\lambda \in \mathcal{T}(G)$, then $\operatorname{supp}(\widehat{\lambda})$ is a non-empty finite set and so (7.112) clearly holds.

Suppose now that $\lambda \notin \mathcal{T}(G)$, in which case $\operatorname{supp}(\widehat{\lambda})$ is infinite (by part (i)). Then there exists an infinite sequence $\{\gamma_n\}_{n=1}^\infty$ of distinct elements in $\operatorname{supp}(\widehat{\lambda})$. Since $\widehat{\lambda} \in c_0(\Gamma)$, it follows that $\lim_{n\to\infty} \widehat{\lambda}(\gamma_n) = 0$ and so (7.112) fails to hold. $\square$

Proposition 7.88. *Let $\lambda \in M(G) \setminus \{0\}$ satisfy*

$$\beta_\lambda := \inf\left\{ |\widehat{\lambda}(\gamma)| : \gamma \in \operatorname{supp}(\widehat{\lambda}) \right\} \,=\, 0. \tag{7.113}$$

(i) *Let $p \in [1, \infty)$ be arbitrary. Then, for every $1 \le r < \infty$, we have $L^r(m_\lambda^{(p)}) \subseteq L^1(m_\lambda^{(p)})$ and the range of the restricted integration operator $I_{m_\lambda^{(p)}} : L^r\big(m_\lambda^{(p)}\big) \to L^p(G)$ is not closed.*

(ii) *The range of $C_\lambda^{(p)} : L^p(G) \to L^p(G)$ is not closed for every $1 \le p < \infty$.*

(iii) $\operatorname{supp}(\widehat{\lambda}) = \Gamma$ *if and only if both $\mathcal{R}\big(C_\lambda^{(p)}\big)$ and $\mathcal{R}\big(I_{m_\lambda^{(p)}}\big)$ are proper dense subspaces of $L^p(G)$ for every $1 \le p < \infty$.*

Proof. (i) Fix $r, p \in [1, \infty)$ and note that always $\mathcal{T}(G) \subseteq L^r\big(m_\lambda^{(p)}\big) \subseteq L^1\big(m_\lambda^{(p)}\big)$. According to (7.113) there is an infinite sequence $\{\gamma_n\}_{n=1}^\infty \subseteq \operatorname{supp}(\widehat{\lambda})$ such that $\lim_{n\to\infty} |\widehat{\lambda}(\gamma_n)| = 0$. Since $\lambda \notin \mathcal{T}(G)$ (by Lemma 7.87(i)) we may suppose that $|\widehat{\lambda}(\gamma_n)| > 0$ for all $n \in \mathbb{N}$. By passing to a subsequence, if necessary, we may assume that $0 < |\widehat{\lambda}(\gamma_n)| < n^{-2}$ for all $n \in \mathbb{N}$. Then $h := \sum_{n=1}^\infty n^{-2}(\cdot, \gamma_n)$ belongs to $C(G) \subseteq L^p(G)$. Suppose that h belongs to the range of the restricted integration operator $I_{m_\lambda^{(p)}} : L^r\big(m_\lambda^{(p)}\big) \to L^p(G)$, in which case there exists $f \in L^r\big(m_\lambda^{(p)}\big)$ such that $I_{m_\lambda^{(p)}} = f * \lambda = h$. Then $\widehat{h} = \widehat{f}\,\widehat{\lambda}$ and so

$$|\widehat{f}(\gamma_n)| \,=\, |\widehat{h}(\gamma_n)| \,/\, |\widehat{\lambda}(\gamma_n)| > 1, \qquad n \in \mathbb{N}.$$

Since, $L^r\big(m_\lambda^{(p)}\big) \subseteq L^1\big(m_\lambda^{(p)}\big) \subseteq L^1(G)$, this contradicts $\widehat{f} \in c_0(\Gamma)$. So, the function $h \notin I_{m_\lambda^{(p)}}\big(L^r(m_\lambda^{(p)})\big)$. However,

$$h \,=\, \lim_{N\to\infty} \sum_{n=1}^N n^{-2}(\cdot, \gamma_n) \,=\, \lim_{N\to\infty} \sum_{n=1}^N \big(n^2 \widehat{\lambda}(\gamma_n)\big)^{-1} I_{m_\lambda^{(p)}}\big((\cdot, \gamma_n)\big), \tag{7.114}$$

with convergence in $L^p(G)$, shows that $h \in \overline{I_{m_\lambda^{(p)}}\big(L^r(m_\lambda^{(p)})\big)}$.

Accordingly, $I_{m_\lambda^{(p)}}\big(L^r(m_\lambda^{(p)})\big)$ is not closed in $L^p(G)$.

(ii) The element h constructed in the proof of part (i) also fails to belong to $\mathcal{R}\big(C_\lambda^{(p)}\big)$, by the same argument (as the domain space $L^p(G)$ of $C_\lambda^{(p)}$ is also contained in $L^1(G)$). Since $I_{m_\lambda^{(p)}}$ and $C_\lambda^{(p)}$ coincide on $L^p(G)$, the formula (7.114) remains valid if we replace $I_{m_\lambda^{(p)}}$ with $C_\lambda^{(p)}$. It follows that $h \in \overline{\mathcal{R}\big(C_\lambda^{(p)}\big)}$ and so $\mathcal{R}\big(C_\lambda^{(p)}\big)$ is also not closed in $L^p(G)$.

(iii) This is a combination of parts (i), (ii) and Corollary 7.86. $\square$

Corollary 7.89. *Let* $\lambda \in M_0(G)$ *satisfy* $\operatorname{supp}(\widehat{\lambda}) = \Gamma$. *Then*

$$\mathcal{R}\big(C_\lambda^{(p)}\big) \subseteq \mathcal{R}(I_{m_\lambda^{(p)}}) \subseteq L^p(G), \quad 1 < p < \infty, \tag{7.115}$$

with both inclusions proper. In particular, neither $C_\lambda^{(p)}$ *nor its optimal extension* $I_{m_\lambda^{(p)}}$ *are surjective.*

Proof. That the containments in (7.115) hold is obvious.

According to Corollary 7.86, both $C_\lambda^{(p)}$ and $I_{m_\lambda^{(p)}}$ are injective. Moreover, Proposition 7.83 ensures that $L^p(G) \subsetneqq L^1\big(m_\lambda^{(p)}\big)$. Since $L^p(G)$ is the domain of $C_\lambda^{(p)}$ and $L^1\big(m_\lambda^{(p)}\big)$ is the domain of the extension $I_{m_\lambda^{(p)}}$ of $C_\lambda^{(p)}$, it follows that the first containment in (7.115) must be proper.

The second containment in (7.115) is proper via Proposition 7.88(iii) above, after noting that our assumptions on λ ensure that (7.113) is valid. $\square$

Remark 7.90. For $\lambda \in M_0(G)$, the set $\operatorname{supp}(\widehat{\lambda})$ is necessarily countable (because $\widehat{\lambda} \in c_0(\Gamma)$). Hence, the equality $\operatorname{supp}(\widehat{\lambda}) = \Gamma$ is only possible if G is metrizable. $\square$

For the Hilbert space setting more precise information is available.

Proposition 7.91. *Let* $\lambda \in M(G)$.

(i) $\ker C_\lambda^{(2)} = \overline{\mathcal{T}\big(G, \Gamma \setminus \operatorname{supp}(\widehat{\lambda})\big)}$.

(ii) *The following assertions are equivalent.*

(a) $\mathcal{R}\big(I_{m_\lambda^{(2)}}\big)$ *is closed in* $L^2(G)$.

(b) $\mathcal{R}\big(C_\lambda^{(2)}\big)$ *is closed in* $L^2(G)$.

(c) *The inequality* (7.112) *holds, that is,* $\beta_\lambda > 0$.

In this case,

$$\mathcal{R}\big(C_\lambda^{(2)}\big) = \mathcal{R}\big(I_{m_\lambda^{(2)}}\big).$$

Proof. (i) The calculation in the proof of (ii) $\Rightarrow$ (i) in Corollary 7.86 shows that $C_\lambda^{(2)}\big((\cdot,\gamma)\big) = 0$ for all $\gamma \in \Gamma \setminus \operatorname{supp}(\widehat{\lambda})$. Hence, $\mathcal{T}\big(G,\ \Gamma \setminus \operatorname{supp}(\widehat{\lambda})\big) \subseteq \ker C_\lambda^{(2)}$ and so also $\overline{\mathcal{T}\big(G,\ \Gamma \setminus \operatorname{supp}(\widehat{\lambda})\big)} \subseteq \ker C_\lambda^{(2)}$.

Given $f \in \ker C_\lambda^{(2)}$ we have $\big(C_\lambda^{(2)}(f)\big)\widehat{\ } = \widehat{f}\,\widehat{\lambda} = 0$ in $\ell^2(\Gamma)$ and so $\widehat{f}(\gamma) = 0$ for all $\gamma \in \operatorname{supp}(\widehat{\lambda})$. So, $\widehat{f} = \sum_{\gamma \notin \operatorname{supp}(\widehat{\lambda})} \widehat{f}(\gamma)\chi_{\{\gamma\}}$ in $\ell^2(\Gamma)$, with the series necessarily a countable sum as $\operatorname{supp}(\widehat{f})$ is countable. By Plancherel's Theorem $f = \sum_{\gamma \notin \operatorname{supp}(\widehat{\lambda})} \widehat{f}(\gamma)(\cdot,\gamma)$ in $L^2(G)$. Accordingly, $f \in \overline{\mathcal{T}\big(G,\ \Gamma \setminus \operatorname{supp}(\widehat{\lambda})\big)}$.

(ii) Proposition 7.88(i) implies that (a) $\Rightarrow$ (c) and Proposition 7.88(ii) implies that (b) $\Rightarrow$ (c). Since $\mathcal{R}\big(C_\lambda^{(2)}\big) \subseteq \mathcal{R}\big(I_{m_\lambda^{(2)}}\big) \subseteq \overline{\mathcal{R}\big(I_{m_\lambda^{(2)}}\big)}$ it follows from Lemma 7.84(ii) that (b) $\Rightarrow$ (a).

(c) $\Rightarrow$ (b). For $h \in \overline{\mathcal{R}\big(C_\lambda^{(2)}\big)}$ we have $\widehat{h}(\gamma) = 0$ for all $\gamma \in \Gamma \setminus \operatorname{supp}(\widehat{\lambda})$; see Lemma 7.84(ii). Since $\beta_\lambda > 0$, the function $\varphi := (\widehat{h}/\widehat{\lambda})\,\chi_{\operatorname{supp}(\widehat{\lambda})} \in \ell^2(\Gamma)$. Moreover, $f := \sum_{\gamma\in\Gamma} \varphi(\gamma)(\cdot,\gamma) \in L^2(G)$ satisfies $(f*\lambda)\widehat{\ } = \widehat{f}\,\widehat{\lambda} = \varphi\widehat{\lambda} = \widehat{h}$, that is, $C_\lambda^{(2)}(f) = h$ and so actually $h \in \mathcal{R}\big(C_\lambda^{(2)}\big)$. This establishes (b).

Finally, the identity $\mathcal{R}\big(C_\lambda^{(2)}\big) = \mathcal{R}\big(I_{m_\lambda^{(2)}}\big)$ follows from (a), (b) and the identity $\overline{\mathcal{R}\big(C_\lambda^{(2)}\big)} = \overline{\mathcal{R}\big(I_{m_\lambda^{(2)}}\big)}$; see Lemma 7.84(ii). $\square$

Observe that (7.113) holds, that is, $\beta_\lambda = 0$ if and only if there exists an infinite sequence $\{\gamma_n\}_{n=1}^\infty \subseteq \operatorname{supp}(\widehat{\lambda})$ satisfying $\lim_{n\to\infty} |\widehat{\lambda}(\gamma_n)| = 0$. In particular, if $\lambda \in M_0(G) \setminus \mathcal{T}(G)$, then $\widehat{\lambda} \in c_0(\Gamma)$ and so this is necessarily the case. However, the following examples show that there also exist measures $\lambda \notin M_0(G)$ which satisfy $\beta_\lambda = 0$.

Example 7.92. (i) Let $G := \mathbb{T}$ be the circle group, in which case $\Gamma = \mathbb{Z}$, with duality $(z,n) = z^n$ for all $z \in \mathbb{T}$ and $n \in \mathbb{Z}$. Let $u \in \mathbb{T}$ be any (fixed) element of infinite order and define $\lambda := \delta_1 - \delta_u$, where $1 \in \mathbb{T}$ is the (multiplicative) identity element. Fix $1 \le p < \infty$. Then $C_\lambda^{(p)} = I - C_{\delta_u}^{(p)} = I - \tau_u$. It is known that the *spectrum* of $C_{\delta_u}^{(p)} = \tau_u \in \mathcal{L}\big(L^p(\mathbb{T})\big)$ is given by $\sigma(C_{\delta_u}^{(p)}) = \mathbb{T}$, [64, Theorem 1], and so, by the Spectral Mapping Theorem, [46, Ch. VII, Theorem 3.11], we have $\sigma\big(C_\lambda^{(p)}\big) = \{1 - z : z \in \mathbb{T}\}$. Note that the range $\widehat{\delta}_u(\mathbb{Z}) = \{u^n : n \in \mathbb{Z}\}$ and so the range $\widehat{\lambda}(\mathbb{Z}) = \{1 - u^n : n \in \mathbb{Z}\}$. Moreover, $\operatorname{supp}(\widehat{\lambda}) = \mathbb{Z} \setminus \{0\}$; this follows from the assumption that u has infinite order. Since $\widehat{\delta}_u(\mathbb{Z})$ is known to be dense in $\mathbb{T}$ (see the proof of Theorem 1 in [64]), it follows that $\widehat{\lambda}(\mathbb{Z})$ is dense in $\sigma\big(C_\lambda^{(p)}\big)$. As $0 \in \sigma\big(C_\lambda^{(p)}\big)$, there exists a sequence from $\widehat{\lambda}(\mathbb{Z} \setminus \{0\})$ which converges to 0. Accordingly, $\beta_\lambda = 0$. However, $\lambda \notin M_0(\mathbb{T})$. To see this, observe that the element $2 \in \sigma\big(C_\lambda^{(p)}\big)$ can be approximated by a sequence of elements $\{z_n\}_{n=1}^\infty \subseteq \widehat{\lambda}(\mathbb{Z})$ and so, for some N large enough, $|z_n| > 1$ for all $n \ge N$.

(ii) Again let $G := \mathbb{T}$ and consider the Riesz product measure given by $\lambda := \prod_{j=1}^{\infty} \big(1 + a_j \cos(n_j t)\big)$ as constructed in [148, Example 2, p. 310], where it is denoted by μ. Set $a_j = 1$ for each $j \in \mathbb{N}$, in which case $\widehat{\lambda}(\mathbb{Z}) = \{1\} \cup \{2^{-k} : k \in \mathbb{N}\}$; see [148, p. 315]. Clearly $\beta_\lambda = 0$. However, as noted on p. 310 of [148], the positive non-atomic measure $\lambda \notin M_0(\mathbb{T})$.

(iii) For a more abstract class of examples, let G_1, G_2 be infinite, separable, compact abelian groups and $G = G_1 \times G_2$ be the *product group*. Then the dual group of G is identified with $\Gamma = \Gamma_1 \times \Gamma_2$ in the sense that

$$\big((x_1, x_2), (\gamma_1, \gamma_2)\big) \;=\; (x_1, \gamma_1) \cdot (x_2, \gamma_2), \qquad (x_1, x_2) \in G, \quad (\gamma_1, \gamma_2) \in \Gamma,$$

[140, Theorem 2.2.2]. The separability condition ensures that $\mathcal{B}(G)$ is precisely the product σ-algebra $\mathcal{B}(G_1) \otimes \mathcal{B}(G_2)$. Normalized Haar measure on G_j is denoted by μ_j, for $j = 1, 2$. Let $\lambda_1 \in M_0(G_1) \setminus \mathcal{T}(G_1)$ be arbitrary and $\xi \in \Gamma_2$ be any fixed element (other than the identity element). Define $\lambda_2 \in M_0(G_2)$ by

$$\lambda_2(A) := \int_A (\cdot, \xi)\, d\mu_2, \qquad A \in \mathcal{B}(G_2).$$

Let $\delta_0^{(j)}$ be the Dirac measure at $0 \in G_j$, for $j = 1, 2$. Then the measure $\lambda \in M(G)$ defined by

$$\lambda := \;(\lambda_1 \times \lambda_2) + \Big(\delta_0^{(1)} \times (\delta_0^{(2)} - \lambda_2)\Big)$$

satisfies

$$\begin{aligned} \widehat{\lambda}(\gamma_1, \gamma_2) &= \widehat{\lambda}_1(\gamma_1)\widehat{\lambda}_2(\gamma_2) + (\delta_0^{(1)})\widehat{\ }(\gamma_1) \cdot (\delta_0^{(2)} - \lambda_2)\widehat{\ }(\gamma_2) \\ &= \widehat{\lambda}_1(\gamma_1)\chi_{\{\xi\}}(\gamma_2) + \big(1 - \chi_{\{\xi\}}(\gamma_2)\big), \end{aligned}$$

for all $(\gamma_1, \gamma_2) \in \Gamma$. That is,

$$\widehat{\lambda}(\gamma_1, \gamma_2) = \begin{cases} \widehat{\lambda}_1(\gamma_1) & \text{if } \gamma_1 \in \Gamma_1 \text{ and } \gamma_2 = \xi, \\ 1 & \text{if } \gamma_1 \in \Gamma_1 \text{ and } \gamma_2 \neq \xi. \end{cases}$$

Clearly $\lambda \notin M_0(G)$, but $\beta_\lambda \leq \inf\big\{|\widehat{\lambda}_1(\gamma_1)| : \gamma_1 \in \operatorname{supp}\big(\widehat{\lambda}_1\big)\big\} = 0$, where we have used the fact that $\operatorname{supp}\big(\widehat{\lambda}_1\big) \times \{\xi\} \subseteq \operatorname{supp}\big(\widehat{\lambda}\big)$. □

7.5 p-th power factorability

Let $1 \leq q < \infty$ and $\lambda \in M(G)$. Then λ is called L^q-*improving* if there exists $r \in (q, \infty)$ such that

$$\lambda * f \in L^r(G), \qquad f \in L^q(G), \tag{7.116}$$

briefly, $\lambda * L^q(G) \subseteq L^r(G)$. Such measures, introduced by E.M. Stein in [152], have become of importance; see [12], [15], [18], [71], [72], [74], [116], [132], [133] for example, and the references therein. The interest in this class of measures, from the viewpoint of this monograph, lies in the fact that they lead in a very natural way to an extensive and non-trivial collection of important operators arising in classical harmonic analysis which are *p-th power factorable* (in the sense of Chapter 5). We begin by summarizing various known results about L^q-improving measures which are relevant to this monograph.

An interpolation argument, [74, p. 295], shows that if λ is L^q-improving for some $1 < q < \infty$, then λ is L^q-improving for every $1 < q < \infty$. So, when we speak of L^q-improving measures we may not specify $1 \le q < \infty$ explicitly. Given $1 \le q < r < \infty$, let $M^{q,r}(G)$ denote the space of all measures $\lambda \in M(G)$ such that $\lambda * L^q(G) \subseteq L^r(G)$. Since $L^r(G) \subseteq L^q(G)$ continuously, a Closed Graph Theorem argument shows that the linear map

$$C_\lambda^{q,r} : L^q(G) \to L^r(G)$$

defined by (7.116) is necessarily continuous, that is, $C_\lambda^{q,r} \in \mathcal{L}\big(L^q(G),\, L^r(G)\big)$. The $M^{q,r}(G)$-norm of λ is defined to be the operator norm of $C_\lambda^{q,r}$. Since $L^q(G) \neq L^r(G)$ whenever $q \neq r$, no Dirac measure δ_x, for $x \in G$, can be L^q-improving. Actually, if λ is L^q-improving, then necessarily $\lambda \in M_c(G)$, [71, p. 80]. Examples of L^q-improving measures are, of course, those of the form $\lambda = \mu_h$ for $h \in L^q(G)$. If $\widehat{\lambda} \in \ell^q(\Gamma)$ and $2 \le q \le r$, then $\lambda \in M^{2,r}(G)$, [72, p. 473]. Hare [74] characterizes L^q-improving measures λ in terms of certain properties of $\widehat{\lambda}$. In particular, if $\widehat{\lambda} \in \ell^r(\Gamma)$ for some $r < \infty$, then λ is L^q-improving, [74, Corollary 1]. There are also many known examples of L^q-improving measures which are μ-singular (usually constructed via Riesz products); see [12], [15], [18], [116], [132], [133] for example.

Proposition 7.93. *Let $1 < q < \infty$ and $\lambda \in M(G)$.*

(i) *If $\lambda \in M^{q,r}(G)$ for some $q < r < \infty$, then each translate of λ belongs to $M^{q,r}(G)$ and has the same $M^{q,r}(G)$-norm as λ.*

(ii) *If $\gamma \in \Gamma$ and $\lambda \in M^{q,r}(G)$ for some $q < r < \infty$, then the measure $A \mapsto \int_A (\cdot, \gamma)\, d\lambda$ on $\mathcal{B}(G)$ belongs to $M^{q,r}(G)$ and has the same $M^{q,r}(G)$-norm as λ.*

(iii) *If $0 \le \lambda \in M^{q,r}(G)$ for some $q < r < \infty$ and $f : G \to \mathbb{C}$ is a bounded, $\mathcal{B}(G)$-measurable function, then the measure $A \mapsto \int_A f\, d\lambda$ on $\mathcal{B}(G)$ belongs to $M^{q,r}(G)$.*

(iv) *If $\lambda \in \bigcap_{1<q<2} M^{q,2}(G)$, then $\lambda \in M_0(G)$.*

For parts (i) and (ii) we refer to [72, Theorem 0.1], for part (iii) to [72, Corollary 1.2], and for part (iv) to [72, Corollary 3.2]. Concerning some interesting "negative results", we record the following facts; see Theorems 1.3 and 1.4 and Remark (iv), p. 487, of [72].

Proposition 7.94. *Let G be a compact abelian group.*

(i) *There exists $\lambda \in M(G)$ such that λ is L^q-improving but, its variation measure $|\lambda|$ is not L^q-improving.*

(ii) *Let $\lambda \in M(G) \setminus \{0\}$ be L^q-improving. Then there exists a probability measure $\eta \in M(G)$ such that λ and η are mutually absolutely continuous and η is not L^q-improving.*

(iii) *There exists $\lambda \in M_0(G)$ such that every non-zero measure $\eta \ll \lambda$ fails to be L^q-improving. In particular, λ itself is not L^q-improving.*

Remark 7.95. (i) An examination of the proof of Theorem 1.3 in [72] shows that the measure λ given in Proposition 7.94(i) can be chosen to be $\mathbb{R}$-valued. Hence, if we decompose the variation measure $|\lambda| = \lambda^+ + \lambda^-$ according to its Hahn decomposition, then at least one of λ^+ or λ^- is not L^q-improving.

(ii) Let $1 < r < \infty$ and $h \in L^r(G)^+$. Define $\lambda := \mu_h$. Then λ is L^q-improving because, for any $1 < q < r$, we have

$$h * L^q(G) \subseteq h * L^1(G) \subseteq L^r(G);$$

see (7.71). Choose η according to Proposition 7.94(ii) and write (via the Radon–Nikodým Theorem)

$$\eta(A) = \int_A g\, d\lambda = \int_A gh\, d\mu = \mu_{gh}(A), \qquad A \in \mathcal{B}(G).$$

This shows that there always exist measures of the form μ_f, for $f \in L^1(G)$, which are *not* L^q-improving. For the circle group $G := \mathbb{T}$ this was noted in [72, Remark (ii), p. 487], via a different argument. The function $f \in L^1(\mathbb{T})$ given by

$$f(t) = \sum_{n=2}^{\infty} \frac{\cos(nt)}{\ln(n)}, \qquad t \in [0, 2\pi],$$

provides a particular example, [72, Remark (i), p. 481].

Note that $f \notin \bigcup_{1<p<\infty} L^p(\mathbb{T})$; just observe that if $f \in L^p(\mathbb{T})$ for some value of $1 < p \leq 2$, then $\widehat{f} \in \ell^{p'}(\mathbb{Z})$ (by the Hausdorff–Young inequality) which is not the case. It is also noted in [72, Remark (i), p. 481] that there exist functions $\varphi \in L^1(\mathbb{T}) \setminus \bigcup_{1<p<\infty} L^p(\mathbb{T})$ such that μ_φ *is* L^q-improving. $\square$

Concerning convolution operators which are p-th power factorable, let us begin with absolutely continuous measures. We recall that all convolution operators $C_\lambda^{(p)}$ with $1 \leq p < \infty$ and $\lambda \in M(G) \setminus \{0\}$ are necessarily μ-determined; see Remark 7.60(i).

Proposition 7.96. *Let $1 < r < p$ and $u \in (1, p)$ satisfy*

$$\frac{1}{u} + \frac{1}{r} = \frac{1}{p} + 1.$$

Let $h \in L^r(G) \setminus L^p(G)$.

(i) *μ_h is L^q-improving and belongs to $M^{u,p}(G)$.*

(ii) *The following inclusions hold:*

$$L^p(G) \subsetneqq L^u(G) \subseteq L^1\big(m_h^{(p)}\big).$$

(iii) *$C_h^{(p)}$ is (p/u)-th power factorable.*

Proof. (i) According to Remark 7.45(iv) we have $|f|*|h| \in L^p(G)$ for all $f \in L^u(G)$. Since $|f*h| \le |f|*|h|$ and $L^p(G)$ is an order ideal it follows that also $f*h \in L^p(G)$. That is, $h * L^u(G) \subseteq L^p(G)$ or, equivalently, $\mu_h \in M^{u,p}(G)$.

(ii) The first inclusion (necessarily proper) is standard (see (7.1)) and the second inclusion is precisely (7.84).

(iii) For $X(\mu) := L^p(G)$ and $E := L^p(G)$ and $T := C_h^{(p)}$ it follows from Theorem 5.7 that $C_h^{(p)}$ is (p/u)-th power factorable if and only if $X(\mu)_{[p/u]} \subseteq L^1\big(m_h^{(p)}\big)$. But, since $X(\mu)_{[p/u]} = L^p(G)_{[p/u]} = L^u(G)$, we see from part (ii) that this is indeed the case. □

In order to formulate an interesting consequence of Proposition 7.96 we need the following fact.

Lemma 7.97. *Let G be an infinite, compact abelian group with normalised Haar measure μ. Then μ is non-atomic.*

Proof. Suppose that $B \in \mathcal{B}(G)$ is an atom for μ. We claim that $\mu(B) = \mu(\{w\})$ for some element $w \in B$. To see this, define

$$\mathcal{K} := \big\{C : C \subseteq B, C \text{ compact}, \mu(C) = \mu(B)\big\}.$$

Then $\mathcal{K}$ has the finite intersection property and, by regularity of μ, we have $\mathcal{K} \neq \emptyset$. So, $A := \bigcap_{C\in\mathcal{K}} C$ is non-empty. Observe that $B \setminus A = \bigcup_{C\in\mathcal{K}}(B\setminus C)$ with $\mu(B\setminus C) = 0$ for all $C \in \mathcal{K}$. By regularity, $\mu(B\setminus A) = 0$ and so $\mu(A) = \mu(B)$ with A compact and $A \subseteq B$. Choose any $w \in A$. Then $\mu(A\setminus\{w\}) + \mu(\{w\}) = \mu(A)$. If it were the case that $\mu(\{w\}) = 0$, then $\mu(A\setminus\{w\}) = \mu(A) = \mu(B)$ and so $A\setminus\{w\}$ is an atom for μ. By the previous argument applied to $A\setminus\{w\}$ in place of B, there is a compact set $D \subseteq A\setminus\{w\} \subseteq B$ such that $\mu(D) = \mu(A\setminus\{w\}) = \mu(A)$. By definition of $\mathcal{K}$ and A we have $A \subseteq D$ and so $w \in D$. However, this is a contradiction because $D \subseteq A\setminus\{w\}$. Therefore, $\mu(\{w\}) = \mu(A)$ and actually, $A = \{w\}$. Moreover, $\mu(B) = \mu(\{w\})$, that is, $\{w\}$ is an atom for μ.

Now, suppose that μ *did* possess an atom. By the previous argument there is an element $w \in G$ such that $\{w\}$ is an atom for μ. Since G is infinite, there exists an infinite sequence $\{x_n\}_{n=1}^\infty$ of distinct elements in G. By translation invariance of μ we have $\mu(\{x_n\}) = \mu(\{w\}) > 0$ for every $n \in \mathbb{N}$. Since the singleton sets $\{x_n\}$, for $n \in \mathbb{N}$, are pairwise disjoint, it follows that $\mu(G) = \infty$, which is a contradiction. Accordingly, μ has no atoms. □

We can now formulate the consequence of Proposition 7.96 alluded to above.

Proposition 7.98. *Let* $1 < p < \infty$.

(i) *The set* $L(G\!:\!p^-) := \big(\bigcap_{1<r<p} L^r(G)\big) \setminus L^p(G)$ *is non-empty.*

(ii) *For each* $h \in L(G\!:\!p^-)$, *the following inclusions hold:*

$$L^p(G) \subsetneqq \bigcup_{1<u<p} L^u(G) \subsetneqq L^1\big(m_h^{(p)}\big) \subsetneqq L^1(G).$$

(iii) *Let* $h \in L(G\!:\!p^-)$. *Then the operator* $C_h^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ *is* v*-th power factorable for every* $v \in [1,p)$, *whereas* $C_h^{(p)}$ *is not* p*-th power factorable.*

Proof. (i) According to Lemma 7.97, there exists a sequence $\big\{A(n)\big\}_{n=1}^\infty \subseteq \mathcal{B}(G)$ of pairwise disjoint sets satisfying $\mu\big(A(n)\big) = 2^{-n}$ for each $n \in \mathbb{N}$. Define the function $f := \sum_{n=1}^\infty \alpha_n \chi_{A(n)}$ with $\alpha_n := 2^{n/p}\, n^{-1/p}$ for $n \in \mathbb{N}$. Then

$$\int_G |f|^p \, d\mu \;=\; \sum_{n=1}^\infty \alpha_n^p \,\mu\big(A(n)\big) \;=\; \sum_{n=1}^\infty n^{-1} \;=\; \infty$$

and so $f \notin L^p(G)$. However, for any $r \in (1,p)$ we have $(1-(r/p)) > 0$ and hence,

$$\begin{aligned}\int_G |f|^r \, d\mu = \sum_{n=1}^\infty \alpha_n^r \,\mu\big(A(n)\big) \;&=\; \sum_{n=1}^\infty \big(1/n^{r/p}\big)2^{n((r/p)-1)} \\ &< \sum_{n=1}^\infty \big(1/2^{1-(r/p)}\big)^n < \infty,\end{aligned}$$

that is, $f \in L^r(G)$.

(ii) Given any $1 < u < p$, let $r \in (1,p)$ be determined by

$$\frac{1}{u} + \frac{1}{r} = \frac{1}{p} + 1.$$

Since $h \in L^r(G) \setminus L^p(G)$, Proposition 7.96(ii) yields $L^u(G) \subseteq L^1\big(m_h^{(p)}\big)$. That is,

$$\bigcup_{1<u<p} L^u(G) \;\subseteq\; L^1\big(m_h^{(p)}\big). \tag{7.117}$$

To prove that this inclusion is proper, assume on the contrary that the equality holds. Select a strictly decreasing sequence $\{u(n)\}_{n=1}^\infty$ in the open interval $(1,p)$ such that $\lim_{n\to\infty} u(n) = 1$. Then we have

$$\bigcup_{n=1}^\infty L^{u(n)}(G) = \bigcup_{1<u<p} L^u(G) = L^1\big(m_h^{(p)}\big); \tag{7.118}$$

see (7.1). With B_n denoting the closed unit ball of $L^{u(n)}(G)$ for $n \in \mathbb{N}$, it follows from (7.118) that

$$\bigcup_{k,n=1}^{\infty} kB_n = \bigcup_{n=1}^{\infty} L^{u(n)}(G) = L^1\big(m_h^{(p)}\big).$$

The Baire Category Theorem, [88, §4,6.(1)], applied to the Banach space $L^1\big(m_h^{(p)}\big)$ then yields that there exist $k, n \in \mathbb{N}$ for which the closure $\overline{kB_n}$ has non-empty interior in $L^1\big(m_h^{(p)}\big)$. But, kB_n is weakly compact in the reflexive Banach space $L^{u(n)}(G)$ and hence, also in $L^1\big(m_h^{(p)}\big)$ because $L^{u(n)}(G)$ is continuously embedded into $L^1\big(m_h^{(p)}\big)$ (see Lemma 2.7, for example) and, consequently, the natural embedding is weakly continuous. In particular, kB_n is closed in $L^1\big(m_h^{(p)}\big)$ and has non-empty interior. So, there exist a function $g \in kB_n$ and a neighbourhood V of 0 in $L^1\big(m_h^{(p)}\big)$ such that $g + V \subseteq kB_n$. Since $V \subseteq kB_n - g \subseteq 2kB_n$, we have

$$L^{u(n+1)}(G) \subseteq L^1\big(m_h^{(p)}\big) = \operatorname{span}(V) \subseteq \operatorname{span}(2kB_n) = L^{u(n)}(G).$$

This contradicts the fact that the inclusion $L^{u(n)}(G) \subseteq L^{u(n+1)}(G)$ is proper as noted immediately after (7.1). Therefore, we conclude that the inclusion (7.117) is proper.

Since $h \notin L^p(G)$, Theorem 7.50 gives the (proper) inclusion

$$L^1\big(m_h^{(p)}\big) \subsetneqq L^1(G).$$

Finally, the (proper) inclusion

$$L^p(G) \subsetneqq \bigcup_{1<u<p} L^u(G)$$

is known; see the discussion immediately after (7.1), for example.

(iii) Fix any $1 < v < p$. Choose $u \in (1,p)$ and $r \in (1,p)$ which satisfy $(p/u) = v$ and $(1/u) + (1/r) = (1/p) + 1$. Since $h \in L^r(G)$, Proposition 7.96(iii) implies that $C_h^{(p)}$ is v-th power factorable.

However, by part (ii) we have

$$L^p(G)_{[p]} = L^1(G) \nsubseteq L^1\big(m_h^{(p)}\big),$$

and so the μ-determined operator $C_h^{(p)}$ is *not* p-th power factorable (by Theorem 5.7). □

In the setting of Proposition 7.98, part (iii) shows that for $h \in L(G : p^-)$ there is no largest number v_0 such that the convolution operator $C_h^{(p)} \in \mathcal{L}\big(L^p(G)\big)$ is v_0-th power factorable.

We now turn to general measures, for which the following fact will be useful.

Proposition 7.99. *Let $\lambda \in M(G)$ be L^q-improving and $1 < u < \infty$. Then, for every $u < r < \infty$ such that $\lambda \in M^{u,r}(G)$ we have*

$$L^u(G) \subseteq L^1\big(m_\lambda^{(r)}\big) \tag{7.119}$$

with a continuous inclusion. Moreover, $C_\lambda^{(r)}$ is p-th power factorable for every $1 \le p \le (r/u)$.

Proof. Observe that $L^r(G) \subseteq L^u(G)$ continuously and the bounded linear operator $C_\lambda^{u,r} : L^u(G) \to L^r(G)$ is an extension of the μ-determined operator $C_\lambda^{(r)}$. By optimality of the space $L^1(m_\lambda^{(r)})$ we necessarily have (7.119).

Let $1 \le p \le (r/u)$. Arguing as in the proof of part (iii) of Proposition 7.96 it suffices to verify that $L^r(G)_{[p]} \subseteq L^1\big(m_\lambda^{(r)}\big)$. But, $1 \le p \le (r/u)$ implies that $u \le (r/p)$ and hence, $L^r(G)_{[p]} = L^{r/p}(G) \subseteq L^u(G)$. Then (7.119) ensures that indeed $L^r(G)_{[p]} \subseteq L^1\big(m_\lambda^{(r)}\big)$, as required. □

Corollary 7.100. *Let $\lambda \in M(G)$ be an L^q-improving measure and $1 < r < \infty$ be arbitrary. Then there exists $1 < s < r$ such that $C_\lambda^{(r)}$ is p-th power factorable for all $1 \le p \le s$.*

Proof. Let $1 < q_1 < q_2 < \infty$ be arbitrary (but fixed). Define

$$p_j(\alpha) := \frac{q_j}{\alpha + (1-\alpha)q_j}, \qquad \alpha \in (0,1),$$

for $j = 1,2$. It is routine to check that $p_1(\alpha) < p_2(\alpha)$ and $1 < p_1(\alpha) < q_1$ for all $\alpha \in (0,1)$. Moreover, given $\alpha \in (0,1)$, it is clear that

$$q_1 < p_2(\alpha) \quad \text{if and only if} \quad \frac{\alpha q_1}{q_2} + (1-\alpha)q_1 < 1.$$

The continuous function $f(\alpha) := \frac{\alpha q_1}{q_2} + (1-\alpha)q_1$ on $[0,1]$ satisfies $f(1) = \frac{q_1}{q_2} < 1$. It follows that there exists $\alpha_0 \in (0,1)$ such that $f(\alpha) < 1$ for all $\alpha \in (\alpha_0, 1)$. That is, $q_1 < p_2(\alpha)$ for all $\alpha \in (\alpha_0,1)$. Now, for each $\alpha \in (\alpha_0,1)$ it follows from [72, Theorem 0.2] that

$$\lambda * L^{p_1(\alpha)}(G) \subseteq L^{p_2(\alpha)}(G) \subseteq L^{q_1}(G), \tag{7.120}$$

that is, $\lambda \in M^{p_1(\alpha),\, q_1}(G)$.

Define now $q_1 := r > 1$ and $u := p_1(\alpha)$ for any $\alpha \in (\alpha_0,1)$, in which case $u < r$. According to (7.120) we have $\lambda * L^u(G) \subseteq L^r(G)$, that is, $\lambda \in M^{u,r}(G)$. Then $s := (r/u)$ has the desired property; see Proposition 7.99. □

Remark 7.101. Suppose that $\lambda \in M(G)$ is *not* an L^q-improving measure; see Proposition 7.94 and Remark 7.95 for the existence of such λ (also, any $\lambda \notin M_c(G)$ will do). Then, for every $1 < r < \infty$ the operator $C_\lambda^{(r)}$ *fails* to be p-th power factorable for every $1 < p < r$. On the contrary, suppose that $C_\lambda^{(r)} \in \mathcal{F}_{[p]}\big(L^r(G), L^r(G)\big)$ for some $1 < p < r$. Then $1 < (r/p) < r$ and $C_\lambda^{(r)}$ has a (unique) continuous $L^r(G)$-valued extension $(C_\lambda^{(r)})_{[p]}$ to $L^r(G)_{[p]} = L^{r/p}(G) \subseteq L^1(G)$. So, $\big(C_\lambda^{(r)}\big)_{[p]}(g) = C_\lambda^{(r)}(g) = \lambda * g$ for all $g \in L^r(G)$. Given any $f \in L^{r/p}(G)$ there exists a sequence $\{f_n\}_{n=1}^\infty \subseteq \mathcal{T}(G) \subseteq L^r(G)$ such that $f_n \to f$ in $L^{r/p}(G)$ as $n \to \infty$. Since $C_\lambda^{(r/p)} \in \mathcal{L}\big(L^{r/p}(G)\big)$, it follows that $f_n * \lambda \to f * \lambda$ in $L^{r/p}(G)$ as $n \to \infty$. But,

$$f_n * \lambda \,=\, C_\lambda^{(r)}(f_n) \,=\, \big(C_\lambda^{(r)}\big)_{[p]}(f_n), \qquad n \in \mathbb{N},$$

and so the continuity of $\big(C_\lambda^{(r)}\big)_{[p]}$ ensures that $\big(C_\lambda^{(r)}\big)_{[p]}(f_n) \to \big(C_\lambda^{(r)}\big)_{[p]}(f)$ in $L^r(G)$ as $n \to \infty$ and hence, also in $L^{r/p}(G)$ as $n \to \infty$ (since $L^r(G) \subseteq L^{r/p}(G)$ continuously). Accordingly,

$$\big(C_\lambda^{(r)}\big)_{[p]}(f) \,=\, f * \lambda, \qquad f \in L^{r/p}(G), \tag{7.121}$$

with the equality as elements of $L^{r/p}(G)$. Since the left-hand side of (7.121) actually belongs to $L^r(G)$ so must the right-hand side. Accordingly, the (unique) continuous extension $\big(C_\lambda^{(r)}\big)_{[p]} : L^{r/p}(G) \to L^r(G)$ is precisely the operator of convolution with λ and hence, $\lambda \in M^{(r/p),\,r}(G)$. This contradicts the hypothesis that λ is not L^q-improving. □

We record formally a fact that was just established in Remark 7.101. In a certain sense it is a "converse" to Proposition 7.99 and Corollary 7.100.

Proposition 7.102. *Let $\lambda \in M(G)$ and $1 < r < \infty$. If $C_\lambda^{(r)}$ is p-th power factorable for some $1 < p < r$, then λ is necessarily L^q-improving and the unique continuous extension $\big(C_\lambda^{(r)}\big)_{[p]} : L^r(G)_{[p]} \to L^r(G)$ of $C_\lambda^{(r)}$ is precisely the convolution operator $C_\lambda^{(r/p),\,r} \in \mathcal{L}\big(L^{r/p}(G), L^r(G)\big)$. In particular, $L^{r/p}(G) \subseteq L^1\big(m_\lambda^{(r)}\big)$.*

Characterizations of L^q-improving measures λ in terms of the decay of $\widehat{\lambda}$ are well known, [74, Theorem], [132, Theorem]. The following result gives a different kind of characterization.

Corollary 7.103. *Let $\lambda \in M(G)$. The following assertions are equivalent.*

(i) *λ is an L^q-improving measure.*

(ii) *There exists $1 < r < \infty$ such that the convolution operator $C_\lambda^{(r)}$ is p-th power factorable for some $p \in (1, \infty)$.*

(iii) *For every* $1 < r < \infty$ *there exists some* $p \in (1, \infty)$ *such that the convolution operator* $C_\lambda^{(r)}$ *is* p*-th power factorable.*

Proof. (i) $\Rightarrow$ (iii). This follows from Corollary 7.100.

(ii) $\Rightarrow$ (i). This is immediate from Proposition 7.102.

(iii) $\Rightarrow$ (ii) is clear. □

Remark 7.104. (i) For measures $\lambda \in M(G)$ which satisfy $\beta_\lambda > 0$ (see (7.112) for the definition of β_λ) the following statements are known to be equivalent.

(a) λ *is* L^q*-improving.*

(b) $\operatorname{supp}(\widehat{\lambda})$ *is a finite subset of* Γ*.*

(c) $\lambda = \mu_h$ *for some* $h \in \mathcal{T}(G)$*.*

For (a) $\Leftrightarrow$ (b) we refer to [74, Corollary 4] and for (b) $\Leftrightarrow$ (c) to Lemma 7.87(i).

What about measures λ for which $\beta_\lambda = 0$? For instance, if $\lambda \in M_0(G)$, then surely $\beta_\lambda = 0$. In this case (b) and (c) remain equivalent (cf. Lemma 7.87(i)) and, when satisfied, surely imply that λ is L^q-improving. However, the converse is false; measures of the form $\lambda = \mu_h$ for suitable $h \in L^1(G) \setminus \mathcal{T}(G)$ belong to $M_0(G)$, are L^q-improving but, fail to satisfy (b) and (c); see Remark 7.95(ii).

The Riesz product measure $\lambda \geq 0$ of Example 7.92(ii) satisfies $\lambda \notin M_0(\mathbb{T})$ and $\beta_\lambda = 0$. Since

$$\sup\{|\widehat{\lambda}(n)| : n \in \mathbb{Z} \setminus \{0\}\} = \frac{1}{2} < 1,$$

it follows from [132, Theorem] that λ is necessarily L^q-improving. According to Proposition 7.93(iv) and the fact that $\lambda \notin M_0(\mathbb{T})$, there must exist $1 < q < 2$ such that $\lambda \notin M^{q,2}(\mathbb{T})$.

(ii) The measure λ of Example 7.92(ii) also exhibits other interesting features. According to Corollary 7.100, whenever $1 < r < \infty$ is given, the operator $C_\lambda^{(r)}$ is p-th power factorable for some $1 < p < r$. However, $C_\lambda^{(r)}$ is not compact (by Proposition 7.58 as $\lambda \notin M_0(\mathbb{T})$) and has totally infinite variation (by Theorem 7.67 as $\lambda \notin M_0(\mathbb{T})$ implies that $\lambda \notin L^r(\mathbb{T})$). Furthermore, both of the inclusions

$$L^r(\mathbb{T}) \subseteq L^1\big(m_\lambda^{(r)}\big) \subseteq L^1(\mathbb{T})$$

are necessarily *proper*. Indeed, since $C_\lambda^{(r)}$ is p-th power factorable for some value of $1 < p < r$ we have (by Proposition 7.102) that $L^{r/p}(\mathbb{T}) \subseteq L^1\big(m_\lambda^{(r)}\big)$ with $(r/p) < r$. Hence, $L^r(\mathbb{T}) \subsetneqq L^{r/p}(\mathbb{T})$ shows that the first inclusion is proper. Since $\lambda \notin M_0(\mathbb{T})$ implies that $\lambda \notin \{\mu_h : h \in L^r(\mathbb{T})\}$, the second inclusion is proper by Theorem 7.67. □

Is was observed after Theorem 7.61 that

$$L^1\big(m_\lambda^{(p)}\big) = N_\lambda^{(p)} := \big\{f \in L^1(G) : |f| * \lambda \in L^p(G)\big\}$$

whenever $\lambda \in M(G)^+$ and that

$$N^{(p)}_{|\lambda|} \subseteq L^1\big(m^{(p)}_\lambda\big) \tag{7.122}$$

for arbitrary $\lambda \in M(G)$; see also Remark 7.45(iv). The following result shows that the inclusion (7.122) can be strict. For the existence of L^q-improving measures λ such that $|\lambda|$ is not L^q-improving we refer to Proposition 7.94 and Remark 7.95.

Proposition 7.105. *Let $\lambda \in M(G)$ be any L^q-improving measure such that $|\lambda|$ is not L^q-improving for some (all) $1 < q < \infty$. Then the containment $N^{(r)}_{|\lambda|} \subseteq L^1\big(m^{(r)}_\lambda\big)$ is proper for every $q < r < \infty$ such that $\lambda * L^q(G) \subseteq L^r(G)$.*

Proof. On the contrary, suppose that $N^{(r)}_{|\lambda|} = L^1\big(m^{(r)}_\lambda\big)$. According to Proposition 7.99 we then have

$$L^q(G) \subseteq L^1\big(m^{(r)}_\lambda\big) = N^{(r)}_{|\lambda|}.$$

That is, for each $f \in L^q(G)$, the function $|f| * |\lambda| \in L^r(G)$. Since $\big|f * |\lambda|\,\big| \leq |f| * |\lambda|$, also $f * |\lambda| \in L^r(G)$. That is, $|\lambda| * L^q(G) \subseteq L^r(G)$, contradicting the fact that the measure $|\lambda|$ is not L^q-improving. □

Remark 7.106. (i) In the setting of Proposition 7.105 we note that $L^s(G) \not\subseteq N^{(r)}_{|\lambda|}$ for every $q \leq s < r$. Indeed, if $L^s(G) \subseteq N^{(r)}_{|\lambda|}$ for some $q \leq s < r$, then the argument used in the proof of Proposition 7.105 shows that $|\lambda| * L^s(G) \subseteq L^r(G)$. This contradicts the fact that $|\lambda|$ is not L^q-improving.

(ii) Suppose that $\lambda \in M(G)$ is *not* an L^q-improving measure. Then, for each $1 < r < \infty$, the space $L^r(G)$ is the *largest* amongst the spaces $\{L^u(G)\}_{1<u<\infty}$ which is contained in $L^1\big(m^{(r)}_\lambda\big)$. That $L^r(G) \subseteq L^1\big(m^{(r)}_\lambda\big)$ follows via optimality. Suppose that $L^u(G) \subseteq L^1\big(m^{(r)}_\lambda\big)$ for some $1 < u < r$. Then Theorem 7.61(vi) implies that

$$\lambda * L^u(G) = I_{m^{(r)}_\lambda}\big(L^u(G)\big) \subseteq I_{m^{(r)}_\lambda}\big(L^1(m^{(r)}_\lambda)\big) \subseteq L^r(G)$$

and hence, λ is L^q-improving, contrary to the choice of λ.

If, in addition to being not L^q-improving, also $\lambda \in M_0(G)$, then $L^r(G) \subsetneqq L^1\big(m^{(r)}_\lambda\big)$ by Proposition 7.83 (for the existence of such λ we refer to Proposition 7.94(iii) and Remark 7.95(ii)). So, even though λ does *not* convolve $L^u(G)$ into $L^r(G)$ for any $u < r$, it *does* convolve the B.f.s $L^1\big(m^{(r)}_\lambda\big)$, which is *genuinely larger* than $L^r(G)$, into $L^r(G)$. From this point of view, for arbitrary $\lambda \in M(G)$, the optimal domain space $L^1\big(m^{(r)}_\lambda\big)$ is also the largest B.f.s. $X(\mu)$ with σ-o.c. norm which contains $L^r(G)$ and such that $\lambda * X(\mu) \subseteq L^r(G)$. If we call λ a *B.f.s.-improving measure* whenever the inclusion $L^r(G) \subseteq L^1\big(m^{(r)}_\lambda\big)$ is proper (for some $1 < r < \infty$), then an examination of the proof of Corollary 7.100 shows that every

L^q-improving measure is necessarily B.f.s.-improving. It was just argued above that every $\lambda \in M_0(G)$ is necessarily B.f.s.-improving but, *not* necessarily L^q-improving. So, the notion of a B.f.s.-improving measure is a genuine extension of that of an L^q-improving measure and hence, may be worthy of further investigation. For an example of a continuous measure in $M_c(G) \setminus M_0(G)$ which is B.f.s.-improving we refer to Remark 7.104(ii). Section 7.4 provides many examples of measures which fail to be B.f.s.-improving. $\square$

As already seen, whenever they are defined, the convolution operators $C_\lambda^{q,r}$ for $1 < q < r < \infty$ play an important role. We note that the vector measure $m_{C_\lambda^{q,r}} : \mathcal{B}(G) \to L^r(G)$ induced by $C_\lambda^{q,r} : L^q(G) \to L^r(G)$ is *precisely* the vector measure $m_\lambda^{(r)}$. Accordingly, every operator $C_\lambda^{q,r}$ is necessarily μ-determined (as $C_\lambda^{(r)}$ is) and $L^1\big(m_\lambda^{q,r}\big) = L^1\big(m_\lambda^{(r)}\big)$, where $m_\lambda^{q,r}$ denotes $m_{C_\lambda^{q,r}}$. So, even though the domain space $L^q(G)$ of $C_\lambda^{q,r}$ is genuinely larger than that of $C_\lambda^{(r)}$, both $C_\lambda^{(r)}$ and $C_\lambda^{q,r}$ have the same *optimal domain*. Of course, this is to be expected as $C_\lambda^{q,r}$ is an extension of $C_\lambda^{(r)}$. In particular, $I_{m_\lambda^{q,r}} = I_{m_\lambda^{(r)}}$. The following result concerning the operators $C_\lambda^{q,r}$ is of some interest; it shows that "L^q-improving" is also "compact-improving".

Proposition 7.107. *Let $\lambda \in M_0(G)$ be an L^q-improving measure. Given $1 < r < \infty$, there exists $1 < u < r$ such that $\lambda \in M^{u,r}(G)$ and $C_\lambda^{s,r} : L^s(G) \to L^r(G)$ is compact for every $u < s \leq r$.*

Proof. An examination of the proof of Corollary 7.100 shows that there exists $1 < u < r$ such that $\lambda \in M^{u,r}(G)$. Proposition 7.58 ensures that $C_\lambda^{(r)} \in \mathcal{L}\big(L^r(G)\big)$ is compact. So, $C_\lambda^{u,r} : L^u(G) \to L^r(G)$ is continuous and $C_\lambda^{(r)} : L^r(G) \to L^r(G)$ is compact. By a standard interpolation result, [90, Theorem 3.10], it follows that $C_\lambda^{s,r} : L^s(G) \to L^r(G)$ is compact for every $u < s \leq r$. $\square$

Remark 7.108. There is also a converse to Proposition 7.107. Namely, if we have $1 < q < r < \infty$ and $\lambda \in M^{q,r}(G)$ has the property that $C_\lambda^{s,r}$ is compact for some $q < s < r$, then necessarily $\lambda \in M_0(G)$. Indeed, let $J : L^r(G) \to L^s(G)$ be the natural inclusion, in which case $C_\lambda^{(r)} = C_\lambda^{s,r} \circ J$. Then $C_\lambda^{(r)}$ is compact and so $\lambda \in M_0(G)$ by Proposition 7.58. $\square$

In conclusion, we point out that the situation for p-th power factorability of the *Fourier transform map* is quite different to that for convolution operators. We restrict attention to $G = \mathbb{T}$ (also $\mathbb{T}^d$ is allowable).

Proposition 7.109. *Let $1 < r < 2$. The Fourier transform map $F_r : L^r(\mathbb{T}) \to \ell^{r'}(\mathbb{Z})$ fails to be p-th power factorable for every $1 < p < \infty$.*

Proof. Let $X(\mu) := L^r(\mathbb{T})$ and $T := F_r$. According to Theorem 5.7 we have $F_r \in \mathcal{F}_{[p]}\big(X(\mu), \ell^{r'}(\mathbb{Z})\big)$ if and only if $X(\mu)_{[p]} \subseteq L^1(m_T)$, that is, if and only if $L^{r/p}(\mathbb{T}) \subseteq \mathbf{F}^r(\mathbb{T})$. Since always $(r/p) < r$, this never happens; see Remark 7.29(ii). $\square$

Getting back to a general G, we may also consider the Fourier transform map as being $c_0(\Gamma)$-valued rather than $\ell^{r'}(\Gamma)$-valued. In this case we have the following

Proposition 7.110. *Let* $1<r<2$. *The Fourier transform map* $F_{r,0}: L^r(G) \to c_0(\Gamma)$ *is* p*-th power factorable if and only if* $1 \leq p \leq r$.

Proof. Let $X(\mu) := L^r(G)$ and $T := F_{r,0}$. According to Theorem 5.7 we have $F_{r,0} \in \mathcal{F}_{[p]}\big(X(\mu),\, c_0(\Gamma)\big)$ if and only if $X(\mu)_{[p]} \subseteq L^1(m_T)$. Since $L^1(m_T) = L^1(G)$ (see Proposition 7.3(ii)), this is the case if and only if $L^{r/p}(G) \subseteq L^1(G)$, i.e., $1 \leq p \leq r$. $\square$

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List of Symbols

$(X_0, X_1)_\rho$, 202
(Ω, Σ, μ), 18
$(\cdot, \gamma)\widehat{\ }$, 298
$(\cdot, \gamma)$, 296
(x, γ), 296
$[-g, g]$, 56
$|U|$, 81
$\dot{+}$, 346
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